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SEMICLASSICAL LIMIT AND WELL-POSEDNESS OF NONLINEAR SCHRÖDINGER-POISSON SYSTEMS

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Abstract. This paper concerns the well-posedness and semiclassical limit of nonlinear Schrödinger-Poisson systems. We show the local well-posedness and the existence of semiclassical limit of the two models for initial data with Sobolev regularity, before shocks appear in the limit system. We establish the existence of a global solution and show the time-asymptotic behavior of a classical solutions of Schrödinger-Poisson system for a fixed re-scaled Planck constant.

1. Introduction

Equations of (nonlinear) Schrödinger type appear in areas of physics such as quantum fluid mechanics (superfluid film), superconductivity, semiconductor, plasma, electromagnetism, etc. [18, 7, 24, 27, 23, 1]. In the present paper, we consider the Cauchy problem for the nonlinear Schrödinger-Poisson (SP) system

$$i\epsilon\,\psi^\epsilon_t + \frac{\epsilon^2}{2}\Delta\psi^\epsilon - (V^\epsilon(x,t) + f'(|\psi^\epsilon|^2))\psi^\epsilon - (\arg\psi^\epsilon)\psi^\epsilon = 0\,, \tag{1.1}$$

$$-\Delta V^\epsilon = |\psi^\epsilon|^2 - \mathcal{C}(x), \quad V \to 0 \text{ as } |x| \to \infty. \tag{1.2}$$

subject to the rapidly oscillating (WKB) initial condition

$$\psi^\epsilon(x,0) = \psi^\epsilon_0(x) = A^\epsilon_0(x)\exp\big(\frac{i}{\epsilon}S_0(x)\big)\,, \tag{1.3}$$

where $f \in C^\infty(\mathbb{R}^+;\mathbb{R})$, $S_0 \in H^s(\mathbb{R}^N)$, $N \geq 1$, for s large enough, and A^ϵ_0 is a function, polynomial in ϵ, with coefficients of Sobolev regularity in x. The scaled Planck constant is here denoted by ϵ. The superscript ϵ in the wave function $\psi^\epsilon(x,t)$ and in the electric potential V^ϵ indicates the ϵ-dependence. The function $\mathcal{C}(x) > 0$ denotes the background ions. The function f depends only on the particle density ρ^ϵ defined by

$$\rho^\epsilon(x,t) = \bar{\psi}^\epsilon(x,t)\psi^\epsilon(x,t), \tag{1.4}$$

where the bar on top, $\bar{\psi}$, denotes complex conjugation. The last nonlinear term serves as a friction damping of phase, used recently in modelling semiconductor

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devices [16], where the argument $\arg \psi^\epsilon = S^\epsilon$ is defined for irrotational flow by

$$\rho^\epsilon \nabla S^\epsilon = \frac{\epsilon}{2i}(\bar{\psi}^\epsilon \nabla \psi^\epsilon - \psi^\epsilon \nabla \bar{\psi}^\epsilon). \tag{1.5}$$

When we introduce the geometric optic ansatz

$$\psi^\epsilon(x,t) = A^\epsilon(x,t) \exp\big(\frac{i}{\epsilon} S^\epsilon(x,t)\big) = \sqrt{\rho^\epsilon(x,t)} \exp\big(\frac{i}{\epsilon} S^\epsilon(x,t)\big), \tag{1.6}$$

the so-called Madelung's transformation and define the hydrodynamical variables ρ^ϵ as (1.4), velocity $\mathbf{u}^\epsilon$ and momentum J^ϵ by

$$\mathbf{u}^\epsilon = \nabla S^\epsilon, \quad J^\epsilon = \rho^\epsilon \mathbf{u}^\epsilon, \tag{1.7}$$

we have the following quantum hydrodynamic form of the Schrödinger-Poisson system (1.1)-(1.2)

$$\partial_t \rho^\epsilon + \operatorname{div}(\rho^\epsilon \mathbf{u}^\epsilon) = 0, \tag{1.8}$$

$$\partial_t(\rho^\epsilon \mathbf{u}^\epsilon) + \operatorname{div}\left(\rho^\epsilon \mathbf{u}^\epsilon \otimes \mathbf{u}^\epsilon\right) + \nabla P(\rho^\epsilon) + \rho^\epsilon \nabla V^\epsilon + \rho^\epsilon \mathbf{u}^\epsilon = \frac{\varepsilon^2}{4} \operatorname{div}\left(\rho^\epsilon \nabla^2 \log \rho^\epsilon\right), \tag{1.9}$$

$$-\Delta V^\epsilon = \rho^\epsilon - \mathcal{C}(x), \tag{1.10}$$

with initial data

$$\rho^\epsilon(x,0) = \rho_0^\epsilon(x), \quad \mathbf{u}^\epsilon(x,0) = \mathbf{u}_0^\epsilon(x). \tag{1.11}$$

Here the hydrodynamics pressure $P(\rho)$ is related to the nonlinear potential $f(\rho)$ by

$$P(\rho) = \rho f'(\rho) - f(\rho). \tag{1.12}$$

Equations (1.8)–(1.9) comprise a closed system governing ρ^ϵ and $\mathbf{u}^\epsilon$ with potential V^ϵ given by the Poisson equation (1.10) which has a form of a perturbation of the Euler-Poisson system. Letting $\varepsilon \to 0+$, we have formally the following Euler-Poisson system (the classical hydrodynamic model of semiconductors)

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.13}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u}\right) + \nabla P(\rho) + \rho \nabla V + \rho \mathbf{u} = 0, \tag{1.14}$$

$$-\Delta V = \rho - \mathcal{C}(x), \tag{1.15}$$

which can be seen formally as the dispersive (semiclassical) limit of the Schrödinger-Poisson system.

The mathematical rigorous analysis of the semiclassical limit for Schrödinger type equations is an issue of much importance and full of complication. The elementary principle of quantum mechanics informs that the (classical) Newton mechanics will dominate in a system when the space and time scale is larger enough than the Planck constant ε (quantum effect). The mathematical analogue of the principle is that as $\varepsilon \to 0$, the system of quantum mechanics becomes the one obeying Newton mechanics, which is called the "semiclassical" limit.

Recently, much progress has been made in such area. For linear Schrödinger equation or Schrödinger-Poisson, the idea of kinetic formulation to solve it global-in-time is the followings. By applying the Wigner transforms, we can obtain a kinetic integral-differential equation–the so-called Wigner equation.

The investigation of kinetic structure of the Wigner equation and the application of the moments methods to its solutions, which yield information of macroscopic densities, help us to pass limit $\varepsilon \to 0+$ in the Wigner equation and the macroscopic densities. We have the Vlasov (Vlasov-Poisson) equation, which is the quantum (hydrodynamic) limiting system of the linear Schrödinger type equations [27, 22, 28, 12, 2, 33]. The analysis of the limiting system gives us the similar macroscopic densities and results to those obtained by the geometric optics approach to the WKB limit of Schrödinger equations and reveals a close relation between the dispersive limit of quantum fluid equations and the kinetic equations [10].

However, it is quite different for nonlinear Schrödinger type equations because the theory of Wigner transforms passing limit $\varepsilon \to 0+$ are still under investigation for nonlinear Schrödinger type equations. Up to now, the mathematical rigorous theory is only established for the one-dimensional defocusing cubic nonlinear Schrödinger equation where the inverse scattering technique for the integrable system was used to obtain the global characterization of the weak limit of the entire nonlinear Schrödinger hierarchy [15], for multi-dimensional nonlinear (including derivative or modified) Schrödinger equations included by the WKB-hierarchy [13, 5, 6, 20, 19] by applying Lax-Friedrich-Kato's quasi-linear symmetric hyperbolic theory in Sobolev space before vortices where due to the spatial vanishing of wave function at infinity a strictly convex entropy was required, and also in [11] for analytic initial data. Moreover, the rigorous incompressible limit analysis of nonlinear Schrödinger equation to incompressible fluids with vortices involved was proven in $\mathbb{R}^2$ [21]. For more detailed review on such topics, one is referred to [9, 32] and the references therein.

For nonlinear Schrödinger-Poisson system (1.1)–(1.2), it is far from well understood on the well-posedness and semiclassical limit. As $\mathcal{C} = 0$ the theory of well-posedness, scattering phenomena, stability of soliton wave, finite time blow-up and so on for (1.16)–(1.17) is well understood [4, 31, 32]. As $\mathcal{C}_0 > 0$ in our case, however, the Poisson coupling requires that $|\psi^\epsilon(x,t)| = \mathcal{C}_0 > 0$ as $|x| \to \infty$, which implies that the value of phase depends on the direction at space infinity. Only the existence of (nonconstant) travelling-wave solutions with nontrivial boundary condition at space infinity is proven for a nonlinear Schrödinger equation in $\mathbb{R}^2$ in terms of conserved Hamiltonian [3]. For Schrödinger-Poisson system (1.1)–(1.2), however, it is quite different since there is no (conserved and nonnegative) Hamiltonian. In fact, even for the following Schrödinger-Poisson system

$$i\epsilon\,\psi_t^\epsilon + \frac{1}{2}\epsilon^2\Delta\psi^\epsilon - (V^\epsilon(x,t) + f'(|\psi^\epsilon|^2))\psi^\epsilon = 0\,, \tag{1.16}$$

$$-\Delta V^\epsilon = |\psi^\epsilon|^2 - \mathcal{C}_0, \tag{1.17}$$

with $\mathcal{C}_0 > 0$ a constant, the conservation laws only hold in the following sense

$$\int_{\mathbb{R}^N} (|\psi^\epsilon(x,t)|^2 - \mathcal{C}_0)dx = \int_{\mathbb{R}^N} (|\psi^\epsilon(x,0)|^2 - \mathcal{C}_0)dx, \tag{1.18}$$

$$\begin{aligned}&\int_{\mathbb{R}^N} \left(\frac{1}{4}|\nabla\psi^\epsilon|^2 + \frac{1}{4}|\nabla V^\epsilon(x,t)|^2 + f(|\psi^\epsilon|^2) - f(\mathcal{C}_0)\right)(x,t)dx\\ &= \int_{\mathbb{R}^N} \left(\frac{1}{4}|\nabla\psi^\epsilon|^2 + \frac{1}{4}|\nabla V^\epsilon(x,0)|^2 + f(|\psi^\epsilon|^2) - f(\mathcal{C}_0)\right)(x,0)dx\end{aligned} \tag{1.19}$$

which does not give the a-priori bounds of energy and density.

