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THE HEAT EQUATION AND THE SHRINKING

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Abstract. This article concerns the Cauchy problem for the partial differential equation

$$\partial_1 u(t,x) - a\partial_2^2 u(t,x) = f(t,x,\partial_2^p u(\mu(t)t,x), \partial_2^q u(t,\nu(t)x))\,.$$

Here t and x are real variables, p and q are positive integers greater than 1, and the *shrinking factors* $\mu(t)$, $\nu(t)$ are positive-valued functions such that their suprema are less than 1.

1. Introduction

The effect of *shrinkings* placed on the independent variables has been investigated in [1, 2, 3, 4]. All of the results obtained so far can be said to be on the same line as the Cauchy-Kovalevskaja theorem. This means that the results obtained are independent of the type of differential equations, such as parabolic or hyperbolic. Therefore, there are possibilities for obtaining some new results in the study of the effect of the shrinking by taking into account the type of differential equation. The present note is our first attempt to pursue such possibilities. To be a bit more exact, we consider here the problem of introducing shrinking factors into the one dimensional heat equation

$$\partial_1 u(t,x) - a\partial_2^2 u(t,x) = f(t,x). \tag{1.1}$$

In this equation $u(t,x)$ is the unknown real-valued function with $(t,x) \in \mathbb{R}^2$, ∂_i denotes partial differentiation with respect to the ith variable, $f(t,x)$ is a given function of (t,x), and a a positive constant.

Next, we recall a well-known result about the solution of (1.1) satisfying the initial condition

$$u(0,x) = \varphi(x). \tag{1.2}$$

When $\varphi(x)$ and $f(t,x)$ are continuous and bounded, if $f(t,x)$ satisfies the Hölder condition with respect to x, then a solution to (1.1)-(1.2), (on the domain $t \geq$

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$0, -\infty < x < \infty$) is given by the formula

$$u(t,x) = \begin{cases} \varphi(x) & \text{if } t = 0, \\ \int_{-\infty}^{\infty} G(t, x-\xi)\varphi(\xi)d\xi + \int_0^t d\tau \int_{-\infty}^{\infty} G(t-\tau, x-\xi) f(\tau,\xi)d\xi & \text{if } t > 0, \end{cases} \tag{1.3}$$

where

$$G(t,x) = \frac{1}{2\sqrt{\pi a t}} e^{-x^2/(4at)} \quad (t > 0,\ -\infty < x < \infty)$$

which is called the heat kernel. It is also well-known that $u(t,x)$ given by (1.3) is the only solution of the Cauchy problem (1.1)-(1.2) among the functions bounded with respect to x.

In particular, when the initial value is

$$u(0,x) = \varphi(x) = 0, \tag{1.4}$$

the formula (1.3) becomes

$$u(t,x) = \int_0^t d\tau \int_{-\infty}^{\infty} G(t-\tau, x-\xi) f(\tau,\xi)d\xi. \tag{1.5}$$

Note that there is no loss of generality if we consider only the homogeneous problem ($\varphi(x) = 0$). In fact, for a non-zero initial condition (1.2), a homogeneous problem is obtained by replacing the unknown function $u(t,x)$ by $u(t,x) - \varphi(x)$.

Now we can state clearly the purpose of the present paper. In this note, we consider the differential equation

$$\partial_1 u(t,x) - a\partial_2^2 u(t,x) = f(t, x, \partial_2^p u(\mu(t)t, x), \partial_2^q u(t, \nu(t)x)) \tag{1.6}$$

instead of the simple heat equation (1.1). In (1.6) t and x are real variables. u is a real-valued unknown function. We assume that the positive-valued function μ and ν satisfy $\sup_t \mu(t) < 1$ and $\sup_t \nu(t) < 1$; μ is called *the time shrinking factor* and ν is called *the space shrinking factor*. p and q denote integers greater than 1. As for the function $f(t,x,v,w)$ we assume, for the sake of our convenience, that it is a continuous function of (t,x,v,w) and is *a Gevrey function* of (x,v,w). For the definition of a Gevrey function see §3.

