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ON NONNEGATIVE ENTIRE SOLUTIONS OF SECOND-ORDER SEMILINEAR ELLIPTIC SYSTEMS

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Abstract. We consider the second-order semilinear elliptic system

$$\Delta u_i = P_i(x)u_{i+1}^{\alpha_i} \quad \text{in } \mathbb{R}^N, \quad i = 1, 2, \dots, m$$

with nonnegative continuous functions P_i. We establish nonexistence criteria of nonnegative nontrivial entire solutions for this system. We also proved a Liouville type theorem for nonnegative entire solutions.

1. Introduction

This paper concerns the second-order semilinear elliptic system

$$\begin{aligned} \Delta u_1 &= P_1(x)u_2^{\alpha_1}, \\ \Delta u_2 &= P_2(x)u_3^{\alpha_2}, \\ &\vdots \\ \Delta u_m &= P_m(x)u_{m+1}^{\alpha_m}, \quad u_{m+1} = u_1, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^N$, $N \geq 1$, $m \geq 2$, and $\alpha_i > 0$, $i = 1, 2, \dots, m$ are constants satisfying $\alpha_1\alpha_2 \cdots \alpha_m > 1$, and the functions $P_i(x)$ are nonnegative continuous functions on $\mathbb{R}^N$.

We are concerned with the problem of existence and nonexistence of nonnegative nontrivial entire solutions of (1.1). By an entire solution of (1.1) we mean a vector function $(u_1, u_2, \dots, u_m) \in (C^2(\mathbb{R}^N))^m$ which satisfies (1.1) at every point of $\mathbb{R}^N$.

The problem of existence and nonexistence of nonnegative entire solutions for the scalar equation

$$\Delta u = f(x, u), \quad x \in \mathbb{R}^N$$

has been investigated by many authors, and numerous results have been obtained (see e.g. [2, 5, 7, 9] and references therein). In particular, when f has the form $f(x, u) = P(x)u^\alpha$ with $\alpha > 0$ and nonnegative function P, critical decay rate of P to admit nonnegative entire solutions has been characterized. On the other hand, very little is known about this problem for elliptic system (1.1) except for the case $m = 2$. For $m = 2$ we refer to [3, 5, 6, 8, 12, 13, 14].

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In [3, 12, 14], the system (1.1) with $m = 2$ has been considered under the conditions $\alpha_i \geq 1$, $i = 1, 2$, and nonexistence criteria of nonnegative nontrivial entire solutions have been obtained. The result is described roughly as follows:

Theorem 1.1. *Let $N \geq 3$, $m = 2$ and $\alpha_i \geq 1, i = 1, 2$. Suppose that P_i, $i = 1, 2$, satisfy*

$$P_i(x) \geq \frac{C_i}{|x|^{\lambda_i}}, \quad |x| \geq r_0 > 0, \quad i = 1, 2, \tag{1.2}$$

where $C_i > 0$ and λ_i, $i = 1, 2$, are constants. If (λ_1, λ_2) satisfies

$$\lambda_1 - 2 + \alpha_1(\lambda_2 - 2) \leq 0 \quad \text{or} \quad \lambda_2 - 2 + \alpha_2(\lambda_1 - 2) \leq 0, \tag{1.3}$$

then the system (1.1) *does not possess any nonnegative nontrivial entire solutions.*

However, if α_1 or α_2 is less than 1, Theorem 1.1 cannot derive any information about the nonnegative nontrivial entire solutions. Recently, Teramoto and Usami [13] have proved a Liouville type theorem for nonnegative entire solutions of (1.1) with $m = 2$ under the condition $\alpha_1\alpha_2 > 1$. The result is described as follows:

Theorem 1.2. *Let $N \geq 3, m = 2, \alpha_1\alpha_2 > 1, 0 < \alpha_1 < 1$. Suppose that P_i, $i = 1, 2$, satisfy* (1.2) *for some constants λ_i, $i = 1, 2$. If (λ_1, λ_2) satisfies*

$$\lambda_1 - 2 + \alpha_1(\lambda_2 - 2) \leq 0,$$

then the system (1.1) *does not possess nonnegative nontrivial entire solutions satisfying*

$$u_1(x) = O(\exp |x|^\rho) \quad \text{as} \quad |x| \to \infty \quad \text{for some } \rho > 0.$$

The aim of this paper is to extend Theorems 1.1 and 1.2 to the system (1.1) with $m \geq 3$.

Let us introduce some notation used throughout this paper. For any sequence $\{s_1, s_2, \dots, s_m\}$, we assume that $s_{m+j} = s_j$, $j = 1, 2, \dots$; that is, the suffixes should be taken in the sense of $\mathbb{Z}/m\mathbb{Z}$. Denote

$$A = \alpha_1\alpha_2 \cdots \alpha_m.$$

For real constants λ_1, $\lambda_2, \dots, \lambda_m$, we put

$$\begin{aligned}\Lambda_i &= \lambda_i - 2 + (\lambda_{i+1} - 2)\alpha_i + (\lambda_{i+2} - 2)\alpha_i\alpha_{i+1} + \dots \\ &\quad + (\lambda_{i+m-1} - 2)\alpha_i\alpha_{i+1}\alpha_{i+2}\dots\alpha_{i+m-2} \\ &= \lambda_i - 2 + \sum_{j=1}^{m-1}\Big\{(\lambda_{i+j} - 2)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\}, \quad i = 1, 2, \dots, m,\end{aligned} \tag{1.4}$$

and

$$\beta_i = \frac{\Lambda_i}{A - 1}, \quad i = 1, 2, \dots, m. \tag{1.5}$$

Since our assumptions imposed on P_i, $1 \leq i \leq m$, essentially take the forms

$$\liminf_{|x|\to\infty} |x|^{\lambda_i}P_i(x) > 0 \quad \text{or} \quad \limsup_{|x|\to\infty} |x|^{\lambda_i}P_i(x) < \infty,$$

all our results are formulated by means of the numbers λ_i, Λ_i, β_i, $1 \leq i \leq m$.

This paper is organized as follows. In Section 2, we give nonexistence criteria of nonnegative nontrivial entire solutions of (1.1). In Section 3, to show the sharpness of our nonexistence criteria we give existence theorems of positive entire solutions for (1.1) under the assumption that P_i have radial symmetry. In the final section (Section 4), we prove a Liouville type theorem for nonnegative entire solutions.

2. A priori estimate and nonexistence results

2.1. **Growth estimate of nonnegative entire solutions.** In this subsection, we study the estimate for nonnegative entire solutions of (1.1) which will play an important role to prove nonexistence theorems for nonnegative nontrivial entire solutions.

For a nonnegative function v defined on $\mathbb{R}^N$, we denote its spherical mean over the sphere $|x| = r,\ r > 0$, $\bar{v}(r)$ by

$$\bar{v}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} v(x)\, dS,$$

where dS denotes the volume element in the surface integral, ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. Moreover we introduce the function $\hat{P}(r), r \ge 0$, by

$$\hat{P}(r) = \begin{cases} \left(\frac{1}{\omega_N r^{N-1}} \int_{|x|=r} P(x)^{-\frac{\alpha'}{\alpha}} dS\right)^{-\alpha/\alpha'}, & \alpha > 1, \\ \min_{|x|=r} P(x), & \alpha = 1, \end{cases} \tag{2.1}$$

where $1/\alpha + 1/\alpha' = 1$. We set $\hat{P}(r) = 0$ if $\int_{|x|=r} P(x)^{-\alpha'/\alpha} dS = \infty$. We note that $\hat{P} = P$ when P has radial symmetry. We have the following well-known result (see [2, p.654], [9, p.508] and [10, p.70]).

Lemma 2.1. *Let $\alpha_i \ge 1$, $i = 1, 2, \dots, m$, and $(u_1, u_2, \dots, u_m)$ be a nonnegative entire solution of (1.1). Then its spherical mean $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m)$ satisfies system of ordinary differential inequalities*

$$\begin{gathered} (r^{N-1}\bar{u}_i'(r))' \ge r^{N-1}\hat{P}_i(r)\bar{u}_{i+1}(r)^{\alpha_i}, \quad r > 0, \\ \bar{u}_i'(0) = 0, \end{gathered} \tag{2.2}$$

where $i = 1, 2, \dots, m$.

Our main result is as follows.

Theorem 2.2. *Let $N \ge 3$, $\alpha_i \ge 1$, $i = 1, 2, \dots, m$, and $A > 1$. Suppose that P_i, $i = 1, 2, \dots, m$, satisfy*

$$\liminf_{|x|\to\infty} |x|^{\lambda_i} P_i(x) > 0, \tag{2.3}$$

where $\lambda_i, i = 1, 2, \dots, m$, are constants. Let $(u_1, u_2, \dots, u_m)$ be a nonnegative entire solution of (1.1). Then u_i, $i = 1, 2, \dots, m$, satisfy

$$u_i(x) \le C_i |x|^{\beta_i} \quad \text{at } \infty\,,$$

where $C_i > 0$ are constants and β_i are defined by (1.5).

Assume that (2.3) holds. Then there are constants $C_i > 0$, $i = 1, 2, \dots, m$, and $R_0 > 0$ such that

$$P_i(x) \ge \frac{C_i}{|x|^{\lambda_i}}, \quad |x| \ge R_0, \quad i = 1, 2, \dots, m.$$

So we can see that $\hat{P}_i$, $i = 1, 2, \dots, m$, defined by (2.1) satisfy

$$\hat{P}_i(r) \ge \frac{C_i}{r^{\lambda_i}}, \quad r \ge R_0. \tag{2.4}$$

Proof of Theorem 2.2. Let $(u_1, u_2, \dots, u_m)$ be a nonnegative entire solution of (1.1). We may assume that $(u_1, u_2, \dots, u_m) \not\equiv (0, 0, \dots, 0)$. Then, by Lemma 2.1, its spherical mean $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m)$ satisfies the system of ordinary differential inequalities (2.2).