Our goals in the present paper are: a) present a local well-posedness theory of Schrödinger-Poisson system (1.1)–(1.2) and a justification of the semiclassical limit from Schrödinger-Poisson (1.1)–(1.2) to Euler-Poisson (1.13)–(1.15); b) establish global existence and long time behavior of (1.1)–(1.2) in $\mathbb{R}^N$ ($N \geq 1$).

To obtain local existence and perform the dispersive limit, the idea here is to transform the Schrödinger-Poisson system as a dispersive perturbation of the Euler-Poisson system in the form of quasi-linear symmetric hyperbolic system by Modified Madelung transform [13] to which the Lax-Friedrich-Kato's theory can be applied as [13, 5, 6, 20]. Notice that the associated potential V^ϵ determined by (1.2) is served as an external force potential and the amplitude of the wave function should be a complex-valued function. Unlike [13, 5, 6, 20] we do not need that the entropy is strictly convex near vacuum. However, due to the hyperbolic nature of the limiting system, it works before the shock singularity.

We show that, for certain initial data, a) solutions of the IVP for (1.1)–(1.3) exist on a time interval $[0, T]$, where T is independent of ϵ; and b) solutions of the IVP for (1.1)–(1.3) converge to solutions of the IVP for (1.13)–(1.15), as $\epsilon \to 0$. Indeed, applying the theory of the quasi-linear symmetric hyperbolic system we will obtain the existence of smooth solutions ψ^ϵ of (1.1)–(1.3) on a time interval $[0, T)$ independent of ϵ. Furthermore, the bounds that we obtained are uniformly bounded in ϵ on the solutions ψ^ϵ will allow to pass to the limit $\epsilon \to 0$ in (2.9)–(2.10) and justify the WKB hierarchy (Theorem 2.1). In addition, to ensure the strong convergence of ψ^ϵ to a classical solution of the Euler-Poisson system (1.13)–(1.15) we require the hypothesis that we are near the solutions of (1.13)–(1.15) initially (Theorem 2.2).

To prove the global existence and long time behavior of Schrödinger-Poisson system (1.1)–(1.2) for fixed $\epsilon > 0$, the idea is to make use of the hydrodynamical form of Schrödinger-Poisson system (1.1)–(1.2) so as to take advantage of the dissipations of nonlinear term and Poisson coupling and establish uniformly a-priori estimates by energy method. These extend local solution globally in time by a continuity argument. The problems to be overcome here are to keep the positivity of density (to make the Madelung's transformation valid) and to control the nonlinear dispersion term in Sobolev space. Instead of fluid equations (1.8)–(1.10), we use another equivalent system (3.5)–(3.7) (see section 3) for variables $(\varrho^\epsilon, \mathbf{u}^\epsilon, V^\epsilon)$ for amplitude, velocity and potential. We show that when initial data are a perturbation of a steady state of (1.1)–(1.2), the classical solution exists globally in time and tends to the steady state exponentially as time grows up (Theorem 3.1).

This paper is arranged as follows. In section 2, we consider the semiclassical limit and local well-posedness of classical solution of IVP (1.1)–(1.3). The global existence and long time behavior of IVP (1.1)–(1.3) is solved in section 3.

2. Well-posedness and semiclassical limit

2.1. **Main results.** Let $H^s(\mathbb{R}^N)$ denotes the usual Sobolev space of order s. First we prove the local existence of solutions to the Cauchy problem for Schrödinger-Poisson system (1.1)–(1.3) for each ϵ, we give sufficient conditions for the well-posedness in Sobolev space $H^s(\mathbb{R}^N)$. Also, we obtain a-priori uniform estimates with respect to ϵ in order to pass limit. For simplicity, we assume that $\mathcal{C}(x) = \mathcal{C}$ is a constant here and after.

Theorem 2.1. *Assume that* $\{|A_0^\epsilon| - \sqrt{\mathcal{C}}\}_\epsilon$ *is a uniformly bounded sequence in* $H^s(\mathbb{R}^N)$ *with compact support,* $S_0 \in H^{s+1}(\mathbb{R}^N)$ *with* ∇S_0 *compact supported,*

$s > (N+4)/2$, and $f \in C^\infty(\mathbb{R}^+, \mathbb{R})$ with $f''(\rho) > 0$ for $\rho > 0$. Then solutions $(\psi^\epsilon, V^\epsilon)$ of the Schrödinger-Poisson system (1.1) – (1.3) exist on a small time interval $[0,T]$, T independent of ϵ. Moreover, $\psi^\epsilon(x,t) = A^\epsilon(x,t)e^{iS^\epsilon(x,t)/\epsilon}$ with $A^\epsilon \in L^\infty([0,T]; H^s(\mathbb{R}^N))$ and $S^\epsilon \in L^\infty([0,T]; H^{s+1}(\mathbb{R}^N))$ uniformly in ϵ and V^ϵ given by (2.13).

To investigate the behavior of $(\psi^\epsilon, V^\epsilon)$ of the Schrödinger-Poisson system (1.1)–(1.2) as $\epsilon \to 0$, we construct a solution of IVP (1.1)–(1.3) with initial data near a classical solution of the Euler-Poisson system (1.13)–(1.15). In fact we have

Theorem 2.2. *Assume that $(\rho, \mathbf{u}, V)$ is a solution of the Euler-Poisson system (1.13)–(1.15) and satisfies $(\rho - \mathcal{C}, \mathbf{u}, V) \in C([0,T], H^{s+2}(\mathbb{R}^N))$, $s \geq (N+4)/2$, with initial condition*

$$\rho_0(x) = \rho(x,0) = |A_0(x)|^2,$$
$$\mathbf{u}_0(x) = \mathbf{u}(x,0) = \nabla S_0(x).$$

Then there exists a critical value of ϵ, ϵ_c dependent of T, such that under the hypothesis

1. *$A_0^\epsilon(x)$ converges strongly to A_0 in $H^s(\mathbb{R}^N)$ as ϵ tends to 0*
2. *$(\sqrt{\rho_0} - \sqrt{\mathcal{C}}, \mathbf{u}_0) \in H^s(\mathbb{R}^N)$ with compact support,*
3. *$0 < \epsilon < \epsilon_c$,*

the IVP for Schrödinger-Poisson system (1.1)–(1.3) has a unique classical solution $(\psi^\epsilon, V^\epsilon)$ on $[0,T]$, the wave function is of the form

$$\psi^\epsilon(x,t) = A^\epsilon(x,t) \exp\big(\frac{i}{\epsilon} S^\epsilon(x,t)\big)$$

with A^ϵ and ∇S^ϵ are bounded in $L^\infty([0,T]; H^s(\mathbb{R}^N))$ uniformly in ϵ and V^ϵ is given by (2.13). Moreover, as $\epsilon \to 0$ $(\rho^\epsilon, \rho^\epsilon \nabla S^\epsilon, V^\epsilon)$ with $\sqrt{\rho^\epsilon} = A^\epsilon$, converges strongly in $C([0,T], H^{s-2}(\mathbb{R}^N))$.

Remark 2.3. 1). The existence of (local or global) classical solution of Euler-Poisson system (1.13)–(1.15) was proved in [26, 8] for $\mathcal{C} = 0$, in [29] for $\mathcal{C} > 0$, and in [14] without frictional damping.

2). If $\mathcal{C}$ is a positive function of x, one can get the same results as Theorem 2.1–2.2 for spatial periodic case.

2.2. **Proof of the main Results.** To study the asymptotic behavior of solutions of the Schrödinger-Poisson system (1.1)–(1.3) as ϵ tends to zero we have to show the existence of a smooth solutions $(\psi^\epsilon, V^\epsilon)$ of (1.1)–(1.3) on a finite time $[0,T]$ independent of ϵ, for initial data $A_0^\epsilon(x)$, $S_0^\epsilon(x)$ and $V_0(x)$ with Sobolev regularity first. For classical solutions, it is convenient to write the Schrödinger-Poisson system (1.1)–(1.3) as a dispersive perturbation of a quasilinear symmetric hyperbolic system instead of quantum hydrodynamics model (1.8)–(1.10). As suggested by Grenier [13] (see also [5, 6, 9, 20]), the modified Madelung's transform can be utilized in the study of the semiclassical limit. More precisely, we will look for wave function ψ^ϵ of the form $\psi^\epsilon(x,t) = A^\epsilon(x,t)\exp(\frac{i}{\epsilon}S^\epsilon(x,t))$, where the complex-valued function $A^\epsilon = a^\epsilon + ib^\epsilon$ represents the amplitude and the real-valued function S^ϵ represents the phase. Considering the change of variable

$$w^\epsilon = \nabla S^\epsilon \tag{2.1}$$

and using the fact that $A^\epsilon = a^\epsilon + ib^\epsilon$ we have the equivalent form of (1.1)–(1.2) or (1.8)–(1.10);

$$\partial_t(a^\epsilon - a_1) + (w^\epsilon \cdot \nabla)(a^\epsilon - a_1) + \frac{1}{2}a^\epsilon \nabla \cdot w^\epsilon = -\frac{\epsilon}{2}\Delta b^\epsilon\,, \tag{2.2}$$

$$\partial_t(b^\epsilon - b_1) + (w^\epsilon \cdot \nabla)(b^\epsilon - b_1) + \frac{1}{2}b^\epsilon \nabla \cdot w^\epsilon = \frac{\epsilon}{2}\Delta a^\epsilon\,, \tag{2.3}$$

$$\partial_t w^\epsilon + (w^\epsilon \cdot \nabla)w^\epsilon + f''\nabla\big((a^\epsilon)^2 + (b^\epsilon)^2\big) + \nabla V^\epsilon + w^\epsilon = 0, \tag{2.4}$$

$$-\Delta V^\epsilon = (a^\epsilon)^2 + (b^\epsilon)^2 - \mathcal{C}. \tag{2.5}$$

with initial data

$$(a^\epsilon, b^\epsilon)(x,0) = (a_0^\epsilon, b_0^\epsilon)(x), \quad (a_0^\epsilon, b_0^\epsilon)(x) \to (a_1, b_1), \quad \text{as } |x| \to \infty, \tag{2.6}$$

$$w^\epsilon(x,0) = w_0^\epsilon(x)\,, \tag{2.7}$$

satisfying

$$\big(a_0^\epsilon(x)\big)^2 + \big(b_0^\epsilon(x)\big)^2 = |A_0^\epsilon(x)|^2\,, \quad a_1^2 + b_1^2 = \mathcal{C}, \quad w_0^\epsilon(x) = \nabla S_0^\epsilon(x)\,. \tag{2.8}$$