We want to solve the Cauchy problem for the equation (1.6) with the homogeneous condition (1.4) only, for the same reason as in the case of the simple heat equation (1.1). Since the unique bounded solution of the Cauchy problem (1.1)-(1.4) is given by (1.5), it is necessary and sufficient for a bounded function $u(t,x)$ to be a solution of the Cauchy problem (1.6)-(1.4) that it satisfies the integral equation

$$u(t,x) = \int_0^t d\tau \int_{-\infty}^{\infty} G(t-\tau, x-s) f(\tau, s, \partial_2^p u(\mu(\tau)\tau, s), \partial_2^q u(\tau, \nu(\tau)s))ds. \tag{1.7}$$

To solve this integral equation, however, we need some preparations to be made in the following two sections. Our final result in this note will be stated and proved in §4. It can be generalized without any essential change to the n-dimensional case. But, for the sake of simplicity of the notation, we refrain from doing so in this note.

Finally in this introduction we make a brief mention on the case where the differential equation (1.6) is linear. As will be seen in the theorem in §4, the domain of t of existence of the solution $u(t,x)$ to the general non-linear Cauchy problem (1.6)-(1.4) may be smaller than the domain of t for definition of the function $f(t,x,v,w)$. If $f(t,x,v,w)$ is linear in v and w, however, it can be shown that the domain of existence of $u(t,x)$ coincides with the domain of definition of $f(t,x,v,w)$. We omit the details.

2. The properties of integrals with the heat kernel

As a preparation for solving the integral equation (1.7) let us recall some fundamental properties of an integral

$$\int_{-\infty}^{\infty} G(t-\tau, x-\xi) g(\tau,\xi) d\xi. \tag{2.1}$$

Proposition 2.1. *Let M, T be positive constants. Let $g(t,x)$ be a real valued continuous function for $(t,x) \in \mathbb{R}^2$ with $0 \le t < T$. Assume that the inequality $|g(t,x)| \le M$ holds. Then*

$$v(t,\tau,x) = \begin{cases} \int_{-\infty}^{\infty} G(t-\tau,x-\xi) g(\tau,\xi) d\xi & \text{if } t > \tau, \\ g(\tau,x) & \text{if } t = \tau \end{cases} \tag{2.2}$$

is continuous for $(t,\tau,x) \in \mathbb{R}^3$ with $0 \le \tau \le t < T$ and satisfies the inequality $|v(t,\tau,x)| \le M$.

When the function $g(t,x)$ in the integral (2.1) is m times differentiable in x, then the function $v(t,\tau,x)$ is also m times differentiable in x. To be exact we have the following proposition; however, we omit its proof.

Proposition 2.2. *Let $g(t,x)$ be a real-valued bounded continuous function for $(t,x) \in \mathbb{R}^2$ with $0 \le t < T$. Assume that $g(t,x)$ is m times differentiable in x and the partial derivatives $\partial_2^k g(t,x)$ $(1 \le k \le m)$ are bounded continuous functions for $(t,x) \in \mathbb{R}^2$ with $0 \le t < T$. Then the function $v(t,\tau,x)$ defined by (2.2) is m times differentiable in x and the partial derivatives $\partial_3^k v(t,\tau,x)$, $1 \le k \le m$, are*

$$\partial_3^k v(t,\tau,x) = \begin{cases} \int_{-\infty}^{\infty} G(t-\tau,x-\xi) \partial_2^k g(\tau,\xi) d\xi & \text{if } t > \tau, \\ \partial_2^k g(\tau,x) & \text{if } t = \tau. \end{cases} \tag{2.3}$$

3. Gevrey functions

As stated in §1, our purpose in the present note is to solve the Cauchy problem (1.6)-(1.4). To be more exact, we assume that the function $f(t,x,v,w)$ in (1.6) is a Gevrey function of (x,v,w) and seek a solution $u(t,x)$ that is a Gevrey function of x. In this section, we shall recall the definition of Gevrey function and state some fundamental properties of Gevrey functions.