Integrating (2.2) over $[0, r]$, we have

$$\bar{u}_i'(r) \geq r^{1-N} \int_0^r s^{N-1} \hat{P}_i(s) \bar{u}_{i+1}(s)^{\alpha_i} ds, \quad i = 1, 2, \dots, m.$$

Hence, we see that $\bar{u}_i'(r) \geq 0$ for $r \geq 0$. Integrating (2.2) twice over $[R, r]$, $R \geq 0$ and $i = 1, 2, \dots, m$, we have

$$\bar{u}_i(r) \geq \bar{u}_i(R) + \frac{1}{N-2} \int_R^r s\big[1 - (\frac{s}{r})^{N-2}\big] \hat{P}_i(s) \bar{u}_{i+1}(s)^{\alpha_i} ds. \tag{2.5}$$

Since $(u_1, u_2, \dots, u_m)$ is nonnegative and nontrivial, there exists a point $x_* \in \mathbb{R}^N$ such that $u_{i_0}(x_*) > 0$ for some $i_0 \in \{1, 2, \dots, m\}$; that is, $\bar{u}_{i_0}(r_*) > 0$, $r_* = |x_*|$. We may assume that $r_* \geq R_0$. Therefore, we see from (2.5) with $R = r_*$ that $\bar{u}_i(r) > 0$ for $r > r_*$, $i = 1, 2, \dots, m$.

First, we will show that

$$\bar{u}_i(r) = O(r^{\beta_i}) \quad \text{as } r \to \infty, \quad i = 1, 2, \dots, m. \tag{2.6}$$

Let us fix $R > r_*$ arbitrarily. Using (2.4) and the inequality

$$s\big[1 - (\frac{s}{r})^{N-2}\big] \geq \frac{N-2}{3^{N-2}}(r - s) \quad \text{for } R \leq r \leq 3R,$$

in (2.5), we have

$$\begin{aligned} \bar{u}_i(r) &\geq \bar{u}_i(R) + \frac{C_i}{3^{N-2}} \int_R^r s^{-\lambda_i}(r - s) \bar{u}_{i+1}(s)^{\alpha_i} ds \\ &\geq \hat{C}_i R^{-\lambda_i} \int_R^r (r - s) \bar{u}_{i+1}(s)^{\alpha_i} ds, \end{aligned}$$

where $R \leq r \leq 3R$ and $\hat{C}_i$ are some positive constants independent of r and R. We put

$$f_i(r; R) = \hat{C}_i R^{-\lambda_i} \int_R^r (r - s) \bar{u}_{i+1}(s)^{\alpha_i} ds, \quad R \leq r \leq 3R. \tag{2.7}$$

For simplicity of notation we write $f_i(r) = f_i(r; R)$ if there is no ambiguity. Clearly, $f_i(r)$, $i = 1, 2, \dots, m$, satisfy

$$\begin{gathered} \bar{u}_i(r) \geq f_i(r), \quad f_i(R) = 0, \\ f_i'(r) \geq 0, \quad f_i'(R) = 0, \end{gathered}$$

and

$$f_i''(r) = \hat{C}_i R^{-\lambda_i} \bar{u}_{i+1}(r)^{\alpha_i} \geq \hat{C}_i R^{-\lambda_i} f_{i+1}(r)^{\alpha_i}, \quad R \leq r \leq 3R. \tag{2.8}$$

From (2.7) and the monotonicity of $\bar{u}_i$, we see that

$$f_i(r; R) \geq \frac{\hat{C}_i}{2} R^{-\lambda_i} \bar{u}_{i+1}(R)^{\alpha_i} (r - R)^2, \quad R \leq r \leq 3R. \tag{2.9}$$

Let us fix $i \in \{1, 2, \dots, m\}$. Multiplying (2.8) by $f_{i+1}'(r) \geq 0$ and integrating by parts of the resulting inequality on $[R, r]$, we have

$$f_{i+1}'(r) f_i'(r) \geq C R^{-\lambda_i} f_{i+1}(r)^{\alpha_i + 1}, \quad R \leq r \leq 3R,$$

where $C = \tilde{C}_i/(\alpha_i + 1)$. For the rest of this article, C denotes various positive constants independent of r and R. Multiplying this inequality by $f'_{i+1}(r) \geq 0$ and integrating by parts, we obtain

$$f'_{i+1}(r)^2 f_i(r) \geq CR^{-\lambda_i} f_{i+1}(r)^{\alpha_i+2}, \quad R \leq r \leq 3R.$$

From (2.8), we see that

$$f'_{i+1}(r)^{2\alpha_{i-1}} f''_{i-1}(r) \geq CR^{-\lambda_i\alpha_{i-1}-\lambda_{i-1}} f_{i+1}(r)^{(\alpha_i+2)\alpha_{i-1}}, \quad R \leq r \leq 3R.$$

Again multiplying this relation by $f'_{i+1}(r) \geq 0$ and integrating by parts on $[R, r]$ twice, we have

$$f'_{i+1}(r)^{2\alpha_{i-1}+2} f_{i-1}(r) \geq CR^{-\lambda_i\alpha_{i-1}-\lambda_{i-1}} f_{i+1}(r)^{(\alpha_i+2)\alpha_{i-1}+2}, \quad R \leq r \leq 3R.$$

From (2.8), we see that for $R \leq r \leq 3R$,

$$\begin{aligned} &f'_{i+1}(r)^{2\alpha_{i-1}\alpha_{i-2}+2\alpha_{i-2}} f''_{i-2}(r) \\ &\geq CR^{-\lambda_i\alpha_{i-1}\alpha_{i-2}-\lambda_{i-1}\alpha_{i-2}-\lambda_{i-2}} f_{i+1}(r)^{\alpha_i\alpha_{i-1}\alpha_{i-2}+2\alpha_{i-1}\alpha_{i-2}+2\alpha_{i-2}}. \end{aligned}$$

By repeating this procedure, we obtain

$$f'_{i+1}(r)^{K_i} f''_{i-(m-1)}(r) = f'_{i+1}(r)^{K_i} f''_{i+1}(r) \geq CR^{-L_i} f_{i+1}(r)^{M_i}, \tag{2.10}$$

where

$$\begin{gathered} K_i = 2\sum_{j=1}^{m-1}\prod_{k=j}^{m-1}\alpha_{i-k}, \\ L_i = \sum_{j=1}^{m-1}\Big\{\lambda_{i-(j-1)}\prod_{k=j}^{m-1}\alpha_{i-k}\Big\} + \lambda_{i+1}, \\ M_i = \prod_{k=0}^{m-1}\alpha_{i-k} + 2\sum_{j=1}^{m-1}\prod_{k=j}^{m-1}\alpha_{i-k} = A + K_i. \end{gathered}$$

Multiplying the inequality (2.10) by $f'_{i+1}(r) \geq 0$ and integrating on $[R, r]$, we obtain

$$f'_{i+1}(r) f_{i+1}(r)^{-\frac{M_i+1}{K_i+2}} \geq CR^{-\frac{L_i}{K_i+2}}, \quad R < r \leq 3R.$$

Since $(M_i + 1)/(K_i + 2) > 1$, we may set $(M_i + 1)/(K_i + 2) = \delta_i + 1$, $\delta_i = (A - 1)/(K_i + 2)$. Integrating this inequality on $[2R, 3R]$ we get

$$f_{i+1}(2R)^{-\delta_i} \geq CR^{-\frac{L_i}{K_i+2}+1}.$$

From (2.9) with $r = 2R$ and this inequality, we have $\bar{u}_{i+2}(R) \leq CR^{\tau_i}$, where

$$\tau_i = \frac{1}{\alpha_{i+1}\delta_i}\Big\{\frac{L_i}{K_i+2} - 1 + (\lambda_{i+1} - 2)\delta_i\Big\}.$$

From the definitions of K_i, L_i, and δ_i, we see that

$$\begin{aligned}
\tau_i &= \frac{1}{\alpha_{i+1}\delta_i(K_i+2)}\Big[\sum_{j=1}^{m-1}\big\{\lambda_{i-j+1}\prod_{k=j}^{m-1}\alpha_{i-k}\big\} - 2\sum_{j=1}^{m-1}\prod_{k=j}^{m-1}\alpha_{i-k} \\
&\quad + (\lambda_{i+1}-2)\prod_{k=0}^{m-1}\alpha_{i-k}\Big] \\
&= \frac{1}{\alpha_{i+1}(A-1)}\Big[\sum_{j=1}^{m-2}\big\{(\lambda_{i-j+1}-2)\prod_{k=j}^{m-1}\alpha_{i-k}\big\} + (\lambda_{i-m+2}-2)\alpha_{i-m+1} \\
&\quad + (\lambda_{i+1}-2)\prod_{k=0}^{m-1}\alpha_{i-k}\Big] \\
&= \frac{1}{\alpha_{i+1}(A-1)}\Big[\sum_{j=0}^{m-2}\big\{(\lambda_{i-j+1}-2)\prod_{k=j}^{m-1}\alpha_{i-k}\big\} + (\lambda_{i+2}-2)\alpha_{i+1}\Big] \\
&= \frac{1}{A-1}\Big[\sum_{j=0}^{m-2}\big\{(\lambda_{i-j+1}-2)\prod_{k=j}^{m-2}\alpha_{i-k}\big\} + \lambda_{i+2}-2\Big] \\
&= \frac{1}{A-1}\Big[(\lambda_{i+1}-2)\alpha_i\alpha_{i-1}\dots\alpha_{i-(m-2)} + (\lambda_i-2)\alpha_{i-1}\alpha_{i-2}\dots\alpha_{i-m+2} + \dots \\
&\quad + (\lambda_{i-m+3}-2)\alpha_{i-m+2} + \lambda_{i+2}-2\Big] \\
&= \frac{1}{A-1}\Big[\lambda_{i+2}-2+\sum_{j=1}^{m-1}\Big\{(\lambda_{i+2+j}-2)\prod_{k=0}^{j-1}\alpha_{i+2+k}\Big\}\Big] = \frac{\Lambda_{i+2}}{A-1}.
\end{aligned}$$

Therefore, we obtain (2.6) by the definition of β_i.