Here f'' is the abbreviation of $f''(\rho^\epsilon)$, $\rho^\epsilon = (a^\epsilon)^2 + (b^\epsilon)^2$. Notice that Eqs. (2.2)–(2.5) are not the same as Eqs. (1.8)–(1.10) where we split into the real and imaginary parts. Here it is split into the order $O(1/\epsilon)$ and $O(1)$ terms. Let us introduce $U^\epsilon = {}^t(a^\epsilon - a_1, b^\epsilon - b_1, w^\epsilon)$ with $w^\epsilon = (w_1^\epsilon, \dots, w_N^\epsilon)$ then this system can be written in the vector form

$$U_t^\epsilon + \sum_{j=1}^N A_j(U^\epsilon)U_{x_j}^\epsilon + U^\epsilon = B^\epsilon + \frac{\epsilon}{2}L(U^\epsilon), \tag{2.9}$$

$$U^\epsilon(x,0) = U_0^\epsilon(x) = {}^t(a_0^\epsilon(x) - a_1, b_0^\epsilon(x) - a_1, w_0^\epsilon(x)) \tag{2.10}$$

where $B^\epsilon = {}^t(a^\epsilon - a_1, b^\epsilon - b_1, -\nabla V^\epsilon)$ and the matrices A_j and L are given respectively by

$$A_j(U^\epsilon) \equiv \begin{pmatrix} w_j^\epsilon & 0 & \frac{1}{2}a^\epsilon e_j \\ 0 & w_j^\epsilon & \frac{1}{2}b^\epsilon e_j \\ 2a^\epsilon f''\ {}^te_j & 2b^\epsilon f''\ {}^te_j & w_j^\epsilon I \end{pmatrix}, \tag{2.11}$$

$$L(U^\epsilon) = \begin{pmatrix} 0 & -\Delta & O \\ \Delta & 0 & O \\ {}^tO & {}^tO & O \end{pmatrix} \begin{pmatrix} a^\epsilon - a_1 \\ b^\epsilon - b_1 \\ {}^tw^\epsilon \end{pmatrix} = \begin{pmatrix} -\Delta(b^\epsilon - b_1) \\ \Delta(a^\epsilon - a_1) \\ {}^tO \end{pmatrix}, \tag{2.12}$$

According to the Poisson equation, the potential is given explicitly in terms of the Newtonian potential

$$V^\epsilon(x,t) = -\int_{\mathbb{R}^N} \frac{\rho^\epsilon(y,t) - \mathcal{C}}{N(2-N)\omega_N|x-y|^{N-2}}dy \tag{2.13}$$

where $\rho^\epsilon(x,t) = |A^\epsilon(x,t)|^2 = (a^\epsilon(x,t))^2 + (b^\epsilon(x,t))^2$ and ω_N denotes the surface area of the unit sphere in $\mathbb{R}^N$. Thus we can rewrite B^ϵ as

$$B^\epsilon := B(\rho^\epsilon) = {}^t(a^\epsilon - a_1, b^\epsilon - b_1, g_1^\epsilon, \dots, g_N^\epsilon) \tag{2.14}$$

where $g_i^\epsilon(x,t), i = 1, \dots, N$, are given by

$$g_i^\epsilon(x,t) = \frac{\partial V^\epsilon}{\partial x_i} = -\int_{\mathbb{R}^N} \frac{x_i - y_i}{N\omega_N|x-y|^N}\big[\rho^\epsilon(y,t) - \mathcal{C}\big]dy. \tag{2.15}$$

Notice that I is a $N \times N$ identity matrix, $e_j = (\delta_{1j}, \delta_{2j}, \dots, \delta_{Nj})$ and L is an antisymmetric matrix. The matrices $A_j(U^\epsilon), j = 1, \dots, N$, can be symmetrized by

$$A_0(U^\epsilon) = \begin{pmatrix} 1 & 0 & O \\ 0 & 1 & O \\ {}^tO & {}^tO & \frac{1}{4f''}I \end{pmatrix} \tag{2.16}$$

which is symmetric and positive if $f'' > 0$, for all $U^\epsilon = {}^t(a^\epsilon - a_1, b^\epsilon - b_1, w^\epsilon)$. This means that f must be a strictly convex function of ρ and corresponds to the *defocusing* Schrödinger-Poisson system. Thus, we write (1.1)–(1.3) as a dispersive perturbation of a quasilinear symmetric hyperbolic system:

$$A_0(U^\epsilon)U_t^\epsilon + \sum_{j=1}^N \mathcal{A}_j(U^\epsilon)U_{x_j}^\epsilon + A_0(U^\epsilon)U^\epsilon = \mathcal{B}(\rho^\epsilon) + \frac{\epsilon}{2}\mathcal{L}(U^\epsilon)\,, \tag{2.17}$$

$$U^\epsilon(x,0) = U_0^\epsilon(x) \tag{2.18}$$

where $\mathcal{A}_j = A_0 A_j$ $(j = 1, \dots, N)$ is symmetric and $\mathcal{B} = A_0 B$. The importance of symmetry is that it leads to simple L^2 and more general H^s estimates which are often related to physical quantities like energy or entropy. The antisymmetric operator $\frac{\epsilon}{2}\mathcal{L} = \frac{\epsilon}{2}A_0 L$ reflects the dispersive nature of the Schrödinger-Poisson system. Moreover, due to the antisymmetry, the energy estimate shows that this term $\mathcal{L}$ contributes nothing to the estimate. The existence of the classical solutions proceeds along the lines of the existence proof for the initial value problem for the quasilinear symmetric hyperbolic system (see [17, 20]) with modifications. As usual, start from the initial data $U^0(x,t) = U_0^\epsilon(x)$ and define $U^{p+1}(x,t;\epsilon)$ inductively as solution of the linear equation; $(p = 1, 2, 3, \dots)$

$$A_0(U^p)U_t^{p+1} + \sum_{j=1}^N \mathcal{A}_j(U^p)U_{x_j}^{p+1} + A_0(U^p)U^{p+1} = \mathcal{B}(\rho^p) + \frac{\epsilon}{2}\mathcal{L}(U^{p+1})\,, \tag{2.19}$$

$$U^{p+1}(x,0) = U_0^\epsilon(x). \tag{2.20}$$

For further reference, we ignore the superscripts p and consider $U \in C^\infty$, $\widetilde{U} \in C^\infty$ satisfying

$$A_0(U)\widetilde{U}_t + \sum_{j=1}^N \mathcal{A}_j(U)\widetilde{U}_{x_j} + A_0(U)\widetilde{U} = G(t) + \frac{\epsilon}{2}\mathcal{L}(\widetilde{U})\,, \tag{2.21}$$

$$\widetilde{U}(x,0) = U_0^\epsilon(x), \tag{2.22}$$

where we rewrite $\mathcal{B}(\rho^p)$ as $G(t)$. Defining the canonical energy by

$$\|\widetilde{U}(t;\epsilon)\|_E^2 := \int \langle A_0\widetilde{U}, \widetilde{U}\rangle dx \tag{2.23}$$

we have the basic energy equality of Friedrich

$$\frac{d}{dt}\|\widetilde{U}(t;\epsilon)\|_E^2 + \|\widetilde{U}(t;\epsilon)\|_E^2 = \int \langle \Gamma\widetilde{U}, \widetilde{U}\rangle dx + 2\int \langle G(t), \widetilde{U}\rangle dx + \epsilon \int \langle \mathcal{L}(\widetilde{U}), \widetilde{U}\rangle dx \tag{2.24}$$

where $\Gamma = \operatorname{div}\vec{A} = (\partial_t, \nabla) \cdot (A_0, \mathcal{A}_1, \dots, \mathcal{A}_N)$. The term $\int \langle \mathcal{L}(\widetilde{U}), \widetilde{U}\rangle dx = 0$ by antisymmetry of $\mathcal{L}$. Assume that the matrices A_0 and $\mathcal{A}_j, j = 1, \dots, N$ together with their derivatives of any desired order are continuous and bounded uniformly in $[0,T]\times\mathbb{R}^N$. Moreover, the matrix A_0 is uniformly positive definite in the sense that there exists a $\mu>0$ such that $\langle A_0U,U\rangle\geq\mu\|U\|^2$, for all U and (x,t). Since Γ is bounded there will exists a constant M such that $|\langle\Gamma\widetilde{U},\widetilde{U}\rangle|\leq M\langle A_0\widetilde{U},\widetilde{U}\rangle$ for all (x,t). From Lemma 2.6 below it foloows that $G\in L^\infty([0,T];H^s(\mathbb{R}^N))\cap C([0,T];H^{s-2}(\mathbb{R}^N))$. Applying Cauchy-Schwarz then Gronwall inequalities, we obtain the energy inequality

$$\max_{0\leq t\leq T}\|\widetilde{U}(t;\epsilon)\|_{L^2(\mathbb{R}^N)}\leq\Big(\|U_0^\epsilon\|_{L^2(\mathbb{R}^N)}+\frac{MT}{\mu^2}\Big)e^{(M+3)T}\,. \tag{2.25}$$