Gevrey functions of one variable. We denote by $\mathbb{Z}_+$ the set of all non-negative integers. Let I be a real interval and λ a constant greater than 1. Let λ be a fix constant greater than 1.

If for a C^∞ function $w : I \to \mathbb{R}$ there are positive constants C, M such that

$$|w^{(k)}(x)| \le CM^k (k!)^\lambda$$

holds for all $x \in I$ and all $k \in \mathbb{Z}_+$, then w is called a Gevrey function on I of order λ.

It is easy to see that a C^∞ function $w : I \to \mathbb{R}$ is a Gevrey function of order λ if and only if there are positive constants C', L such that

$$|w^{(k)}(x)| \le 2^{-5} C' L^k (k!)^\lambda (1+k)^{-2}$$

holds for all $x \in I$ and all $k \in \mathbb{Z}_+$. So we write, according to Yamanaka [5],

$$\Gamma_\lambda(k) = 2^{-5}(k!)^\lambda(1+k)^{-2}$$

for $k = 0, 1, 2, \dots$ and define

$$|w|_L = \sup\Big\{\frac{|w^{(k)}(x)|}{L^k\Gamma_\lambda(k)} : x \in I,\ k \in \mathbb{Z}_+\Big\}$$

for each C^∞ function $w : I \to \mathbb{R}$. We denote by $\gamma_L(I)$ the family of all C^∞ functions $w : I \to \mathbb{R}$ such that $|w|_L < \infty$.

Besides this family, we need another type of Gevrey family. For a C^∞ function $w : I \to \mathbb{R}$, we write

$$\lvert w\rvert = \sup_{x\in I}|w(x)|, \quad \|w\|_L = \max\{2^6\lvert w\rvert, 2^3L^{-1}|w'|_L\} \tag{3.1}$$

and define

$$\mathcal{G}_L(I) = \{w \in C^\infty(I,\mathbb{R}) : \|w\|_L < \infty\}.$$

Between the two types of Gevrey families $\gamma_L(I)$ and $\mathcal{G}_L(I)$ there is the following relation which proof can be found in [4, Proposition 2.1] and in [6, Lemma 5.2].

Proposition 3.1. *If $0 < L < M$, then $\gamma_L(I) \subset \mathcal{G}_M(I) \subset \gamma_M(I)$ and the inclusion maps are linear and bounded.*

The norm $\|\cdot\|_L$ has the following useful property which proof can be found in [4, Proposition 2.2] and in [6, Theorem 5.4].

Proposition 3.2. *If v and w are in $\mathcal{G}_L(I)$, then the product vw is again in $\mathcal{G}_L(I)$ and the inequality $\|vw\|_L \le \|v\|_L\|w\|_L$ holds.*

As for the result of differentiation of a function belonging to the family $\gamma_L(I)$ there is the following fact which proof can be found in [4, Proposition 2.3].

Proposition 3.3. *Let L be a positive constant, α be a constant greater than 1, and q be a positive integer. Assume that $w \in \gamma_L(I)$. Then the qth derivative $w^{(q)}$ of w is in the family $\gamma_{\alpha L}(I)$ and*

$$|w^{(q)}|_{\alpha L} \le (\alpha L)^q\big(\frac{\lambda q}{\log\alpha}\big)^{\lambda q}|w|_L\,.$$

For us the following modification of the above proposition is useful. Its proof can be found in [4, Proposition 2.4].

Proposition 3.4. *Let L and M be positive constants such that $L < M$. Assume that w is in $\gamma_L(I)$ and q is a positive integer. Then $w^{(q)}$ is in $\gamma_M(I)$ and*

$$|w^{(q)}|_M \le M^{(1+\lambda)q}\Big(\frac{\lambda q}{M-L}\Big)^{\lambda q}|w|_L\,.$$

As for composition of two Gevrey type functions there is the following fact. For its proof see [4, Proposition 2.5] and [5, Theorem 3.1].