Put $B_\rho(x) = \{y \in \mathbb{R}^N : |y-x| \le \rho\}$. Since u_i, $i = 1, 2, \dots, m$, are subharmonic functions in $\mathbb{R}^N$, we have

$$\begin{aligned}
u_i(x) &\le \frac{1}{|B_{|x|/2}(x)|}\int_{B_{|x|/2}(x)} u_i(y)dy \\
&\le \frac{C}{|x|^N}\int_{B_{3|x|/2}(0)\setminus B_{|x|/2}(0)} u_i(y)dy \\
&= \frac{C}{|x|^N}\int_{|x|/2}^{3|x|/2}\int_{|y|=r} u_i(y)dSdr \\
&= \frac{C}{|x|^N}\int_{|x|/2}^{3|x|/2} r^{N-1}\bar{u}_i(r)dr \\
&\le \frac{C}{|x|^N}\int_{|x|/2}^{3|x|/2} r^{N-1+\beta_i}dr \\
&= \frac{C}{|x|^N}\big[\big(\frac{3|x|}{2}\big)^{N+\beta_i} - \big(\frac{|x|}{2}\big)^{N+\beta_i}\big] \\
&= C|x|^{\beta_i} \quad \text{at } \infty,
\end{aligned}$$

where $C > 0$ is a constant. Thus the proof is complete. □

Remark 2.3. In [1], M-F. Bidaut-Veron and P. Grillot have obtained important estimates of solutions on singularities for the case $m = 2$. In the case $m = 2$, by using Kelvin transformation, the estimates which they obtained become the same as those which we got in Theorem 2.2. Furthermore, it is important that these estimates hold without assumptions $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$.

2.2. **Radially symmetric system.** In this subsection we study the nonexistence of nonnegative nontrivial radial entire solutions of (1.1). Through this subsection we always assume that P_i, $i = 1, 2, \dots, m$, have radial symmetry.

Theorem 2.4. *Let $N \geq 3$. Suppose that P_i, $i = 1, 2, \dots, m$, satisfy*

$$P_i(r) \geq \frac{C_i}{r^{\lambda_i}}, \quad r \geq R_0 > 0, \tag{2.11}$$

where $C_i > 0$ and λ_i are constants. Moreover, Λ_i defined by (1.4) *satisfy*

$$\Lambda_i \leq 0 \quad \textit{for some } i \in \{1, 2, \dots, m\}. \tag{2.12}$$

If $(u_1, u_2, \dots, u_m)$ is a nonnegative radial entire solution of (1.1)*, then*

$$(u_1, u_2, \dots, u_m) \equiv (0, 0, \dots, 0)\,.$$

Theorem 2.5. *Let $N = 2$. Suppose that P_i, $i = 1, 2, \dots, m$, satisfy*

$$P_i(r) \geq \frac{C_i}{r^2(\log r)^{\lambda_i}}, \quad r \geq R_0 > 1, \tag{2.13}$$

where $C_i > 0$ and λ_i, $i = 1, 2, \dots, m$, are constants. Moreover

$$\Lambda_i \leq A - 1 \quad \textit{for some } i \in \{1, 2, \dots, m\}\,. \tag{2.14}$$

If $(u_1, u_2, \dots, u_m)$ is a nonnegative radial entire solution of (1.1)*, then*

$$(u_1, u_2, \dots, u_m) \equiv (0, 0, \dots, 0).$$

Theorem 2.6. *Let $N = 1$. Suppose that P_i, $i = 1, 2, \dots, m$, satisfy* (2.11) *with some constants $C_i > 0$ and λ_i, $i = 1, 2, \dots, m$. Moreover*

$$\Lambda_i \leq A - 1 \quad \textit{for some } i \in \{1, 2, \dots, m\}.$$

If $(u_1, u_2, \dots, u_m)$ is a nonnegative radial entire solution of (1.1)*, then*

$$(u_1, u_2, \dots, u_m) \equiv (0, 0, \dots, 0).$$

Proof of Theorem 2.4. Let $(u_1, u_2, \dots, u_m)$ be a nonnegative nontrivial radial entire solution of (1.1). Then $(u_1, u_2, \dots, u_m)$ satisfies the system of ordinary differential equations

$$\begin{gathered} (r^{N-1}u_i'(r))' = r^{N-1}P_i(r)u_{i+1}(r)^{\alpha_i}, \quad r > 0, \\ u_i'(0) = 0, \end{gathered} \qquad i = 1, 2, \dots, m. \tag{2.15}$$

Integrating (2.15) over $[0, r]$, we have

$$u_i'(r) = r^{1-N} \int_0^r s^{N-1} P_i(s) u_{i+1}(s)^{\alpha_i} ds, \quad i = 1, 2, \dots, m.$$

Hence, we see that u_i, $i = 1, 2, \dots, m$, are nondecreasing on $r \geq 0$. Integrating (2.15) twice over $[R, r]$, for $R \geq 0$ and $i = 1, 2, \dots, m$, we have

$$u_i(r) \geq u_i(R) + \frac{1}{N-2} \int_R^r s\big[1 - (\frac{s}{r})^{N-2}\big] P_i(s) u_{i+1}(s)^{\alpha_i} ds, \tag{2.16}$$

Since u_i, $i=1,2,\dots,m$, are nonnegative, nontrivial and nondecreasing functions, there exists an $r_*>0$ such that $u_{i_0}(r_*)>0$ for some $i_0\in\{1,2,\dots,m\}$. We may assume that $r_*\ge R_0$. We see from (2.16) with $R=r_*$ that $u_i(r)>0$ for $r>r_*,\ i=1,2,\dots,m$.

Using similar arguments as in the proof of Theorem 2.2, we obtain

$$u_i(r)\le C_i r^{\beta_i}\quad \text{at } \infty,\quad i=1,2,\dots,m, \tag{2.17}$$

where $C_i>0$ are constants and β_i are defined by (1.5). Note that our assumption (2.12) shows $\beta_i\le 0$ for some $i\in\{1,2,\dots,m\}$.

If there exists an $i_0\in\{1,2,\dots,m\}$ such that $\Lambda_{i_0}<0$, then we see that $\beta_{i_0}<0$ in (2.17). This shows that u_{i_0} tends to 0 as $r\to\infty$. On the other hand, from (2.16) with $R=r_*$ we see that

$$u_{i_0}(r)>u_{i_0}(r_*)>0,\quad r>r_*+1.$$

This is a contradiction. It remains only to discuss the case that $\Lambda_i\ge 0$, $i=1,2,\dots,m$. From the assumption of Λ_i, there exists an $i_0\in\{1,2,\dots,m\}$ such that $\Lambda_{i_0}=0$. Without loss of generality we may assume that $i_0=m$, that is,

$$\Lambda_i\ge 0,\quad i=1,2,\dots,m-1\quad\text{and}\quad \Lambda_m=0\,.$$

From the definition of β_i it follows that $\beta_i\ge 0$ and $\beta_m=0$.

We first observe that

$$\lambda_{m-1}\le 2 \tag{2.18}$$

and

$$\lambda_i\le -\sum_{j=1}^{m-i-1}\Big\{(\lambda_{i+j}-2)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\}+2,\quad i=1,2,\dots,m-2. \tag{2.19}$$

In fact, from the definition of Λ_i, we obtain

$$\begin{aligned}
\lambda_i &\ge -\sum_{j=1}^{m-1}\Big\{(\lambda_{i+j}-2)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\}+2\\
&=-\Big(\sum_{j=1}^{m-i-1}+\sum_{j=m-i+1}^{m-1}\Big)\Big\{(\lambda_{i+j}-2)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\}-(\lambda_m-2)\prod_{k=0}^{m-i-1}\alpha_{i+k}+2\\
&\equiv -S_1-S_2-S_3+2\,.
\end{aligned}$$

From the assumption on Λ_m, we have

$$\lambda_m-2=-\sum_{j=1}^{m-1}\Big\{(\lambda_{m+j}-2)\prod_{k=1}^{j-1}\alpha_{m+k}\Big\}.$$

Substituting this relation to S_3 we have

$$\begin{aligned}
S_3 &= -\sum_{j=1}^{m-1}\Big\{(\lambda_{m+j}-2)\prod_{k=0}^{j-1}\alpha_{m+k}\Big\}\prod_{k=0}^{m-i-1}\alpha_{i+k}\\
&= -\sum_{j=1}^{m-1}\Big\{(\lambda_{m+j}-2)\prod_{k=0}^{j+m-i-1}\alpha_{i+k}\Big\}\\
&= -\Big(\sum_{j=m-i+1}^{m-1}+\sum_{j=m}^{2m-i-1}\Big)\Big\{(\lambda_{i+j}-2)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\}\\
&= -S_2-\sum_{j=0}^{m-i-1}\Big\{(\lambda_{i+m+j}-2)\prod_{k=0}^{j+m-1}\alpha_{i+k}\Big\}\\
&= -S_2-S_1A-(\lambda_i-2)A.
\end{aligned}$$