Higher derivative estimates for $\widetilde{U}$ are obtained by differentiating (2.21), taking the inner product of the resulting equation with the corresponding derivative of $\widetilde{U}$, and applying the above procedure. We define $\widetilde{U}_\alpha$ by $\widetilde{U}_\alpha:=D^\alpha\widetilde{U}$ for $|\alpha|\leq s$ then

$$A_0(U)\frac{\partial\widetilde{U}_\alpha}{\partial t}+\sum_{j=1}^N\mathcal{A}_j(U)\frac{\partial\widetilde{U}_\alpha}{\partial x_j}+A_0(U)\widetilde{U}_\alpha=G^\alpha(t)+\frac{\epsilon}{2}\mathcal{L}(\widetilde{U}_\alpha)\,, \tag{2.26}$$

$$\widetilde{U}_\alpha(x,0)=D^\alpha U_0^\epsilon(x). \tag{2.27}$$

with G^α defined by the commutator terms as

$$G^\alpha=A_0(U)\Big(D^\alpha B-\sum_{j=1}^N\big[D^\alpha,A_j(U)\big]\frac{\partial\widetilde{U}}{\partial x_j}\Big) \tag{2.28}$$

Since $\frac{\partial\widetilde{U}_\alpha}{\partial x_j},\frac{\partial A_j(U)}{\partial x_i}\in H^s(\mathbb{R}^N)$, we can apply the Moser-type calculus inequality to estimate the commutator terms;

$$\big\|D^\mu\frac{\partial\widetilde{U}}{\partial x_j}D^\nu\frac{\partial A_j(U)}{\partial x_j}\big\|_{L^2(\mathbb{R}^N)}\leq C\big\|\frac{\partial\widetilde{U}}{\partial x_j}\big\|_{H^s(\mathbb{R}^N)}\big\|\frac{\partial A_j(U)}{\partial x_j}\big\|_{H^s(\mathbb{R}^N)}\leq CM_0^2 \tag{2.29}$$

provided $\|\widetilde{U}\|_{H^s(\mathbb{R}^N)}\leq 2M_0$. (Note that $\|A_j(U)\|_{H^s(\mathbb{R}^N)}\leq C\|U\|_{H^s(\mathbb{R}^N)}$.) Thus we have

$$\|G^\alpha\|_{L^2(\mathbb{R}^N)}\leq\|D^\alpha G\|_{L^2(\mathbb{R}^N)}+C_2M_0^2 \tag{2.30}$$

as long as $\|U\|_{H^s(\mathbb{R}^N)}\leq 2M_0$. This implies

$$\|\widetilde{U}(t)\|_{H^s(\mathbb{R}^N)}\leq(M_0+(C_3M_0^2+M)T)e^{C_3M_0T}\leq 2M_0 \tag{2.31}$$

for $t\in[0,T]$ provided that T is so small that the last inequality holds. The result is a solution U^ϵ on a time interval $[0,T]$ with T independent of ϵ satisfying

$$\|U^p(t;\epsilon)\|_{H^s(\mathbb{R}^N)}\leq C\,,\quad t\in[0,T] \tag{2.32}$$

as soon as $U_0^\epsilon\in H^s(\mathbb{R}^N)$. It follows from (2.19) and (2.32) that

$$\|\partial_tU^p(t;\epsilon)\|_{H^{s-2}(\mathbb{R}^N)}\leq C\,,\quad t\in[0,T]. \tag{2.33}$$

Therefore, for any fixed ϵ, we have constructed a sequence $\{U^p\}_{p=0}^\infty$ belonging to

$$C([0,T];H^s(\mathbb{R}^N))\cap C^1([0,T];H^{s-2}(\mathbb{R}^N)) \tag{2.34}$$

satisfying (2.19) and (2.20) as well as the uniform estimates

$$\max_{0\leq t\leq T}\big(\|\partial_tU^p(t;\epsilon)\|_{H^{s-2}(\mathbb{R}^N)}+\|U^p(t;\epsilon)\|_{H^s(\mathbb{R}^N)}\big)\leq C. \tag{2.35}$$

It follows from the Arzela-Ascoli theorem that there exists

$$U\in L^\infty([0,T];H^s(\mathbb{R}^N))\cap\mathrm{Lip}([0,T];H^{s-2}(\mathbb{R}^N)) \tag{2.36}$$

such that

$$\max_{0\le t\le T} \|U^p - U\|_{H^{s-2}(\mathbb{R}^N)} \to 0\,, \quad \text{as } p\to\infty. \tag{2.37}$$

Furthermore, for $0<\theta<2$ we have the convergence

$$U^p \to U \quad \text{in } C([0,T];H^{s-\theta}(\mathbb{R}^N)) \tag{2.38}$$

by the standard interpolation inequality. Choosing s such that $s-\theta-2>[N/2]$, then the space $H^s(\mathbb{R}^N)$ becomes an algebra. Indeed, we can show that

$$U\in C([0,T];H^s(\mathbb{R}^N))\cap C^1([0,T];H^{s-2}(\mathbb{R}^N)) \hookrightarrow C^1([0,T]\times\mathbb{R}^N)) \tag{2.39}$$

by Sobolev embedding theorem. Thus the solutions we construct are classical. The uniqueness of the classical solutions to the IVP for (2.19) and (2.20) follows from a straightforward energy estimate for the difference of two solutions. To show that $\rho^\epsilon(x,t)=(a^\epsilon(x,t))^2+(b^\epsilon(x,t))^2>0$ for all $0\le t<\infty$, we will employ the polar coordinates:

$$A^\epsilon = a^\epsilon + ib^\epsilon = \sqrt{\rho^\epsilon}e^{i\theta^\epsilon}. \tag{2.40}$$

Applying the chain rule to obtain

$$a^\epsilon\Delta b^\epsilon - b^\epsilon\Delta a^\epsilon = \operatorname{div}(\rho^\epsilon\nabla\theta^\epsilon) \tag{2.41}$$

then from (2.2)–(2.3) we derive the continuity equation for ρ^ϵ

$$\partial_t\rho^\epsilon + \operatorname{div}(\rho^\epsilon w^\epsilon + \epsilon\rho^\epsilon\nabla\theta^\epsilon) = 0 \tag{2.42}$$

which has an extra term of order $O(\epsilon)$ comparing with the usual continuity equation. We can interpret this as a classical transport equation disturbed by the quantum fluctuation. Let (ξ,τ) be an arbitrary fixed space-time point in $\mathbb{R}^N\times[0,T]$. Since $w^\epsilon+\epsilon\nabla\theta^\epsilon\in C^1([0,T];H^s(\mathbb{R}^N))$, the well-known theorem for ordinary differential equations guarantees that the problem

$$\frac{dx}{dt} = w^\epsilon(x,t)+\epsilon\nabla\theta^\epsilon(x,t), \quad x|_{t=\tau}=\xi \tag{2.43}$$

has a unique solution $x=\Psi(t)\in C^1([0,T];\mathbb{R}^N)$. Equation (2.42) implies

$$\frac{d}{dt}\rho^\epsilon(\Psi(t),t) = -\operatorname{div}(w^\epsilon+\epsilon\nabla\theta^\epsilon)\rho^\epsilon \tag{2.44}$$

Integrating over $[0,\tau]$ we have

$$\rho^\epsilon(\xi,\tau)=\rho^\epsilon(\Psi(0),0)\exp\Big[-\int_0^\tau \operatorname{div}\big(w^\epsilon(\Psi(t),t)+\epsilon\nabla\theta^\epsilon(\Psi(t),t)\big)dt\Big]. \tag{2.45}$$

Thus $\rho^\epsilon(\xi,\tau)\ge 0$ if $\rho^\epsilon(\Psi(0),0)=\rho_0^\epsilon(\Psi(0))\ge 0$. Denote $R\{u\}=\sup\{|x|:u(x)\neq 0\}$ for $u\in C(\mathbb{R}^N)$. If $\rho^\epsilon(\xi,\tau)\neq 0$ then $\rho_0^\epsilon(\Psi(0))\neq 0$ so that $|\Psi(0)|\le R\{\rho_0^\epsilon\}$, and

$$\begin{aligned} |\xi| = |\Psi(\tau)| &= \Big|\Psi(0)+\int_0^\tau w^\epsilon(\Psi(t),t)+\epsilon\nabla\theta^\epsilon(\Psi(t),t)dt\Big| \\ &\le |\Psi(0)|+\int_0^\tau |w^\epsilon|_\infty+\epsilon|\nabla\theta^\epsilon|_\infty dt \\ &\le R\{\rho_0^\epsilon\}+(1+\epsilon)CT. \end{aligned} \tag{2.46}$$

The same proof can be applied to

$$\partial_t(\rho^\epsilon-\mathcal{C})+(w^\epsilon+\epsilon\rho^\epsilon\nabla\theta^\epsilon)\nabla(\rho^\epsilon-\mathcal{C})+\operatorname{div}(w^\epsilon+\epsilon\rho^\epsilon\nabla\theta^\epsilon)(\rho^\epsilon-\mathcal{C})=0 \tag{2.47}$$

which yields

$$|\xi|=|\Psi(\tau)|\le R\{\rho_0^\epsilon-\mathcal{C}\}+(1+\epsilon)CT. \tag{2.48}$$

Therefore, we have proven the existence and uniqueness of the classical solution of the dispersive perturbation of the quasilinear symmetric hyperbolic system (2.19)–(2.20).