Proposition 3.5. *Let I, J be open intervals and L, M be positive constants. Assume that $w : J \to \mathbb{R}$ is a C^∞ function such that $w' \in \gamma_L(J)$ and $v : I \to J$ is a C^∞ function such that $v' \in \gamma_M(I)$. Assume further that the inequality*

$$|v'|_M \le L^{-1}M \tag{3.2}$$

holds. Then the derivative $(w \circ v)'$ *of the composite function* $w \circ v : I \to \mathbb{R}$ *belongs to the family* $\gamma_M(I)$ *and*

$$|(w \circ v)'|_M \le L^{-1} M |w'|_L \,.$$

In terms of the norms $\| \cdot \|_L$ and $\| \cdot \|_M$ the above proposition is modified as follows. For its proof see [4, Proposition 2.6] and [6, Theorem 5.3].

Proposition 3.6. *Let* I, J *be open intervals and* L, M *be positive constants. Assume that* $w : J \to \mathbb{R}$ *is in the family* $\mathcal{G}_L(J)$ *and* $v : I \to J$ *is in the family* $\mathcal{G}_M(I)$*. Assume further that the inequality* (3.2) *holds. Then the composite function* $w \circ v : I \to \mathbb{R}$ *belongs to the family* $\mathcal{G}_M(I)$ *and*

$$\|w \circ v\|_M \le \|w\|_L \,.$$

Gevrey functions of several variables. For a function of m variables we denote by ∂_j the partial differentiation with respect to the jth variable and write $\partial = (\partial_1, \dots, \partial_m)$. Further, if $k = (k_1, \dots, k_m)$ is an element of $\mathbb{Z}_+^m$, then we write $\partial^k = \partial_1^{k_1} \cdots \partial_m^{k_m}$.

Let U be an open set of $\mathbb{R}^m$. If $f : U \to \mathbb{R}$ is a C^∞ function and there are positive constants C, M such that the inequality

$$|\partial^k f(x)| \le C M^{|k|} (k!)^\lambda,$$

where $|k| = k_1 + \cdots + k_m$ and $k! = k_1! \cdots k_m!$, holds everywhere in U for any m dimensional index $k = (k_1 \dots, k_m)$, then f is called a Gevrey function on U of order λ.

A C^∞ function $f : U \to \mathbb{R}$ is a Gevrey function of order λ, if and only if there are positive constants C', L such that the inequality

$$|\partial^k f(x)| \le C' L^{|k|} \Gamma_\lambda(|k|)$$

holds everywhere in U for any m dimensional index k. For this reason we write

$$|f|_L = \sup_{x,k} \frac{|\partial^k f(x)|}{L^{|k|} \Gamma_\lambda(|k|)}$$

for any C^∞ function $f : U \to \mathbb{R}$ and define

$$\gamma_L(U) = \{ f \in C^\infty(U, \mathbb{R}) : |f|_L < \infty \}.$$

Further we write, like (3.1),

$$|\!|w|\!| = \sup_{x \in U} |w(x)|, \quad \|w\|_L = \max\{ 2^6 |\!|w|\!|, 2^3 L^{-1} \max_i |\partial_i w|_L \}$$

and define

$$\mathcal{G}_L(U) = \{ w \in C^\infty(U, \mathbb{R}) : \|w\|_L < \infty \}.$$

It is necessary for us to know what comes out when m Gevrey functions $g_1(x)$, ..., $g_m(x)$ of one variable x are substituted for the last m variables $y_1, \dots, y_m$ in a Gevrey function $f(x, y_1, \dots, y_m)$ of $m+1$ variables $x, y_1, \dots, y_m$.

Proposition 3.7. *Let* $J_1, \dots, J_m$ *and* I *be open intervals and* L, M *be positive constants. Write* $U = I \times J_1 \times \cdots \times J_m$*. Let* f *be an element of the family* $\mathcal{G}_L(U)$ *and* $g_i : I \to J_i$, $i = 1, \dots, m$, *be in the family* $\mathcal{G}_M(I)$*. Assume that*

$$M \ge L(1 + \max_i |g_i'|_M). \tag{3.3}$$

Put $\varphi(x) = f(x, g_1(x), \dots, g_m(x))$ *for* $x \in I$*. Then* φ *is in* $\mathcal{G}_M(I)$ *and* $\|\varphi\|_M \le \|f\|_L$*.*

The proof of this proposition can be found in [4, Proposition 2.7] and [5, Lemma 8.1].