Thus we obtain $\lambda_i \geq S_1(A-1)+(\lambda_i-2)A+2$, namely

$$\begin{aligned}
0 &\geq (A-1)(\lambda_i-2+S_1)\\
&= (A-1)\Big[\lambda_i-2+\sum_{j=1}^{m-i-1}\Big\{(\lambda_{i+j}-2)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\}\Big].
\end{aligned}$$

Since $A>1$, we see that (2.19) holds. Similarly we can get (2.18). From the above computation we see that

$$\lambda_i < -\sum_{j=1}^{m-i-1}\Big\{(\lambda_{i+j}-2)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\}+2 \quad \text{if } \Lambda_i>0$$

and

$$\lambda_i = -\sum_{j=1}^{m-i-1}\Big\{(\lambda_{i+j}-2)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\}+2 \quad \text{if } \Lambda_i=0\,.$$

For the rest of this article C denotes various positive constants. Integrating (2.15) twice over $[r_*, r]$, from (2.11), we have

$$\begin{aligned}
u_i(r) &\geq u_i(r_*)+\frac{1}{N-2}\int_{r_*}^{r} s\big[1-(\frac{s}{r})^{N-2}\big]P_i(s)u_{i+1}(s)^{\alpha_i}\,ds,\\
&\geq u_i(r_*)+\frac{C_i}{N-2}\big[1-(\frac{1}{2})^{N-2}\big]\int_{r_*}^{r/2} sP_i(s)u_{i+1}(s)^{\alpha_i}\,ds \qquad (2.20)\\
&\geq C\int_{r_*}^{r/2} s^{1-\lambda_i}u_{i+1}(s)^{\alpha_i}ds,
\end{aligned}$$

where $r\geq 2r_*$, $i=1,2,\dots,m$. We first consider the case that $\Lambda_{m-1}=0$. From (2.18) we see that $\lambda_{m-1}=2$. From (2.20) with $i=m-1$, we have

$$u_{m-1}(r)\geq Cu_m(r_*)^{\alpha_{m-1}}\int_{r_*}^{r/2} s^{-1}ds \geq C\log r, \quad r\geq r_1>2r_*.$$

On the other hand, we can see that $\beta_{m-1}=0$ in (2.17); that is, u_{m-1} is bounded near infinity. This is a contradiction.

Next we consider the case that $\Lambda_{m-2} = 0$. Then we see from (2.18) and (2.19) with $i = m-2$ that

$$\lambda_{m-1} < 2 \quad \text{and} \quad \lambda_{m-2} = -(\lambda_{m-1} - 2)\alpha_{m-2} + 2.$$

From (2.20) with $i = m-1$ we have

$$u_{m-1}(r) \geq C u_m(r_*)^{\alpha_{m-1}} \int_{r_*}^{r/2} s^{1-\lambda_{m-1}} ds \geq C r^{2-\lambda_{m-1}}, \quad r \geq r_1 > 2r_*.$$

From this estimate and (2.20) with $i = m-2$ we obtain

$$\begin{aligned} u_{m-2}(r) &\geq C \int_{r_1}^{r/2} s^{1-\lambda_{m-2}+(2-\lambda_{m-1})\alpha_{m-2}} ds \\ &= C \int_{r_1}^{r/2} s^{-1} ds \\ &\geq C \log r, \quad r \geq r_2 > 2r_1 . \end{aligned}$$

On the other hand, we can see that $\beta_{m-2} = 0$ in (2.17); that is, u_{m-2} is bounded near infinity. This is a contradiction.

Similarly, suppose that there exists an $i_0 \in \{1, 2, \dots, m\}$ such that $\Lambda_{i_0} = 0$ and $\Lambda_i > 0$, $i = i_0 + 1, \dots, m-1$. Then we see from (2.18) and (2.20) with $i = m-1$ that

$$u_{m-1}(r) \geq C r^{2-\lambda_{m-1}}, \quad r \geq r_1 > 2r_*.$$

From this estimate, (2.19) with $i = m-2$, (2.20) with $i = m-2$, we have

$$\begin{aligned} u_{m-2}(r) &\geq C \int_{r_*}^{r} s^{1-\lambda_{m-2}+\alpha_{m-2}(2-\lambda_{m-1})} ds \\ &\geq C r^{2-\lambda_{m-2}+\alpha_{m-2}(2-\lambda_{m-1})}, \quad r \geq r_2 > 2r_1. \end{aligned}$$

By repeating the above procedure, we get a sequence $\{r_j\}_{j=2}^{m-i_0-1}$ such that

$$u_i(r) \geq C r^{\tau_i}, \quad r \geq r_j > 2r_{j-1}, \quad i = m-2, m-3, \dots, i_0 + 1,$$

where

$$\begin{aligned} \tau_i &= 2 - \lambda_i + \alpha_i \tau_{i+1} \\ &= 2 - \lambda_i + \sum_{j=1}^{m-i-1} \Big\{ (2 - \lambda_{i+j}) \prod_{k=0}^{j-1} \alpha_{i+k} \Big\} > 0. \end{aligned}$$

From (2.19) with $i = i_0$ and (2.20) with $i = i_0$, we have

$$\begin{aligned} u_{i_0}(r) &\geq C \int_{r_{m-i_0-1}}^{r/2} s^{1-\lambda_{i_0}+\alpha_{i_0}\tau_{i_0+1}} ds \\ &= C \int_{r_{m-i_0-1}}^{r/2} s^{-1} ds \\ &\geq C \log r, \quad r \geq r_{m-i_0} > 2r_{m-i_0-1}. \end{aligned}$$

On the other hand, since $\Lambda_{i_0} = 0$, we have $\beta_{i_0} = 0$ in (2.17). This yields a contradiction. Thus the proof of Theorem 2.4 is complete. $\square$

Proof of Theorem 2.5. Suppose to the contrary that (1.1) has a nonnegative nontrivial radial entire solution $(u_1, u_2, \dots, u_m)$. Then $(u_1, u_2, \dots, u_m)$ satisfies (2.15). Integrating (2.15) twice over $[0, r]$, we have

$$u_i(r) = u_i(0) + \int_0^r s\log(\frac{r}{s})P_i(s)u_{i+1}(s)^{\alpha_i}ds, \quad i = 1, 2, \dots, m. \tag{2.21}$$

Let $r \ge e$. Then from (2.21), we have

$$\begin{aligned} u_i(r) = u_i(0) &+ \int_0^1 s\log(\frac{r}{s})P_i(s)u_{i+1}(s)^{\alpha_i}ds \\ &+ \int_1^e s\log(\frac{r}{s})P_i(s)u_{i+1}(s)^{\alpha_i}ds + \int_e^r s\log(\frac{r}{s})P_i(s)u_{i+1}(s)^{\alpha_i}ds \\ &\ge u_i(0) + u_{i+1}(0)^{\alpha_i}\int_0^1 sP_i(s)ds\log r + \int_e^r s\log(\frac{r}{s})P_i(s)u_{i+1}(s)^{\alpha_i}ds \\ &\ge \tilde{C}_i\log r + \int_e^r s\log(\frac{r}{s})P_i(s)u_{i+1}(s)^{\alpha_i}ds, \quad r \ge e, \end{aligned} \tag{2.22}$$

where $i = 1, 2, \dots, m$ and $\tilde{C}_i \ge 0$ are constants.

Let $u_i(r) = v_i(r)\log r$. Then from (2.22), we have

$$v_i(r) \ge \tilde{C}_i + \int_e^r s\left(1 - \frac{\log s}{\log r}\right)P_i(s)(\log s)^{\alpha_i}v_{i+1}(s)^{\alpha_i}ds. \tag{2.23}$$

Let $t = \log s$, $\eta = \log r$, and $v_i(r) = v_i(e^\eta) = \tilde{v}_i(\eta)$. Then (2.23) becomes

$$\tilde{v}_i(\eta) \ge \tilde{C}_i + \int_1^\eta t\big(1 - \frac{t}{\eta}\big)\tilde{P}_i(t)\tilde{v}_i(t)^{\alpha_i}dt, \quad i = 1, 2, \dots, m,$$

where $\tilde{P}_i$, $i = 1, 2, \dots, m$, are given by $\tilde{P}_i(t) = e^{2t}P_i(e^t)t^{\alpha_i-1}$. From (2.13), we have

$$\tilde{P}_i(t) \ge e^{2t}\frac{C_i}{e^{2t}(\log e^t)^{\lambda_i}}t^{\alpha_i-1} = \frac{C_i}{t^{\lambda_i-\alpha_i+1}}, \quad t \ge \log R_0, \quad i = 1, 2, \dots, m.$$

From (2.14) and the definition of Λ_i,

$$\begin{aligned} \lambda_i - \alpha_i + 1 &= \Lambda_i + 2 - \sum_{j=1}^{m-1}\Big\{(\lambda_{i+j} - 2)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\} - \alpha_i + 1 \\ &\le 2 - \sum_{j=1}^{m-1}\Big\{\big((\lambda_{i+j} - \alpha_{i+j} + 1) - 2\big)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\} \\ &\quad + A - \alpha_i - \sum_{j=1}^{m-1}\Big\{(\alpha_{i+j} - 1)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\} \\ &= 2 - \sum_{j=1}^{m-1}\Big\{\big((\lambda_{i+j} - \alpha_{i+j} + 1) - 2\big)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\}, \end{aligned}$$

namely, for some $i \in \{1, 2, \dots, m\}$,

$$(\lambda_i - \alpha_i + 1) - 2 + \sum_{j=1}^{m-1}\Big\{((\lambda_{i+j} - \alpha_{i+j} + 1) - 2)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\} \le 0\,.$$

Using similar arguments as in the proof of Theorem 2.4, we obtain a contradiction. Thus the proof is complete. ∎

Proof of Theorem 2.6. Let $(u_1, u_2, \dots, u_m)$ be a nonnegative nontrivial radial entire solution of (1.1). Then by integrating (1.1) over $[0, r]$, we have

$$\begin{aligned} u_i(r) &= u_i(0) + \int_0^1 (r-s)P_i(s)u_{i+1}(s)^{\alpha_i}ds + \int_1^r (r-s)P_i(s)u_{i+1}(s)^{\alpha_i}ds \\ &\geq u_i(0) + u_{i+1}(0)^{\alpha_i}\int_0^1 r\Big(1-\frac{s}{r}\Big)P_i(s)ds + \int_1^r (r-s)P_i(s)u_{i+1}(s)^{\alpha_i}ds \\ &\geq \tilde{C}_i r + \int_1^r (r-s)P_i(s)u_{i+1}(s)^{\alpha_i}ds, \end{aligned}$$

where $i = 1, 2, \dots, m$, $r \geq 2$, and $\tilde{C}_i \geq 0$ are constants.