Theorem 2.4. *Let $f \in C^\infty(\mathbb{R}^+, \mathbb{R})$ with $f''(\rho) > 0$ for $\rho > 0$, and $s > [N/2] + 3$. Assume that the initial data*

$$U_0^\epsilon = {}^t(a_0^\epsilon - a_1, b_0^\epsilon - b_1, w_0^\epsilon) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \times (H^s(\mathbb{R}^N))^N \tag{2.49}$$

are compact supported and satisfies the uniform bound

$$\|U_0^\epsilon\|_{H^s(\mathbb{R}^N)} = \|a_0^\epsilon - a_1\|_{H^s(\mathbb{R}^N)} + \|b_0^\epsilon - b_1\|_{H^s(\mathbb{R}^N)} + \|w_0^\epsilon\|_{H^s(\mathbb{R}^N)} < C_0, \tag{2.50}$$

and $B^\epsilon \in C([0,T]; H^s(\mathbb{R}^N)) \cap C^1([0,T]; H^{s-2}(\mathbb{R}^N))$ with $\|B^\epsilon\|_{H^s(\mathbb{R}^N)} \le C_1$. Then there exists a time interval $[0,T]$ with $T > 0$, so that the IVP for the (2.19)–(2.20) *has a unique classical solution $U^\epsilon = {}^t(a^\epsilon - a_1, b^\epsilon - b_1, w^\epsilon)$;*

$$(a^\epsilon - a_1, b^\epsilon - b_1) \in C^1([0,T] \times \mathbb{R}^N) \cap C^1([0,T]; C^2(\mathbb{R}^N)) \tag{2.51}$$

$$w^\epsilon \in C^1([0,T] \times \mathbb{R}^N) \tag{2.52}$$

Furthermore,

$$U^\epsilon \in C([0,T]; H^s(\mathbb{R}^N)) \cap C^1([0,T]; H^{s-2}(\mathbb{R}^N)) \tag{2.53}$$

and T depends on the bound C in (2.50) *and in particular, not on ϵ. The solution $U^\epsilon = {}^t(a^\epsilon - a_1, b^\epsilon - b_1, w^\epsilon)$ satisfies the estimate*

$$\|U^\epsilon\|_{H^s(\mathbb{R}^N)} = \|a^\epsilon - a_1\|_{H^s(\mathbb{R}^N)} + \|b^\epsilon - b_1\|_{H^s(\mathbb{R}^N)} + \|w^\epsilon\|_{H^s(\mathbb{R}^N)} < C \tag{2.54}$$

for all $t \in [0,T]$. The constant C is also independent of ϵ. In addition, if $\rho_0^\epsilon(x) = (a_0^\epsilon)^2 + (b_0^\epsilon)^2 > 0$ then $\rho^\epsilon(x,t) > 0$ for all $t \ge 0$; if ρ_0^ϵ has a compact support, then $\rho^\epsilon(\cdot,t)$ does too for any $t \in [0,T]$ and

$$R\{\rho^\epsilon(\cdot,t)\} \le R\{\rho_0^\epsilon\} + (1+\epsilon)MT.$$

Proof of Theorem 2.1. Since $A^\epsilon = a^\epsilon + ib^\epsilon$ and $w^\epsilon = \nabla S^\epsilon$, it follows from (2.51)–(2.52) that

$$A^\epsilon \in C([0,T]; H^s(\mathbb{R}^N)) \cap C^1([0,T]; H^{s-2}(\mathbb{R}^N)) \tag{2.55}$$

$$S^\epsilon \in C([0,T]; H^{s+1}(\mathbb{R}^N)) \cap C^1([0,T]; H^s(\mathbb{R}^N)) \tag{2.56}$$

and thus

$$A^\epsilon \in C^1([0,T] \times \mathbb{R}^N) \cap C^1([0,T]; C^2(\mathbb{R}^N)), \quad S^\epsilon \in C^1([0,T]; C^2(\mathbb{R}^N)) \tag{2.57}$$

by Sobolev embedding theorem. The wave function $\psi^\epsilon = A^\epsilon e^{iS^\epsilon/\epsilon}$ has the same regularity as A^ϵ, thus

$$\begin{aligned} A^\epsilon &\in C([0,T]; H^s(\mathbb{R}^N)) \cap C^1([0,T]; H^{s-2}(\mathbb{R}^N)) \\ &\hookrightarrow C^1([0,T] \times \mathbb{R}^N) \cap C^1([0,T]; C^2(\mathbb{R}^N)). \end{aligned} \tag{2.58}$$

For classical solutions, the Schrödinger-Poisson system (1.1) (1.2) is equivalent to the dispersive quasilinear hyperbolic system (2.9) (2.10). Applying this equivalent relation, Theorem 2.1 follows immediately. ☐

The limiting system of (2.9)–(2.10) is the quasilinear symmetric hyperbolic system (formally letting $\epsilon \to 0$)

$$U_t + \sum_{j=1}^{N} A_j(U)U_{x_j} + U = B(t), \quad U(x,t) = {}^t(a, b, w)\,, \tag{2.59}$$

$$U(x,0) = U_0(x) = {}^t(a_0(x) - a_1, b_0(x) - b_1, w_0(x))\,, \tag{2.60}$$

which is equivalent to (1.13)–(1.15) as long as the solutions are smooth. As a corollary we also prove the existence and uniqueness of the local smooth solutions of the Euler-Poisson system (1.13)–(1.15).

Corollary 2.5. *Assume the hypothesis of Theorem 2.4. Given $U_0^\epsilon, U_0 \in H^s(\mathbb{R}^N)$ and $U_0^\epsilon(x)$ converges to $U_0(x)$ in $H^s(\mathbb{R}^N)$ as ϵ tends to 0. Let $[0,T]$ be the fixed interval determined in Theorem 2.4. Then as $\epsilon \to 0$ there exists $U \in L^\infty\big([0,T]; H^s(\mathbb{R}^N)\big)$ such that*

$$U^\epsilon \to U \quad \text{in } C([0,T]; H^{s-\sigma}(\mathbb{R}^N)) \text{ for all } \sigma > 0 \tag{2.61}$$

The function $U(x,t)$ belongs to $C([0,T]; H^s(\mathbb{R}^N) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^N))$ and is a classical solution of (2.59)–(2.60) with initial data $U(x,0) = U_0(x)$.

Proof. By a classical compactness argument, Arzela-Ascoli theorem (applied in time variable), the Rellich lemma (applied in the space variables), we deduce from (2.53) the existence of a subsequence of $\{U^\epsilon\}$ such that

$$U^\epsilon \text{ converges strongly in } C([0,T]; H^{s-\sigma}(\mathbb{R}^N)) \text{ to a function } U \tag{2.62}$$

for $\sigma > 0$. Furthermore, from the equation itself we also have

$$U^\epsilon \to U \text{ strongly in } C^1([0,T]; H^{s-2-\sigma}(\mathbb{R}^N)). \tag{2.63}$$

Since $U_0^\epsilon(x)$ converges strongly to $U_0(x)$ in $H^s(\mathbb{R}^N)$, this limiting solution has initial data $U_0(x)$. Also $\mathcal{L}(U^\epsilon)$ is uniformly bounded in $H^s(\mathbb{R}^N)$ therefore the perturbation term $\frac{\epsilon}{2}\mathcal{L}(U^\epsilon)$ tends to zero as $\epsilon \to 0$. This system admits a unique solution. It follows that the convergence to U takes place without passing to subsequence. This complete the proof of the corollary. $\square$

Proof of Theorem 2.2. As usual we consider the difference of (2.9) and (2.59). Setting $\tilde{U}^\epsilon = U^\epsilon - U$ then we have

$$\tilde{U}_t^\epsilon + \sum_{j=1}^{N} A_j(U)\tilde{U}_{x_j}^\epsilon + \tilde{U}^\epsilon = \tilde{B}^\epsilon + F^\epsilon + \frac{\epsilon}{2}\big(L(\tilde{U}^\epsilon) + L(U)\big) \tag{2.64}$$

$$\tilde{U}^\epsilon(x,0) = U_0^\epsilon(x) - U_0(x) \tag{2.65}$$

where

$$F^\epsilon = -\sum_{j=1}^{N} \big(A_j(U^\epsilon) - A_j(U)\big)U_{x_j}^\epsilon\,. \tag{2.66}$$

Since the symmetrizer $A_0(U)$ is positive definite, the previous energy estimates is applicable to (2.64). The matrix $A_j(U), j = 1, 2, \dots, N$, is symmetrizable. The energy associated with (2.64) is

$$\|\tilde{U}^\epsilon(t)\|_E^2 \equiv \int \langle A_0(U)\tilde{U}^\epsilon, \tilde{U}^\epsilon\rangle dx \tag{2.67}$$

and the Friedrich energy equality becomes

$$\begin{aligned}\frac{d}{dt}\|\tilde{U}^\epsilon(t)\|_E^2 + \|\tilde{U}^\epsilon(t)\|_E^2 &= \int \langle \Gamma^\epsilon \tilde{U}^\epsilon, \tilde{U}^\epsilon\rangle dx + 2\int \langle A_0(U)(\tilde{B}^\epsilon + F^\epsilon), \tilde{U}^\epsilon\rangle dx \\ &\quad + \frac{\epsilon}{2}\int \langle A_0(U)L(\tilde{U}^\epsilon) + L(U), \tilde{U}^\epsilon\rangle dx\end{aligned} \tag{2.68}$$

where $\tilde{B}^\epsilon = B^\epsilon - B$ and

$$\Gamma^\epsilon = \operatorname{div}\vec{A}(U) = \partial_t A_0(U) + \partial_{x_1}\mathcal{A}_1(U) + \cdots + \partial_{x_N}\mathcal{A}_N(U) \tag{2.69}$$

The antisymmetry of L yields

$$\frac{\epsilon}{2}\int \langle A_0(U)L(\tilde{U}^\epsilon), \tilde{U}^\epsilon\rangle dx = 0\,. \tag{2.70}$$

The Cauchy-Schwarz inequality implies

$$\frac{\epsilon}{2}\int \langle A_0(U)L(U), \tilde{U}^\epsilon\rangle dx \le \epsilon C\|U\|_{H^2(\mathbb{R}^N)}\|\tilde{U}^\epsilon\|_{L^2(\mathbb{R}^N)}\,. \tag{2.71}$$

Thus we only need to estimate the nonhomogeneous term $F^\epsilon + \tilde{B}^\epsilon$. Indeed,

$$\|F^\epsilon + \tilde{B}^\epsilon\|_{H^s(\mathbb{R}^N)} \le C\|\tilde{U}^\epsilon\|_{H^s(\mathbb{R}^N)} \tag{2.72}$$

By applying Gronwall inequality and the strict positivity of $A_0(U)$, we deduce the inequality

$$\|\tilde{U}^\epsilon\|_{H^s(\mathbb{R}^N)} \le (C(\epsilon) + \|A_0^\epsilon - A_0\|_{H^s(\mathbb{R}^N)})e^{cT} \tag{2.73}$$

with $C(\epsilon) \to 0$ as $\epsilon \to 0$. This completes the proof of Theorem 2.2. □

Let $\rho^\epsilon - \mathcal{C} \in H^2(\mathbb{R}^N)$ have compact support. Then the Newtonian potential V^ϵ defined by (2.13) is well-defined. We can estimate the Sobolev norms of $g^\epsilon = (g_1^\epsilon, g_2^\epsilon, \dots, g_N^\epsilon) = \nabla V^\epsilon$ as follows.