Partial Gevrey functions. It is necessary for us to consider functions of $m+1$ variables which are in a Gevrey class with respect to the last m variables only. We call them partial Gevrey functions. For a function $f(t, y_1, \dots, y_m)$ of $m+1$ variables $t, y_1, \dots, y_m$ we write $\tilde{\partial} = (\partial_2, \dots, \partial_{m+1})$. For a non-negative integer j_0 we write $\mathbb{Z}_+(j_0) = \{j \in \mathbb{Z}_+ : j \le j_0\}$. Let U be an open set of $\mathbb{R}^m$, I a real interval and j_0 a non-negative integer. Then we denote by $C^{(j_0,\infty)}(I, U)$ the set of all functions $f : I \times U \to \mathbb{R}$ such that the partial derivative $\partial_1^j \tilde{\partial}^k f : I \times U \to \mathbb{R}$ exists and is continuous for each $(j, k) \in \mathbb{Z}_+(j_0) \times \mathbb{Z}_+^m$. Further, if $h(t)$ is a positive valued function of $t \in I$, we write

$$\begin{aligned} &\widetilde{\mathcal{G}}_h^{(j_0)}(I, U) \\ &= \{f \in C^{(j_0,\infty)}(I, U) : \text{ if } t \in I \text{ and } 0 \le j \le j_0, \text{ then } \partial_1^j f(t, \cdot, \dots, \cdot) \in \mathcal{G}_{h(t)}(U)\}. \end{aligned}$$

We shall need, however, the case where $j_0 = 0$ only. We simply write $\widetilde{\mathcal{G}}_h(I, U)$ instead of $\widetilde{\mathcal{G}}_h^{(0)}(I, U)$. We shall need the following proposition.

Proposition 3.8. *Let T be a positive constant and $h(t)$ a positive valued function of $t \in [0, T)$. Let g be an element of $\widetilde{\mathcal{G}}_h([0, T), \mathbb{R})$ such that*

$$\|g(t, \cdot)\|_{h(t)} \le C,$$

where C is a positive constant. Then a function $\psi(t, x)$ of $(t, x) \in [0, T) \times \mathbb{R}$ is defined by

$$\psi(t, x) = \int_0^t d\tau \int_{-\infty}^{\infty} G(t - \tau, x - \xi) g(\tau, \xi)\, d\xi, \tag{3.4}$$

where G denotes the heat kernel. ψ belongs to the family $\widetilde{\mathcal{G}}_h([0, T), \mathbb{R})$ and satisfies the inequality

$$\|\psi(t, \cdot)\|_{h(t)} \le Ct. \tag{3.5}$$

Proof. Write $\Delta = \{(t, \tau, x) \in \mathbb{R}^3 : 0 \le \tau \le t < T\}$ and define $v(t, \tau, x)$ for $(t, \tau, x) \in \Delta$ by (2.2). Since g is a bounded continuous function, we can use Proposition 2.1 and see that $v(t, \tau, x)$ is a continuous function for $(t, \tau, x) \in \Delta$ and

$$|v(t, \tau, x)| \le \sup_{\xi \in \mathbb{R}} |g(\tau, \xi)| = |g(\tau, \cdot)| \le 2^{-6} \|g(\tau, \cdot)\|_{h(\tau)} \le 2^{-6} C\,. \tag{3.6}$$