Setting $u_i(r) = rv_i(r)$ for $r \geq 2$ and $i = 1, 2, \dots, m$, we obtain

$$v_i(r) \geq \tilde{C}_i + \int_1^r s\Big(1-\frac{s}{r}\Big)\tilde{P}_i(s)v_{i+1}(s)^{\alpha_i}ds,$$

where $\tilde{P}_i(s) = P_i(s)s^{\alpha_i - 1}$. From (2.11), we have

$$\tilde{P}_i(s) \geq \frac{C_i}{s^{\lambda_i - \alpha_i + 1}}, \quad s \geq R_0, \quad i = 1, 2, \dots, m.$$

Using the same computation as in the proof of Theorem 2.5, we can see that for some $i \in \{1, 2, \dots, m\}$,

$$(\lambda_i - \alpha_i + 1) - 2 + \sum_{j=1}^{m-1}\Big\{\big((\lambda_{i+j} - \alpha_{i+j} + 1) - 2\big)\prod_{k=0}^{j-1}\alpha_{i+k}\Big\} \leq 0\,.$$

From the proof of Theorem 2.4, we get a contradiction. Thus the proof is complete. □

2.3. **System** (1.1) **without radial symmetry.** In this subsection we consider the nonexistence of nonnegative nontrivial entire solutions of (1.1) without radial symmetry. Through this subsection we always assume that $\alpha_i \geq 1$, $i = 1, 2, \dots, m$, and $A > 1$.

Theorem 2.7. *Let $N \geq 3$. Suppose that P_i, $i = 1, 2, \dots, m$, satisfy*

$$\liminf_{|x|\to\infty} |x|^{\lambda_i} P_i(x) > 0, \tag{2.24}$$

where λ_i, $i = 1, 2, \dots, m$, are constants. Also $\Lambda_i \leq 0$ for some $i \in \{1, 2, \dots, m\}$. If $(u_1, u_2, \dots, u_m)$ is nonnegative entire solution of (1.1)*, then*

$$(u_1, u_2, \dots, u_m) \equiv (0, 0, \dots, 0).$$

Theorem 2.8. *Let $N = 2$. Suppose that P_i, $i = 1, 2, \dots, m$, satisfy*

$$\liminf_{|x|\to\infty} |x|^2(\log|x|)^{\lambda_i} P_i(x) > 0, \tag{2.25}$$

where λ_i are constants. Moreover $\Lambda_i \leq A - 1$ for some $i \in \{1, 2, \dots, m\}$. If $(u_1, u_2, \dots, u_m)$ is nonnegative entire solution of (1.1)*, then*

$$(u_1, u_2, \dots, u_m) \equiv (0, 0, \dots, 0).$$

Theorem 2.9. *Let $N = 1$. Suppose that P_i, satisfy* (2.24) *with some constants λ_i, $i = 1, 2, \dots, m$. Moreover $\Lambda_i \leq A - 1$ for some $i \in \{1, 2, \dots, m\}$. If $(u_1, u_2, \dots, u_m)$ is nonnegative entire solution of* (1.1)*, then*

$$(u_1, u_2, \dots, u_m) \equiv (0, 0, \dots, 0).$$

Suppose that (2.24) holds. Then there exist some constants $C_i > 0$ and $R_0 > 0$ such that

$$P_i(x) \ge \frac{C_i}{|x|^{\lambda_i}}, \quad |x| \ge R_0, \quad i = 1, 2, \dots, m\,.$$

So we can see that $\hat{P}_i$ defined by (2.1) satisfy

$$\hat{P}_i(r) \ge \frac{C_i}{r^{\lambda_i}}, \quad r \ge R_0\,.$$

Similarly, suppose that (2.25) holds. Then $\hat{P}_i$ satisfy

$$\hat{P}_i(r) \ge \frac{C_i}{r^2(\log r)^{\lambda_i}}, \quad r \ge R_0 > 1,$$

where $i = 1, 2, \dots, m$, and $C_i > 0$ are some constants.

The proof of Theorem 2.7 follows from Lemma 2.1, Theorem 2.2 and the proof of Theorem 2.4. Similarly, the proofs of Theorems 2.8 and 2.9 follow from Lemma 2.1 and the proofs of Theorems 2.5 and 2.6, respectively.

Remark 2.10. When $m = 2$, our nonexistence results (Theorems 2.7–2.9) reduce to those obtained in [12]. However, the proofs presented here are simpler than in [12].

3. Existence results

In this section we consider existence of positive radial entire solutions of the semilinear elliptic system

$$\begin{gathered} \Delta u_1 = P_1(|x|)u_2^{\alpha_1}, \\ \Delta u_2 = P_2(|x|)u_3^{\alpha_2}, \\ \vdots \\ \Delta u_m = P_m(|x|)u_{m+1}^{\alpha_m}, \quad u_{m+1} = u_1\,. \end{gathered} \tag{3.1}$$

Through this section, we assume that $P_i(r)$, $r = |x|$, $i = 1, 2, \dots, m$, are nonnegative continuous functions and $\alpha_i > 0$ are constants satisfying $A > 1$.

Theorem 3.1. *Let $N \ge 3$. Suppose that P_i satisfy*

$$P_i(r) \le \frac{C_i}{r^{\lambda_i},} \quad r \ge R_0 > 0, \tag{3.2}$$

where $i = 1, 2, \dots, m$, and $C_i > 0$, λ_i are constants. Moreover $\Lambda_i > 0$, $i = 1, 2, \dots, m$. Then (3.1) *has infinitely many positive radial entire solutions.*

Theorem 3.2. *Let $N = 2$. Suppose that P_i satisfy*

$$P_i(r) \le \frac{C_i}{r^2(\log r)^{\lambda_i}}, \quad r \ge R_0 > 1, \tag{3.3}$$

where $i = 1, 2, \dots, m$, and $C_i > 0$ and λ_i are constants. Moreover

$$\Lambda_i > A - 1, \quad i = 1, 2, \dots, m\,. \tag{3.4}$$

Then (3.1) *has infinitely many positive radial entire solutions.*

Theorem 3.3. *Let $N = 1$. Suppose that P_i satisfy* (3.2) *with some constants $C_i > 0$ and λ_i, $i = 1, 2, \dots, m$. Moreover $\Lambda_i > A - 1$, $i = 1, 2, \dots, m$. Then* (3.1) *has infinitely many positive entire solutions.*

We give an example that shows the sharpness of our results.

Example. Let us consider the elliptic system

$$\begin{aligned}
\Delta u_1 &= \frac{1}{(1+|x|)^{\lambda_1}} u_2^{\alpha_1}, \\
\Delta u_2 &= \frac{1}{(1+|x|)^{\lambda_2}} u_3^{\alpha_2}, \\
&\vdots \\
\Delta u_m &= \frac{1}{(1+|x|)^{\lambda_m}} u_1^{\alpha_m},
\end{aligned} \tag{3.5}$$

where $x \in \mathbb{R}^N$, $N \geq 3$, and $\alpha_i > 0, i = 1, 2, \dots, m$, are constants satisfying $\alpha_1 \alpha_2 \cdots \alpha_m > 1$. We can completely characterize the existence of positive radial entire solutions of this system in terms of α_i and λ_i, $i = 1, 2, \dots, m$. In fact, we can see that the inequalities

$$\frac{C_i}{|x|^{\lambda_i}} \leq \frac{1}{(1+|x|)^{\lambda_i}} \leq \frac{\tilde{C}_i}{|x|^{\lambda_i}}, \quad |x| \geq 1, \quad i = 1, 2, \dots, m$$

hold for some constants $C_i > 0$ and $\tilde{C}_i > 0$, $i = 1, 2, \dots, m$. Then, from Theorem 2.4 and Theorem 3.1, a necessary and sufficient condition for (3.5) to have positive radial entire solution is

$$\Lambda_i > 0, \quad i = 1, 2, \dots, m.$$

Proof of Theorem 3.1. Without loss of generality, we assume that $R_0 = 1$ in (3.2). We first observe that $(u_1, u_2, \dots, u_m)$ is a positive radial entire solution of (3.1) if and only if the function $(v_1(r), v_2(r), \dots, v_m(r)) = (u_1(x), u_2(x), \dots, u_m(x))$, $r = |x|$, satisfies the system of second order ordinary differential equations

$$\begin{gathered}
r^{1-N}(r^{N-1}v_i')' = P_i(r) v_{i+1}^{\alpha_i}, \quad r > 0, \\
v_i'(0) = 0,
\end{gathered} \tag{3.6}$$