Lemma 2.6. *Assume s is a nonnegative integer. If $\rho_0^\epsilon - \mathcal{C} \in H^s(\mathbb{R}^N)$ has compact support, then $g^\epsilon \in H^{s+1}(\mathbb{R}^N)$ and*

$$\|g^\epsilon\|_{H^{s+1}(\mathbb{R}^N)} \le C(s)\big(1 + (R\{\rho^\epsilon - \mathcal{C}\})^{(2+N)/2}\big)\|\rho^\epsilon - \mathcal{C}\|_{H^s(\mathbb{R}^N)} \tag{2.74}$$

Here the constant $C(s)$ depends only on s.

Proof. Let $(\rho^\epsilon - \mathcal{C}) \in H^2(\mathbb{R}^N)$ satisfy $R\{\rho^\epsilon - \mathcal{C}\} \le 1$ at this moment, we have

$$|g^\epsilon(x,t)| \le \|\rho^\epsilon - \mathcal{C}\|_{L^\infty(\mathbb{R}^N)} \int_{|y|\le 1} \frac{dy}{|x-y|^{N-1}} \le C_1\|\rho^\epsilon - \mathcal{C}\|_{L^\infty(\mathbb{R}^N)} \frac{1}{1+|x|^{N-1}}$$

Hence $g^\epsilon \in L^2(\mathbb{R}^N)$ and $\|g^\epsilon\|_{L^2(\mathbb{R}^N)} \le C_2\|\rho^\epsilon - \mathcal{C}\|_{L^\infty(\mathbb{R}^N)}$. If $R\{\rho^\epsilon - \mathcal{C}\} = \eta > 0$ is arbitrary, then, by applying the above estimate to $\rho^\epsilon(x/\eta) - \mathcal{C}$, we obtain

$$\|g^\epsilon\|_{L^2(\mathbb{R}^N)} \le C_2\|\rho^\epsilon - \mathcal{C}\|_{L^\infty(\mathbb{R}^N)}\eta^{(2+N)/2}$$

Since g^ϵ is an L^2-solution of the Poisson equation $(1-\Delta)g^\epsilon = g^\epsilon + \nabla(\rho^\epsilon - \mathcal{C})$ we know that $g^\epsilon \in H^2(\mathbb{R}^N)$ and

$$\begin{aligned}\|g^\epsilon\|_{H^2(\mathbb{R}^N)} &= \|g^\epsilon + \nabla(\rho^\epsilon - \mathcal{C})\|_{L^2(\mathbb{R}^N)} \\ &\le C_2\|\rho^\epsilon - \mathcal{C}\|_{L^\infty(\mathbb{R}^N)}\eta^{(2+N)/2} + \|\rho^\epsilon - \mathcal{C}\|_{H^1(\mathbb{R}^N)} \\ &\le C(1+\eta^{(2+N)/2})\|\rho^\epsilon - \mathcal{C}\|_{H^1(\mathbb{R}^N)}\end{aligned} \tag{2.75}$$

Using the above equation again, we get the iteration scheme

$$\|g^\epsilon\|_{H^3(\mathbb{R}^N)} = \|g^\epsilon + \nabla(\rho^\epsilon - \mathcal{C})\|_{H^1(\mathbb{R}^N)} \le C(1+\eta^{(2+N)/2})\|\rho^\epsilon - \mathcal{C}\|_{H^2(\mathbb{R}^N)}\,. \tag{2.76}$$

This is the estimate claimed for $s = 2$. We can prove the general case by induction on s. This completes the proof of the lemma. $\square$

Remark 2.7. It follows immediately from the above lemma and the explicit form of g^ϵ that if $\rho^\epsilon - \mathcal{C} \in C([0,T]; H^2(\mathbb{R}^N))$ then $g^\epsilon \in C([0,T]; H^3(\mathbb{R}^N))$, and if $\rho^\epsilon - \mathcal{C} \in L^\infty([0,T]; H^s(\mathbb{R}^N)) \cap C([0,T]; H^{s-2}(\mathbb{R}^N))$, then

$$g^\epsilon \in L^\infty([0,T]; H^{s+1}(\mathbb{R}^N)) \cap C([0,T]; H^{s-1}(\mathbb{R}^N)).$$

Remark 2.8. In addition to the transport equation (2.42), we can also obtain the Hamilton-Jacobi equation for the phase function θ^ϵ (see (3.7) below)

$$\partial_t \theta^\epsilon + w^\epsilon \cdot \nabla \theta^\epsilon + \frac{\epsilon}{2} |\nabla \theta^\epsilon|^2 = \frac{\epsilon}{2} \frac{\Delta \sqrt{\rho^\epsilon}}{\sqrt{\rho^\epsilon}} \tag{2.77}$$

where the $O(\epsilon)$ term occurs due to the quantum effect and it converges to the pure transport equation as ϵ tends to zero. In fact, by (2.2)–(2.4) and (2.41) one obtains that $(\rho^\epsilon, \theta^\epsilon, w^\epsilon)$ satisfies an IVP for

$$\partial_t \rho^\epsilon + \nabla \cdot (\rho^\epsilon w^\epsilon + \epsilon \rho^\epsilon \nabla \theta^\epsilon) = 0, \tag{2.78}$$

$$\partial_t \theta^\epsilon + w^\epsilon \cdot \nabla \theta^\epsilon + \frac{\epsilon}{2} |\nabla \theta^\epsilon|^2 = \frac{\epsilon}{2} \frac{\Delta \sqrt{\rho^\epsilon}}{\sqrt{\rho^\epsilon}}, \tag{2.79}$$

$$\partial_t w^\epsilon + (w^\epsilon \cdot \nabla) w^\epsilon + \nabla f'(\rho^\epsilon) + \nabla V^\epsilon + w^\epsilon = 0, \tag{2.80}$$

$$-\Delta V^\epsilon = \rho^\epsilon - \mathcal{C}, \tag{2.81}$$

$$\rho^\epsilon(x,0) = \rho_0^\epsilon(x), \quad \rho_0^\epsilon - \mathcal{C} \text{ has compact support}, \tag{2.82}$$

$$\theta^\epsilon(x,0) = 0, \quad w^\epsilon(x,0) = w_0^\epsilon(x). \tag{2.83}$$

By Theorem 2.4 and (2.41), we conclude that $(\rho^\epsilon, \theta^\epsilon, w^\epsilon)$ satisfying

$$(\rho^\epsilon - \mathcal{C}, w^\epsilon, V^\epsilon) \in C([0,T],\ H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \times H^{s+2}(\mathbb{R}^N)), \tag{2.84}$$

$$\theta^\epsilon \in C([0,T],\ H^s(\mathbb{R}^N)), \quad \nabla \theta^\epsilon \in C([0,T],\ H^{s-1}(\mathbb{R}^N)). \tag{2.85}$$

is the classical solution of IVP (2.78)–(2.83) for $0 \le t \le T$ and is bounded with respect to ϵ. By passing limit in (2.78)–(2.83), one has

$$\partial_t \rho + \nabla \cdot (\rho w) = 0, \tag{2.86}$$

$$\partial_t \theta + w \cdot \nabla \theta = 0, \tag{2.87}$$

$$\partial_t w + (w \cdot \nabla) w + \nabla f'(\rho) + \nabla V + w = 0, \tag{2.88}$$

$$-\Delta V = \rho - \mathcal{C}. \tag{2.89}$$

$$\rho(x,0) = \rho_0(x), \quad \rho_0 - \mathcal{C} \text{ has compact support}, \tag{2.90}$$

$$\theta(x,0) = 0, \quad w(x,0) = w_0(x). \tag{2.91}$$

It follows immediately from (2.87) and (2.91) that $\theta(x,t)$ satisfies

$$\theta(x(t),t) = 0, \quad \text{along} \quad \frac{dx}{dt} = w(x(t),t),\ x(0) = x_0 \in \mathbb{R}^N; \tag{2.92}$$

hence the velocity $v = \nabla \theta$ is zero all the time. Also, we conclude that Eqs. (2.86), (2.88)–(2.89) are equivalent to the Euler-Poisson system (1.13)–(1.15).

Remark 2.9. There is a very interesting stochastic analogue of the characteristic equation (2.43). Replacing the quantum fluctuation by the Brownian motion W, the Wiener process, then (2.43) becomes the Itô stochastic differential equation

$$dx = w^\epsilon(x,t)dt + \epsilon dW \tag{2.93}$$

Thus we can also serve the quantum hydrodynamics equations (2.78) – (2.81) as the stochastic counterpart to the Euler-Poisson system in classical fluid mechanics (see [30] and the references therein).

3. Existence of a global solution and long time behavior

With the help of madelung transform for irrotational fluid, it is possible to extend the local solution (given by Theorem 2.4) of Cauchy problem (1.1)–(1.3) globally in time and analyze its asymptotic behavior in Sobolev space for fixed $\epsilon > 0$ by applying energy method to the hydrodynamic equations (1.8)–(1.10) and obtaining the a-priori estimate on the correspond macroscopic variables (density, velocity and potential). Let

$$\psi_0^\epsilon = |A_0^\epsilon(x)| \exp\big(\frac{i}{\epsilon} S_0(x)\big).$$

Theorem 3.1. *Let $S_c = -f'(\mathcal{C})$ and $\epsilon > 0$ fixed. Assume that $(|A_0^\epsilon| - \sqrt{\mathcal{C}}, S_0 - S_c) \in H^s(\mathbb{R}^N)$ with compact support. Then there is a $\eta_1 > 0$ such that if $\|(|A_0^\epsilon| - \sqrt{\mathcal{C}}, S_0 - S_c)\|_{H^s(\mathbb{R}^N)} \le \eta_1$ there exists a global solution*

$$\psi^\epsilon(x,t) = A^\epsilon(x,t) \exp\big(\frac{i}{\epsilon} S^\epsilon(x,t)\big)$$

of IVP (1.1)–(1.3) *such that*

$$\|\psi^\epsilon - \psi_c\|_{H^s(\mathbb{R}^N)} + \|V^\epsilon\|_{H^s(\mathbb{R}^N)} \le C\|(|A_0^\epsilon| - \sqrt{\mathcal{C}}, S_0 - S_c)\|_{H^s(\mathbb{R}^N)} e^{-\beta t}. \tag{3.1}$$

where $\psi_c = \sqrt{\mathcal{C}} \exp\big(\frac{i}{\epsilon} S_c\big)$ and $\beta > 0$ is a constant.