where $i = 1, 2, \dots, m$, and $' = d/dr$. Integrating (3.6) twice, we obtain the following system of integral equations equivalent to (3.6):

$$v_i(r) = a_i + \frac{1}{N-2} \int_0^r s\big[1 - (\frac{s}{r})^{N-2}\big] P_i(s) v_{i+1}(s)^{\alpha_i} ds\,, \tag{3.7}$$

where $r \geq 0$, $i = 1, 2, \dots, m$, and $a_i = v_i(0)$. Therefore, it suffices to solve (3.7). Choose constants $a_i > 0$, $i = 1, 2, \dots, m$, so that

$$\begin{gathered}
\frac{(2a_{i+1})^{\alpha_i}}{N-2} \int_0^1 s P_i(s) ds \leq \frac{a_i}{2}, \\
\frac{C_i (2a_{i+1})^{\alpha_i}}{(N-2)(2 - \lambda_i + \alpha_i \beta_{i+1})} \leq \frac{a_i}{2},
\end{gathered} \tag{3.8}$$

where β_i, $i = 1, 2, \dots, m$, are defined by (1.5). It is possible to choose such a_i's by the assumption $A > 1$. We note that $2 - \lambda_i + \alpha_i \beta_{i+1} = \beta_i$ by the definitions of Λ_i and β_i. Define the functions F_i, $i = 1, 2, \dots, m$, by

$$F_i(r) = \begin{cases} 2a_i & \text{for } 0 \leq r \leq 1, \\ 2a_i r^{\beta_i} & \text{for } r \geq 1. \end{cases}$$

We regard the space $(C[0,\infty))^m$ as a Fréchet space equipped with the topology of uniform convergence of functions on each compact subinterval of $[0,\infty)$. Let $X \subset (C[0,\infty))^m$ denotes the subset defined by

$$X = \{(v_1, v_2, \dots, v_m) \in (C[0,\infty))^m : a_i \le v_i(r) \le F_i(r),\ r \ge 0,\ 1 \le i \le m\}.$$

Clearly, X is a non-empty closed convex subset of $(C[0,\infty))^m$. Define the mapping $\mathcal{F} : X \to (C[0,\infty))^m$ by $\mathcal{F}(v_1, v_2, \dots, v_m) = (\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_m)$, where

$$\tilde{v}_i(r) = a_i + \frac{1}{N-2} \int_0^r s\big[1 - (\frac{s}{r})^{N-2}\big] P_i(s) v_{i+1}(s)^{\alpha_i} ds, \quad r \ge 0\,.$$

To apply the Schauder-Tychonoff fixed point theorem, we show that $\mathcal{F}$ is a continuous mapping from X into itself such that $\mathcal{F}(X)$ is relatively compact.
(I) $\mathcal{F}$ maps X into itself. Let $(v_1, v_2, \dots, v_m) \in X$. Clearly, $\tilde{v}_i \ge a_i$, $i = 1, 2, \dots, m$. For $0 \le r \le 1$, we have

$$\begin{aligned}
\tilde{v}_i(r) &\le a_i + \frac{1}{N-2} \int_0^r s P_i(s) v_{i+1}(s)^{\alpha_i} ds \\
&\le a_i + \frac{(2a_{i+1})^{\alpha_i}}{N-2} \int_0^1 s P_i(s) ds \\
&\le a_i + \frac{a_i}{2} < 2a_i, \quad i = 1, 2, \dots, m.
\end{aligned}$$

For $r \ge 1$, from (3.2), we have

$$\begin{aligned}
\tilde{v}_i(r) &\le a_i + \frac{1}{N-2} \int_0^1 s P_i(s) v_{i+1}(s)^{\alpha_i} ds + \frac{1}{N-2} \int_1^r s P_i(s) v_{i+1}(s)^{\alpha_i} ds \\
&\le \frac{3a_i}{2} + \frac{(2a_{i+1})^{\alpha_i} C_i}{N-2} \int_1^r s^{1-\lambda_i+\alpha_i\beta_{i+1}} ds \\
&\le \frac{3a_i}{2} + \frac{(2a_{i+1})^{\alpha_i} C_i}{(N-2)(2-\lambda_i+\alpha_i\beta_{i+1})} r^{2-\lambda_i+\alpha_i\beta_{i+1}} \\
&\le \frac{3a_i}{2} + \frac{a_i}{2} r^{\beta_i} \le 2a_i r^{\beta_i}, \quad i = 1, 2, \dots, m.
\end{aligned}$$

Therefore, $\mathcal{F}(X) \subset X$.
(II) $\mathcal{F}$ is continuous. Let $\{(v_{1,l}, v_{2,l}, \dots, v_{m,l})\}_{l=1}^{\infty}$ be a sequence in X which converges to $(v_1, v_2, \dots, v_m) \in X$ uniformly on each compact subinterval of $[0,\infty)$. Then

$$\begin{aligned}
|\tilde{v}_{i,l}(r) - \tilde{v}_i(r)| &\le \frac{1}{N-2} \int_0^r s\big[1 - (\frac{s}{r})^{N-2}\big] P_i(s) |v_{i+1,l}(s)^{\alpha_i} - v_{i+1}(s)^{\alpha_i}| ds \\
&\le \frac{1}{N-2} \int_0^r s P_i(s) |v_{i+1,l}(s)^{\alpha_i} - v_{i+1}(s)^{\alpha_i}| ds, \quad i = 1, 2, \dots, m\,.
\end{aligned}$$

Since the functions $h_{i,l}(s) = sP_i(s)|v_{i+1,l}(s)^{\alpha_i} - v_{i+1}(s)^{\alpha_i}|$, $l \in \mathbb{N}$, $1 \le i \le m$, satisfy $h_{i,l}(s) \le 2sP_i(s)F_{i+1}(s)^{\alpha_i}$, $s \ge 0$, and $\{h_{i,l}(s)\}_{l=1}^{\infty}$, $i = 1, 2, \dots, m$, converge to 0 at every point s, the Lebesgue dominated convergence theorem implies that $\{\tilde{v}_{i,l}\}_{l=1}^{\infty}$, $i = 1, 2, \dots, m$, converge to $\tilde{v}_i$ uniformly on each compact subinterval of $[0,\infty)$. These imply the continuity of $\mathcal{F}$.
(III) $\mathcal{F}(X)$ is relatively compact. It suffices to show the local equicontinuity of $\mathcal{F}(X)$, since $\mathcal{F}(X)$ is locally uniformly bounded by the fact that $\mathcal{F}(X) \subset X$. Let

$(v_1, v_2, \dots, v_m) \in X$ and $R > 0$. Then we have

$$\tilde{v}_i'(r) = \int_0^r (\frac{s}{r})^{N-1} P_i(s) v_{i+1}(s)^{\alpha_i} ds \le \int_0^R P_i(s) F_{i+1}(s)^{\alpha_i} ds\,.$$

These imply the local boundedness of the set $\{(\tilde{v}_1', \tilde{v}_2', \dots, \tilde{v}_m');\ (v_1, v_2, \dots, v_m) \in X\}$. Hence the relative compactness of $\mathcal{F}(X)$ is shown by the Ascoli-Arzelà theorem.

Therefore, applying the Schauder-Tychonoff fixed point theorem, there exists an element $(v_1, v_2, \dots, v_m) \in X$ such that $(v_1, v_2, \dots, v_m) = \mathcal{F}(v_1, v_2, \dots, v_m)$, that is, $(v_1, v_2, \dots, v_m)$ satisfies the system of integral equations (3.7). The function $(u_1(x), u_2(x), \dots, u_m(x)) = (v_1(|x|), \dots, v_m(|x|))$ then gives a solution of (3.6). Since infinitely many $(a_1, a_2, \dots, a_m)$ satisfy (3.8), we can construct an infinitude of positive radial entire solutions of (3.1). This completes the proof. □

Proof of Theorem 3.2. Without loss of generality, we may assume that $R_0 = e$ in (3.3). As before, it suffices to solve the following system of integral equations:

$$v_i(r) = a_i + \int_0^r s \log(\frac{r}{s}) P_i(s) v_{i+1}(s)^{\alpha_i} ds, \quad r \ge 0, \quad i = 1, 2, \dots, m,$$

where $a_i = v_i(0)$. Choose constants $a_i > 0$ so that

$$\begin{gathered}(2a_{i+1})^{\alpha_i} e \int_0^e P_i(s) ds \le \frac{a_i}{2},\\ \frac{(2a_{i+1})^{\alpha_i} C_i}{1 - \lambda_i + \alpha_i \beta_{i+1}} \le \frac{a_i}{2},\end{gathered}$$

where β_i, $i = 1, 2, \dots, m$, are defined by (1.5). It is possible to choose such a_i's by the assumption $A > 1$. We notice that $\beta_i > 1$ by the assumption (3.4). Define the functions F_i, $i = 1, 2, \dots, m$, by

$$F_i(r) = \begin{cases} 2a_i & \text{for } 0 \le r \le e,\\ 2a_i (\log r)^{\beta_i} & \text{for } r \ge e. \end{cases}$$

Consider the set

$$Y = \{(v_1, v_2, \dots, v_m) \in (C[0,\infty))^m : a \le v_i(r) \le F_i(r),\ r \ge 0,\ 1 \le i \le m\},$$

which is a closed convex subset of $(C[0,\infty))^m$. Define the mapping $\mathcal{F} : Y \to (C[0,\infty))^m$ by $\mathcal{F}(v_1, v_2, \dots, v_m) = (\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_m)$, where

$$\tilde{v}_i(r) = a_i + \int_0^r s \log(\frac{r}{s}) P_i(s) v_{i+1}(s)^{\alpha_i} ds, \quad r \ge 0, \quad i = 1, 2, \dots, m\,.$$

We will verify that $\mathcal{F}$ is a continuous mapping from Y into itself such that $\mathcal{F}(Y)$ is relatively compact.

We first show that $\mathcal{F}$ maps Y into itself. Let $(v_1, v_2, \dots, v_m) \in Y$. It is clear that $\tilde{v}_i \ge a_i$, $i = 1, 2, \dots, m$. Let $0 \le r \le e$. Then, using the inequality $0 \le s \log(r/s) \le r/e$ for $0 \le s \le r$, we have

$$\begin{aligned}\tilde{v}_i(r) &\le a_i + \frac{r}{e} \int_0^r P_i(s) v_i(s)^{\alpha_i} ds\\ &\le a_i + (2a_{i+1})^{\alpha_i} \int_0^e P_i(s) ds\\ &\le a_i + \frac{a_i}{2} < 2a_i,\ i = 1, 2, \dots, m.\end{aligned}$$

Let $r \ge e$. Then we write

$$\tilde{v}_i(r) = a_i + \Big(\int_0^1 + \int_1^e + \int_e^r\Big) s\log(\frac{r}{s}) P_i(s) v_{i+1}(s)^{\alpha_i} ds \equiv a_i + I_1 + I_2 + I_3.$$

The inequality $0 \le s\log(r/s) \le \log r$ for $0 \le s \le 1$ implies that

$$I_1 \le \int_0^1 P_i(s) v_{i+1}(s)^{\alpha_i} ds \log r \le (2a_{i+1})^{\alpha_i} e \int_0^1 P_i(s) ds \log r. \tag{3.9}$$

The integrals I_2 and I_3 are estimated as follows:

$$\begin{aligned} I_2 &\le \int_1^e sP_i(s) v_{i+1}(s)^{\alpha_i} ds \log r \\ &\le (2a_{i+1})^{\alpha_i} \int_1^e sP_i(s) ds \log r \\ &\le (2a_{i+1})^{\alpha_i} e \int_1^e P_i(s) ds \log r \ ; \end{aligned} \tag{3.10}$$

$$\begin{aligned} I_3 &\le \int_e^r sP_i(s) v_{i+1}(s)^{\alpha_i} ds \log r \\ &\le (2a_{i+1})^{\alpha_i} C_i \int_e^r s^{-1} (\log s)^{-\lambda_i + \alpha_i\beta_{i+1}} ds \log r \\ &= (2a_{i+1})^{\alpha_i} C_i \int_1^{\log r} t^{-\lambda_i + \alpha_i\beta_{i+1}} dt \log r \\ &\le \frac{(2a_{i+1})^{\alpha_i} C_i}{1 - \lambda_i + \alpha_i\beta_{i+1}} (\log r)^{2-\lambda_i+\alpha_i\beta_{i+1}} \\ &\le \frac{a_i}{2} (\log r)^{\beta_i}. \end{aligned} \tag{3.11}$$

From (3.9) and (3.10), we have

$$I_1 + I_2 \le (2a_{i+1})^{\alpha_i} e \int_0^e P_i(s) ds \log r \le \frac{a_i}{2} (\log r)^{\beta_i}. \tag{3.12}$$

Thus by (3.11) and (3.12) we obtain $\tilde{v}_i(r) \le 2a_i (\log r)^{\beta_i}$, $i = 1, 2, \dots, m$. Therefore, $\mathcal{F}(v_1, v_2, \dots, v_m) \in Y$.

The continuity of $\mathcal{F}$ and the relative compactness of $\mathcal{F}(Y)$ can be verified in a routine manner. Thus there exists an element $(v_1, v_2, \dots, v_m) \in Y$ such that $(v_1, v_2, \dots, v_m) = \mathcal{F}(v_1, v_2, \dots, v_m)$ by the Schauder-Tychonoff fixed point theorem. It is clear that this $(v_1, v_2, \dots, v_m)$ gives rise to a positive radial entire solution $(u_1(x), u_2(x), \dots, u_m(x)) = (v_1(|x|), v_2(|x|), \dots, v_m(|x|))$ of (3.1). $\square$

The proof of Theorem 3.3 is the same as that of Theorem 3.1. So we leave the proof to the reader.

4. Liouville type theorem

Consider the semilinear elliptic system

$$\begin{aligned} \Delta u_1 &= P_1(x) u_2^{\alpha_1}, \\ \Delta u_2 &= P_2(x) u_3^{\alpha_2}, \\ &\vdots \\ \Delta u_m &= P_m(x) u_{m+1}^{\alpha_m}, \ u_{m+1} = u_1, \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^N$, $N \geq 3$ and $m \geq 2$ are integers and $\alpha_i > 0$, $i = 1, 2, \dots, m$, are constants satisfying $\alpha_1 \alpha_2 \cdots \alpha_m > 1$. Suppose that

$$P_i(x) \geq \frac{C_i}{|x|^{\lambda_i}}, \quad |x| \geq x_0 > 0, \quad i = 1, 2, \dots, m,$$

hold for some constants $C_i > 0$ and $\lambda_i \in \mathbb{R}$, satisfying $\Lambda_i \leq 0$ for some $i \in \{1, 2, \dots, m\}$. If, in addition, $\alpha_i \geq 1$, $i = 1, 2, \dots, m$, then as studied in Sections 2.2 and 2.3 one can conclude from Theorems 2.4 and 2.7 that system (4.1) has no nonnegative nontrivial entire solutions. However, if at least one of α_i, $i \in \{1, 2, \dots, m\}$, is less than 1, then one cannot derive any information about the nonnegative nontrivial entire solutions without radial symmetry. When $\alpha_1 \alpha_2 \cdots \alpha_m > 1$ and the same hypothesis of Theorem 2.7 hold, does not (4.1) possess a nonnegative nontrivial entire solutions? To give a partial answer this question we prove a Liouville type theorem for nonnegative entire solutions of (4.1). Our result is as follows:

Theorem 4.1. *Let $N \geq 3$. Suppose that*

$$\liminf_{|x| \to \infty} |x|^{\lambda_i} P_i(x) > 0, \quad i = 1, 2, \dots, m, \tag{4.2}$$

hold for some constants $\lambda_i, i = 1, 2, \dots, m$, and there exists an $i_0 \in \{1, 2, \dots, m\}$ such that $\Lambda_{i_0} \leq 0$. If $(u_1, u_2, \dots, u_m)$ is a nonnegative entire solution of (4.1) satisfying

$$u_{i_0}(x) = O(\exp |x|^\rho) \text{ as } |x| \to \infty \quad \text{for some } \rho > 0, \tag{4.3}$$

then $(u_1, u_2, \dots, u_m) \equiv (0, 0, \dots, 0)$.

The next lemma is needed in proving Theorem 4.1.

Lemma 4.2. *Let $(u_1, u_2, \dots, u_m)$ be a nonnegative entire solution of (4.1), and $b \in (0, 1)$ be a constant. Then its spherical mean $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m)$ satisfies the ordinary differential inequalities*

$$\begin{aligned} \bar{u}_i'(r) \geq \tilde{C}_i r P_{i*}(r) \bar{u}_{i+1}(br)^{\alpha_i}, \ r > 0, \\ \bar{u}_i'(0) = 0, \end{aligned} \quad i = 1, 2, \dots, m, \tag{4.4}$$

where $\tilde{C}_i = \tilde{C}_i(N, \alpha_i, b) > 0$, $i = 1, 2, \dots, m$, are constants and

$$P_{i*}(r) = \min_{|x| \leq r} P_i(x), \quad r \geq 0, \quad i = 1, 2, \dots, m.$$

To prove this lemma, we present the following lemma; see [4, p.244] or [11, p.225].

Lemma 4.3. *Let D be a domain in $\mathbb{R}^N$. Suppose that $\sigma > 0$ is a constant, and $x_0 \in D$ and $r > 0$ satisfy $B_{2r}(x_0) \equiv \{x \in \mathbb{R}^N; |x - x_0| \leq 2r\} \subset D$. Then, we can find a constant $C = C(N, \sigma) > 0$ satisfying*

$$\Big(\max_{B_r(x_0)} u \Big)^\sigma \leq \frac{C}{r^N} \int_{B_{2r}(x_0)} u^\sigma dx,$$

for any function $u \in C^2(D)$ satisfying $u \geq 0$, $\Delta u \geq 0$ in D.