By Theorem 2.1 and theorem 2.4 with modifications, we obtain the local existence of IVP (1.1)–(1.3) under the assumption of Theorem 3.1. To extend the local solution globally in time, the uniformly a-priori estimates are to be established. Note that

$$\|\psi^\epsilon - \psi_c\|_{H^s(\mathbb{R}^N)} \le C\|(|A^\epsilon| - \sqrt{\mathcal{C}}, S^\epsilon - S_c)\|_{H^s(\mathbb{R}^N)}, \tag{3.2}$$

and that

$$\psi^\epsilon(x,t) = \varrho^\epsilon(x,t) \exp\big(\frac{i}{\epsilon} S^\epsilon(x,t)\big)$$

solves Cauchy problem (1.1)–(1.3), it is sufficient to prove

$$\|(\varrho^\epsilon - \sqrt{\mathcal{C}}, \mathbf{u}^\epsilon)\|_{H^s(\mathbb{R}^N)\times H^{s-1}(\mathbb{R}^N)} + \|V^\epsilon\|_{H^s(\mathbb{R}^N)} \le C r_0 e^{-\beta t} \tag{3.3}$$

with

$$r_0 = \|(\varrho_0^\epsilon - \sqrt{\mathcal{C}}, \mathbf{u}_0^\epsilon)\|_{H^s(\mathbb{R}^N)\times H^{s-1}(\mathbb{R}^N)}, \tag{3.4}$$

for $(\varrho^\epsilon, \mathbf{u}^\epsilon, V^\epsilon)$, which satisfies the following initial value problems:

$$2\varrho^\epsilon \cdot \varrho_t^\epsilon + \operatorname{div}((\varrho^\epsilon)^2 \mathbf{u}^\epsilon) = 0, \tag{3.5}$$

$$\mathbf{u}_t^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)\mathbf{u}^\epsilon + \nabla P((\rho^\epsilon)^2) + \mathbf{u}^\epsilon = \nabla V^\epsilon + \frac{\epsilon^2}{2} \nabla\big(\frac{\Delta \varrho^\epsilon}{\varrho^\epsilon}\big), \tag{3.6}$$

$$\Delta V^\epsilon = (\varrho^\epsilon)^2 - \mathcal{C}, \tag{3.7}$$

$$\varrho^\epsilon(x,0) = \varrho_0^\epsilon := |A_0^\epsilon(x)|, \quad \mathbf{u}^\epsilon(x,0) = \mathbf{u}_0^\epsilon := \nabla S_0(x), \tag{3.8}$$

with the velocity defined for irrotational fluids by $\mathbf{u}^\epsilon = \nabla S^\epsilon$.

For IVP (3.5)–(3.8) we have the following theorem.

Theorem 3.2. *Assume that* $(\varrho_0^\epsilon - \sqrt{\mathcal{C}}, \mathbf{u}_0^\epsilon) \in H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$ *with compact support. Then there is a* $\eta_0 > 0$ *such that if* $\|\varrho_0^\epsilon - \sqrt{\mathcal{C}}\|_{H^s(\mathbb{R}^N)} + \|\mathbf{u}_0^\epsilon\|_{H^{s-1}(\mathbb{R}^N)} \le \eta_0$ *there exists a global classical solution* $(\rho^\epsilon, \mathbf{u}^\epsilon, V^\epsilon)$ *of IVP* (3.5)–(3.8) *such that*

$$\|(\varrho^\epsilon - \sqrt{\mathcal{C}})(t)\|^2_{H^s(\mathbb{R}^N)} + \|\mathbf{u}^\epsilon(t)\|_{H^{s-1}(\mathbb{R}^N)} + \|V^\epsilon(t)\|_{H^s(\mathbb{R}^N)} \le Cr_0 e^{-\alpha_0 t}, \tag{3.9}$$

with $\alpha_0 > 0$ *a constant and* r_0 *is given by* (3.4).

Since the transformation $\psi^\epsilon = \varrho^\epsilon \exp\left(\frac{i}{\epsilon} S^\epsilon(x,t)\right)$ gives for S^ϵ that

$$S_t^\epsilon + \frac{1}{2}|\nabla S^\epsilon|^2 + (f'((\varrho^\epsilon)^2) + V^\epsilon) + S^\epsilon = \frac{\epsilon^2}{2}\frac{\Delta \varrho^\epsilon}{\varrho^\epsilon}, \tag{3.10}$$

from which we can obtain

$$\begin{aligned}
&\|S^\epsilon - S_c\|^2_{L^2(\mathbb{R}^N)} \\
&\le C\big(\|(\varrho^\epsilon - \sqrt{\mathcal{C}}\|^2_{H^2(\mathbb{R}^N)} + \|(\mathbf{u}^\epsilon, V^\epsilon)\|^2_{L^2(\mathbb{R}^N)}\big) + C\|S^\epsilon - S_c\|^2_{L^2(\mathbb{R}^N)} e^{-t}.
\end{aligned}$$

This and Theorem 3.2 yield theorem 3.1.

Remark 3.3. Theorem 3.2 also implies the global existence and large time behavior for IVP (1.8)–(1.11) by setting $(\rho^\epsilon, \mathbf{u}^\epsilon, V^\epsilon) = ((\varrho^\epsilon)^2, \mathbf{u}^\epsilon, V^\epsilon)$.

Proof of Theorem 3.2. The key point is to obtain the uniform a-priori estimates in Sobolev space for $(w, \mathbf{u}^\epsilon, V^\epsilon)$ with $w = \varrho^\epsilon - \sqrt{\mathcal{C}}$ for the time period $T > 0$ when the local solution $(\varrho^\epsilon, \mathbf{u}^\epsilon, V^\epsilon)$ exists.

A computation shows that the perturbation $(w, \mathbf{u}^\epsilon, V^\epsilon)$ satisfies the the Cauchy problem

$$w_{tt} + w_t + \frac{1}{4}\varepsilon^2\Delta^2 w - P'(\mathcal{C})\Delta w + \mathcal{C}w = f_1 \tag{3.11}$$

$$\mathbf{u}_t^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla)\mathbf{u}^\epsilon + Q'(\sqrt{\mathcal{C}})\nabla w + \mathbf{u}^\epsilon = f_2, \tag{3.12}$$

$$\Delta V^\epsilon = (2\sqrt{\mathcal{C}} + w)w, \tag{3.13}$$

where

$$\begin{aligned}
f_1(x,t) =& -(w + \sqrt{\mathcal{C}})^{-1} w_t^2 - \frac{1}{2}(3\sqrt{\mathcal{C}} + w)w^2 - \nabla w \cdot \nabla V^\epsilon \\
&+ \frac{1}{2(w + \sqrt{\mathcal{C}})}\Delta P((w + \sqrt{\mathcal{C}})^2) - P'(\mathcal{C})\Delta w \\
&+ \overset{2}{\operatorname{div}}\Big((\sqrt{\mathcal{C}} + w)^2 \mathbf{u}^\epsilon \otimes \mathbf{u}^\epsilon\Big) + \frac{\varepsilon^2}{4(w + \sqrt{\mathcal{C}})}|\Delta w|^2, \\
f_2(x,t) =& \nabla V^\epsilon - (Q'(\sqrt{\mathcal{C}} + w) - Q'(\sqrt{\mathcal{C}}))\nabla w + \frac{1}{2}\varepsilon^2 \nabla\Big(\frac{\Delta w}{w + \sqrt{\mathcal{C}}}\Big)
\end{aligned}$$

with $Q(\rho) = H(\rho^2)$ and $\rho H'(\rho) = P'(\rho)$. The corresponding initial values are

$$w(x,0) = w_0 =: \varrho_0^\epsilon - \sqrt{\mathcal{C}}, \;\; w_t(x,0) = -\mathbf{u}_0^\epsilon \cdot \nabla w_0 - \frac{1}{2}(\sqrt{\mathcal{C}} + w_0)\operatorname{div}\mathbf{u}_0^\epsilon, \tag{3.14}$$

$$\mathbf{u}^\epsilon(x,0) = \mathbf{u}_0^\epsilon. \tag{3.15}$$

Then w and $\mathbf{u}$ are balanced through

$$2w_t + 2\mathbf{u}^\epsilon \cdot \nabla w + (\sqrt{\mathcal{C}} + w)\,\nabla\cdot \mathbf{u}^\epsilon = 0, \tag{3.16}$$

Applying energy method to Cauchy Problem (3.11)–(3.16), we have, after a tedious computation (we omit the details here), the following a-priori estimates.