Proof of Lemma 4.2. Let $(u_1, u_2, \dots, u_m)$ be a nonnegative entire solution of (4.1). By taking the mean value of (4.1), we have

$$(r^{N-1} \bar{u}_i'(r))' = \frac{1}{\omega_N} \int_{|x| = r} P_i(x) u_{i+1}(x)^{\alpha_i} dS, \quad r \geq 0, \quad i = 1, 2, \dots, m. \tag{4.5}$$

Since an integration of (4.5) shows that $\bar{u}_i(r)$, $i = 1, 2, \dots, m$, are nondecreasing on $[0, \infty)$, we may assume that $b > 1/2$ in (4.4). Put $b = 1 - a$, $a \in (0, 1/2)$. Integrating (4.5) over $[0, r]$, we have

$$\bar{u}_i'(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|\le r} P_i(x) u_{i+1}(x)^{\alpha_i} dx \ge \frac{P_{i*}(r)}{\omega_N r^{N-1}} \int_{|x|\le r} u_{i+1}(x)^{\alpha_i} dx\,. \tag{4.6}$$

Let $r > 0$ be fixed. We take $y_{i+1} \in \mathbb{R}^N$, $i = 1, 2, \dots, m$, such that

$$u_{i+1}(y_{i+1}) = \max_{|x|=(1-a)r} u_{i+1}(x) \quad \Big(= \max_{|x|\le(1-a)r} u_{i+1}(x)\Big),$$

and take $z_{i+1} \in \mathbb{R}^N$, $i = 1, 2, \dots, m$, such that $z_{i+1} = My_{i+1}$, $0 < M < 1$, and $|y_{i+1} - z_{i+1}| = ar$. Then we can see that

$$\int_{|x|\le r} u_{i+1}(x)^{\alpha_i} dx \ge \int_{|x-z_{i+1}|\le 2ar} u_{i+1}(x)^{\alpha_i} dx, \quad i = 1, 2, \dots, m,$$

and using Lemma 4.3, we obtain

$$\begin{aligned}
\int_{|x-z_{i+1}|\le 2ar} u_{i+1}(x)^{\alpha_i} dx &\ge C_i r^N \Big(\max_{|x-z_{i+1}|\le ar} u_{i+1}(x) \Big)^{\alpha_i} \\
&= C_i r^N \{u_{i+1}(y_{i+1})\}^{\alpha_i} \\
&= C_i r^N \Big(\max_{|x|=(1-a)r} u_{i+1}(x) \Big)^{\alpha_i} \\
&\ge C_i r^N \bar{u}_{i+1}((1-a)r)^{\alpha_i},
\end{aligned}$$

where $i = 1, 2, \dots, m$ and $C_i = C_i(N, \alpha_i, a) > 0$ are constants. ¿From this estimate and (4.6) we obtain (4.4). Thus the proof is complete. □

Proof of Theorem 4.1. Assume that (4.2) holds. Then there exist positive constants $C_i > 0$, $i = 1, 2, \dots, m$, and $R_0 > 0$ such that

$$P_i(x) \ge \frac{C_i}{|x|^{\lambda_i}} \quad \text{for } |x| \ge R_0.$$

So that

$$P_{i*}(r) \ge \frac{C_i}{r^{\lambda_i}} \quad \text{for } r \ge R_0. \tag{4.7}$$

Without loss of generality we may assume that $i_0 = 1$. Suppose to the contrary that (4.1) has a nonnegative nontrivial entire solution $(u_1, u_2, \dots, u_m)$ satisfying (4.3) with $i_0 = 1$. Then, by Lemma 4.2, its spherical mean $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m)$ satisfies (4.4).

We choose the constant $b < 1$ in (4.4) such that $1 < b^{-2m} < A^{1/\rho}$, where ρ is the number appearing in (4.3). We first show that

$$\lim_{r\to\infty} \bar{u}_1(r) = \infty, \tag{4.8}$$

and

$$\bar{u}_1(lr) \ge L\bar{u}_1(r)^A \quad \text{near } +\infty \tag{4.9}$$

where $L > 0$ is some constant and $l = b^{-2m}$.

Integrating (4.4) on $[0, r]$, we have

$$\bar{u}_i(r) \ge \bar{u}_i(0) + \tilde{C}_i \int_0^r sP_{i*}(s)\bar{u}_{i+1}(bs)^{\alpha_i} ds, \quad r \ge 0, \quad i = 1, 2, \dots, m. \tag{4.10}$$

Since $(u_1, u_2, \dots, u_m)$ is nonnegative and nontrivial, for some point $x_* \in \mathbb{R}^N$ we have $u_i(x_*) > 0$ for some $i \in \{1, 2, \dots, m\}$; that is $\bar{u}_i(r_*) > 0$, $r_* = |x_*|$. We may assume that $r_* \geq R_0$. Therefore, we see from (4.10) that $\bar{u}_i(r) > 0$ for $r > r_*$.

Let $r \geq r_*/b$ be large enough. Integrating (4.4) over $[br, r]$, from (4.7) and the monotonicity of u_i we have

$$\begin{aligned}\bar{u}_i(r) - \bar{u}_i(br) &\geq \tilde{C}_i \int_{br}^{r} s P_{i*}(s) \bar{u}_{i+1}(bs)^{\alpha_i} ds \\ &\geq \tilde{C}_i \bar{u}_{i+1}(b^2 r)^{\alpha_i} \int_{br}^{r} s^{1-\lambda_i} ds \\ &= \tilde{C}_i \frac{1 - b^{2-\lambda_i}}{2 - \lambda_i} \bar{u}_{i+1}(b^2 r)^{\alpha_i} r^{2-\lambda_i},\end{aligned}$$

namely,

$$\bar{u}_i(r) \geq C r^{2-\lambda_i} \bar{u}_{i+1}(b^2 r)^{\alpha_i}, \quad i = 1, 2, \dots, m, \tag{4.11}$$

where C is some positive constant. Notice that (4.11) is still valid even though $\lambda_i = 2$ (with $C = \tilde{C}_i \log b^{-1}$).

From (4.11), by iteration, it follows that

$$\bar{u}_1(r) \geq C r^{-\Lambda_1} \bar{u}_1(b^{2m} r)^A, \quad r > \frac{r_*}{b^{2m}},$$

where $C > 0$ is some constant. From the assumption $\Lambda_1 \leq 0$, we obtain (4.9).

The inequality (4.4) with $i = 1$ and (4.11) imply

$$\bar{u}_1'(r) \geq C r^{\tau} P_{1*}(r) \bar{u}_1(b^{2(m-1)+1} r)^A, \tag{4.12}$$

where

$$\tau = 1 + \sum_{j=1}^{m-1} \Big\{ (2 - \lambda_{1+j}) \prod_{k=0}^{j-1} \alpha_{1+k} \Big\} = \lambda_1 - 1 - \Lambda_1.$$

Integrating (4.12) over $[r_1, r]$, $b^{2(m-1)+1} r_1 > r_*$, we have

$$\begin{aligned}\bar{u}_1(r) &\geq \bar{u}_1(r_1) + C \int_{r_1}^{r} s^{\tau} P_{1*}(s) \bar{u}_1(b^{2(m-1)+1} s)^A ds \\ &\geq \bar{u}_1(r_1) + C \bar{u}_1(b^{2(m-1)+1} r_1)^A \int_{r_1}^{r} s^{\tau - \lambda_1} ds.\end{aligned}$$

From the assumption $\Lambda_1 \leq 0$, we can see that $\tau - \lambda_1 \geq -1$, which implies that (4.8) holds. Let $\tilde{r}$ be large so that

$$L^{\frac{1}{A-1}} \bar{u}_1(\tilde{r}) \geq e, \tag{4.13}$$

and

$$\bar{u}_1(lr) \geq L \bar{u}_1(r)^A, \quad r \geq \tilde{r}\,, \tag{4.14}$$

where $L > 0$ is the constant appearing in (4.9). It is possible to choose such an $\tilde{r}$ by (4.8) and (4.9). For $k \in \mathbb{N}$, from (4.14) we obtain

$$\begin{aligned}
\bar{u}_1(l^k\tilde{r}) &\geq L\bar{u}_1(l^{k-1}\tilde{r})^A \\
&\geq L^{1+A}\bar{u}_1(l^{k-2}\tilde{r})^{A^2} \\
&\geq \dots \\
&\geq L^{1+A+\dots+A^{k-1}}\bar{u}_1(\tilde{r})^{A^k} \\
&= L^{-\frac{1}{A-1}}\left[L^{\frac{1}{A-1}}\bar{u}_1(\tilde{r})\right]^{A^k}.
\end{aligned}$$

Hence we see from (4.13) that

$$\bar{u}_1(l^k\tilde{r}) \geq L^{-\frac{1}{A-1}} \exp A^k. \tag{4.15}$$

Let $r \geq l\tilde{r}$. Then we can find that there exists a unique positive integer $k = k(r)$ such that $l^k\tilde{r} \leq r < l^{k+1}\tilde{r}$. Thus k satisfies

$$k > \frac{\log r - \log\tilde{r}}{\log l} - 1.$$

It follows therefore from (4.15) that

$$\begin{aligned}
\bar{u}_1(r) \geq \bar{u}_1(l^k\tilde{r}) &\geq L^{-\frac{1}{A-1}} \exp A^k \\
&\geq L^{-\frac{1}{A-1}} \exp\left\{A^{-\frac{\log\tilde{r}}{\log l}-1} \cdot A^{\frac{\log r}{\log l}}\right\} \\
&= L^{-\frac{1}{A-1}} \exp\left\{A^{-\frac{\log\tilde{r}}{\log l}-1} r^{\frac{\log A}{\log l}}\right\}.
\end{aligned} \tag{4.16}$$

On the other hand, because $u_1(x) = O(\exp|x|^\rho)$ as $|x| \to \infty$, we obviously have

$$\bar{u}_1(r) = O(\exp r^\rho) \text{ as } r \to \infty.$$

Since $\log A/\log l = \log A/\log b^{-2m} > \rho$ from our choice of b, (4.16) gives a contradiction. The proof is complete. $\square$

Remark 4.4. (i) When $m = 2$, Theorem 4.1 reduces to [13, Theorem 1]. However, the proof given here is simpler than in [13].
(ii) As described in Remark 2.3, in the case $m = 2$, the nonnegative entire solution (u_1, u_2) of (4.1) satisfies

$$u_1(x) \leq C|x|^{\beta_1} \text{ and } u_2(x) \leq C|x|^{\beta_2} \quad \text{at } \infty$$

without the assumptions $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$ under the condition (4.2). From this fact and (4.8), we can see that if (λ_1, λ_2) satisfies $\Lambda_1 \leq 0$, then the system (4.1) does not have nonnegative nontrivial entire solutions. Therefore, we find that Theorem 2.7 holds without the assumptions $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$. So we conjecture that the conclusion of Theorem 2.7 holds without the assumptions $\alpha_i \geq 1$, $i = 1, 2, \dots, m$.

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