Lemma 3.4. *Let $T > 0$. Assume that the local solutions $(w, \mathbf{u}^\epsilon, V^\epsilon)$ of the Cauchy problem* (3.11)–(3.16) *belong to $H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N) \times H^{s-2}(\mathbb{R}^N)$ and satisfy*

$$N(T) =: \max_{0 \le t \le T} \|(w, \mathbf{u}^\epsilon)(t)\| e^{\gamma t} \ll 1, \tag{3.17}$$

with γ chosen to be arbitrary small. Then it holds

$$\|(\varrho^\epsilon - \sqrt{\mathcal{C}}, \mathbf{u}^\epsilon, V^\epsilon)(t)\|^2_{H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N) \times H^s(\mathbb{R}^N)} \le C r_0 e^{-\alpha t}. \tag{3.18}$$

Here $\alpha > 0$ and C are constants independent of γ, and r_0 is given by (3.4)*.*

In terms of Lemma 3.4 we prove that the a-priori bounds (3.17) is true for the local classical solution provided that $\|(\varrho_0^\epsilon - \sqrt{\mathcal{C}}, \mathbf{u}_0^\epsilon)\|_{H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)}$ is small enough and $\gamma \ll \alpha$. The continuity argument shows that the classical solution $(\rho^\epsilon, \mathbf{u}^\epsilon, V^\epsilon)$ exists global in time. Thus, the proof of Theorem 3.2 is completed. $\square$

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References

[1] F. Bass, A. Nasanov; *Nonlinear electromagnetic spin waves.* Phys. Rep. **1990**, *189*, 165–223.
[2] P. Bechouche, N. J. Mauser, F. Poupaud; *Semiclassical limit for the Schrödinger-Poisson equation in a crystal.* Comm. Pure Appl. Math. **2001**, *54*, 852–890.
[3] F. Bethuel, J. C. Saut; *Traveling waves for the Gross-Pitaevskii equation. I.* Ann. Inst. H. Poincarè Phys. Thèor. **1999**, *70*, no. 2, 147–238.
[4] T. Cazanava; *An introduction to nonlinear Schrödinger equations.* Testos de Métodos Matemáticos, Rio de Janeiro 1989.
[5] B. Desjardins, C.-K. Lin, T.-C. Tso; *Semiclassical limit of the derivative nonlinear Schrödinger equation.* Math. Models Methods Appl. Sci. **2000**, *10*, 261–285.
[6] B. Desjardins, C.-K. Lin; *On the semiclassical limit of the general modified NLS equation.* J. Math. Anal. Appl. **2001**, *260*, 546–571.
[7] R. Feynman; *Statistical Mechanics*, A set of lectures, Frontiers in Physics. New York 1972.
[8] P. Gamblin; *Solution reguliere a temps petit pour l'equation d'Euler-Poisson*, Commun. in Partial Differential Equations **1993**, *18*, 731–745.
[9] I. Gasser, C.-K. Lin, P. Markowich; *A review of dispersive limit of the (non)linear Schrödinger-type equation.* Taiwanese J. of Mathematics. **2000**, *4*, 501–529.
[10] I. Gasser, P. Markowich; *Quantum hydrodynamics, Wigner transforms and the classical limit.* Asymptotic Anal. **1997**, *14*, 97–116.
[11] P. Gérard; *Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire. Séminarie EDP de l'École Polytechnique*, lecture n^o XIII, Palasiseau 1992.
[12] P. Gérard, P. A. Markowich, N. J. Mauser, F. Poupaud; *Homogenization limits and Wigner transforms.* Comm. Pure and Appl. Math. **1990**, *50*, 321–377.
[13] E. Grenier; *Semiclassical limit of the nonlinear Schrödinger equation in small time*; Proc. Amer. Math. Soc. **1998**, *126*, 523–530.

[14] Y. Guo; *Smooth irrotational flows in the large to the Euler-Poisson system in* $\mathbb{R}^{3+1}$. Comm. Math. Phys. **1998**, *195*, 249–265.
[15] S. Jin, C. D. Levermore, D. W. McLaughlin; *The semiclassical limit of the defocusing NLS hierarchy*. Comm. Pure Appl. Math. **1999**, *52*, 613–654.
[16] A. Jüngel, M. C. Mariani, D. Rial; *Local existence of solutions to the transient quantum hydrodynamic equations*. Math. Models Meth. Appl. Sci. to appear 2001.
[17] S. Klainerman, A. Majda; *Singular limits and quasilinear systems with large parameter and the incompressible limit of compressible fluids*, Comm. Pure and Appl. Math. **1981**, *34*, 481–524.
[18] S. Kurihura; *Large-amplitude quasi-solitons in superfluid films*; J. Phys. Soc. Japan **1981**, *50*, 3262–3267.
[19] J. H. Lee, C.-K. Lin; *The behavior of solutions of NLS equation of derivative type in the semiclassical limit*; Chaos, Solitons & Fractals, **2002**, *13*, 1475–1492.
[20] C.-K. Lin; *Singular limit of the modified nonlinear Schrödinger equation*, In *Nonlinear Dynamics and Renormalization Group*, Edited by Sigal, I.M. and Sulem, C. CRM Proceeding & Lecture Note Vol. 27, 97–109, Amer. Math. Soc. 2001.
[21] F.-H. Lin, J. K. Xin; *On the incompressible fluid limit and the vortex motion law of the nonlinear Schroedinger equation*; Comm. Math. Phys. **1999**, *200*, 249-274.
[22] P. L. Lions, T. Paul; *Sur les mesures de Wigner*. Rev. Mat. Iberomaerica **1993**, *9*, 553–618.
[23] A. Litvak, V. Sergeev; *One dimensional collapse of plasma waves*; J. Experiment. Theoret. Phys. Lett. **1978**, *27*, 517–520.
[24] M. Loffredo, L. Morato; *On the creation of quantum vortex lines in rotating HeII*. I1 nouvo cimento **1993**, *108 B*, 205–215.
[25] A. Majda; *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag 1984.
[26] T. Makino, S. Ukai; *Sur l'existence des solutions locales de l'équation d'Euler-Poisson pour l'évolution d'étoiles gazeuses*; J. Math. Kyoto Univ. **1987**, *27*, 387–399.
[27] P. A. Markowich, C. Ringhofer, C. Schmeiser; *Semiconductor Equations*; Springer-Verlag 1989.
[28] P. A. Markowich, N. J. Mauser; *The classical limit of a self-consistent quantum-Vlasov equation in* 3*d*; Math. Models Methods Appl. Sci. **1993**, *3*, 109–124.
[29] L. Marcati, M. Mei; *Asymptotic convergence to steady-state solutions for solutions of the initial boundary problem to a hydrodynamic model for semiconductors*; preprint 1998.
[30] M. Nagasawa; *Stochastic Processes in Quantum Physics*; Monographs in Mathematics 94, Birkhäuser Verlag 2000.
[31] W. Strauss; *Nonlinear wave equations*, CBMS regional conference series in mathematics 73, AMS, Providence 1989.
[32] C. Sulem, P.-L. Sulem; *The Nonlinear Schrödinger Equation*; Springer-Verlag 1999.
[33] P. Zhang, Y.-X Zheng, N. Mauser; *The limit from the Schrödinger-Poisson to Vlasov-Poisson equations with general data in one dimension*; Comm. Pure Appl. Math. **55** (2002), no. 5, 582–632.

4. Addendum: Posted August 17, 2006

The authors want to make the following two corrections:

On the sixth line of Theorem 2.1, the expression $A^\epsilon \in L^\infty([0,T]; H^s(\mathbb{R}^N))$ should be replaced by $|A^\epsilon| - \sqrt{\mathcal{C}} \in L^\infty([0,T]; H^s(\mathbb{R}^N))$.

On the fourteenth line of Theorem 2.2, the expression A^ϵ should be replaced by $|A^\epsilon| - \sqrt{\mathcal{C}} \in L^\infty([0,T]; H^s(\mathbb{R}^N))$.

After these corrections, the two theorem will read:

Theorem 2.1. *Assume that* $\{|A_0^\epsilon| - \sqrt{\mathcal{C}}\}_\epsilon$ *is a uniformly bounded sequence in* $H^s(\mathbb{R}^N)$ *with compact support,* $S_0 \in H^{s+1}(\mathbb{R}^N)$ *with* ∇S_0 *compact supported,* $s > (N+4)/2$*, and* $f \in C^\infty(\mathbb{R}^+, \mathbb{R})$ *with* $f''(\rho) > 0$ *for* $\rho > 0$*. Then solutions* $(\psi^\epsilon, V^\epsilon)$ *of the Schrödinger-Poisson system* $(1.1)-(1.3)$ *exist on a small time interval* $[0,T]$*,* T *independent of* ϵ*. Moreover,* $\psi^\epsilon(x,t) = A^\epsilon(x,t)e^{iS^\epsilon(x,t)/\epsilon}$ *with*

$|A^\epsilon| - \sqrt{\mathcal{C}} \in L^\infty([0,T]; H^s(\mathbb{R}^N))$ and $S^\epsilon \in L^\infty([0,T]; H^{s+1}(\mathbb{R}^N))$ uniformly in ϵ and V^ϵ given by (2.13).

Theorem 2.2. *Assume that $(\rho, \mathbf{u}, V)$ is a solution of the Euler-Poisson system (1.13)–(1.15) and satisfies $(\rho - \mathcal{C}, \mathbf{u}, V) \in C([0,T], H^{s+2}(\mathbb{R}^N))$, $s \geq (N+4)/2$, with initial condition*

$$\rho_0(x) = \rho(x,0) = |A_0(x)|^2,$$
$$\mathbf{u}_0(x) = \mathbf{u}(x,0) = \nabla S_0(x).$$

Then there exists a critical value of ϵ, ϵ_c dependent of T, such that under the hypothesis

(1) *$A_0^\epsilon(x)$ converges strongly to A_0 in $H^s(\mathbb{R}^N)$ as ϵ tends to* 0
(2) *$(\sqrt{\rho_0} - \sqrt{\mathcal{C}}, \mathbf{u}_0) \in H^s(\mathbb{R}^N)$ with compact support,*
(3) *$0 < \epsilon < \epsilon_c$,*

the IVP for Schrödinger-Poisson system $(1.1)-(1.3)$ *has a unique classical solution $(\psi^\epsilon, V^\epsilon)$ on $[0,T]$, the wave function is of the form*

$$\psi^\epsilon(x,t) = A^\epsilon(x,t) \exp\big(\frac{i}{\epsilon} S^\epsilon(x,t)\big)$$

with $|A^\epsilon| - \sqrt{\mathcal{C}}$ and ∇S^ϵ are bounded in $L^\infty([0,T]; H^s(\mathbb{R}^N))$ uniformly in ϵ and V^ϵ is given by (2.13)*. Moreover, as $\epsilon \to 0$ $(\rho^\epsilon, \rho^\epsilon \nabla S^\epsilon, V^\epsilon)$ with $\sqrt{\rho^\epsilon} = A^\epsilon$, converges strongly in $C([0,T], H^{s-2}(\mathbb{R}^N))$.*

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