Using the notation $v(t, \tau, x)$, the definition (3.4) of $\psi(t, x)$ is rewritten as

$$\psi(t, x) = \int_0^t v(t, \tau, x) d\tau. \tag{3.7}$$

By (3.6) and (3.7) we have

$$|\psi(t, \cdot)| \le 2^{-6} Ct. \tag{3.8}$$

Further, since $g(t, x)$ has bounded continuous partial derivatives $\partial_2^k g(t, x)$, $k = 1, 2, \cdots$, we can use Proposition 2.2 and see that $v(t, \tau, x)$ is infinitely differentiable in x and the partial derivatives $\partial_3^k v(t,\tau,x)$, $k=1,2,\cdots$, are given by (2.3). By Proposition 2.1, 2.2 and (2.3) we see that

$$\begin{aligned}|\partial_3^k v(t,\tau,x)| &\le \sup_{\xi\in\mathbb{R}} |\partial_2^k g(\tau,\xi)|\\ &\le |\partial_2 g(\tau,\cdot)|_{h(\tau)} h(\tau)^{k-1}\Gamma_\lambda(k-1)\\ &\le 2^{-3}\|g(\tau,\cdot)\|_{h(\tau)} h(\tau)^k \Gamma_\lambda(k-1)\\ &\le 2^{-3} C h(t)^k \Gamma_\lambda(k-1)\,.\end{aligned} \tag{3.9}$$

By (3.9) and (3.7) we see that

$$|\partial_2^k \psi(t,x)| \le 2^{-3} C h(t)^k \Gamma_\lambda(k-1) t$$

for $k=1,2,\ldots$, and that

$$|\partial_2 \psi(t,\cdot)|_{h(t)} \le 2^{-3} C h(t) t\,. \tag{3.10}$$

By (3.8) and (3.10) we see that the inequality (3.5) holds. □

4. Main result and its proof

In this secion, as in §3, λ denotes a fixed constant greater than 1. Using the notation introduced in the preceding section, we can state our result on the Cauchy problem (1.6)-(1.4). It is stated as the following theorem.

Theorem 4.1. *Let L, R and T_0 be positive constants. In the differential equation* (1.6) *suppose that $\mu(t)$ and $\nu(t)$ are positive valued continuous functions for $t\in[0,T_0)$. Assume that*

$$\sup_{0\le t<T_0} \max\{\mu(t),\nu(t)\} < 1.$$

p and q denote positive integers greater than 1. *Put $M=\max\{1,2L\}$, $s=(2\lambda p)^{-1}$ and $h(t)=M(1+t^s)$ for $t\ge 0$. Write*

$$U=\{(x,v,w)\in\mathbb{R}^3 : \max\{|v|,|w|\}<R\}.$$

Assume that the function f in (1.6) *as well as its partial derivatives $\partial_3 f$ and $\partial_4 f$ belong to the family $\widetilde{\mathcal{G}}_L([0,T_0),U)$ and that*

$$C_f := \sup_{0\le t<T_0} \max\{\|f(t,\cdot,\cdot,\cdot)\|_L,\ \|\partial_3 f(t,\cdot,\cdot,\cdot)\|_L,\ \|\partial_4 f(t,\cdot,\cdot,\cdot)\|_L\} < \infty.$$

Then there exists a positive number $T^*\le T_0$ such that the initial-value problem (1.6)-(1.4) *has a unique solution $u\in\widetilde{\mathcal{G}}_h([0,T^*),\mathbb{R})\bigcap C^{(1,\infty)}([0,T^*),\mathbb{R})$ such that*

$$\|u(t,\cdot)\|_{h(t)} \le C_f t, \quad 0\le t<T^*. \tag{4.1}$$

Proof. As in §1 it is sufficient for our purpose of solving the Cauchy problem (1.6)-(1.4) to solve the integral equation (1.7). To this end, for any given $T>0$, we write

$$\mathcal{H}(T)=\{v\in\widetilde{\mathcal{G}}_h([0,T),\mathbb{R}) : \sup_{0<t<T}\frac{\|v(t,\cdot)\|_{h(t)}}{t}<\infty\},$$

which is a Banach space with the norm

$$\mathbb{N}_{\mathcal{H}(T)}[v]=\sup_{0<t<T}\frac{\|v(t,\cdot)\|_{h(t)}}{t}, \quad v\in\mathcal{H}(T).$$

Further, for an arbitrary $K>0$, we set

$$\mathcal{F}(T,K)=\{v\in\mathcal{H}(T) : \mathbb{N}_{\mathcal{H}(T)}[v]\le K\},$$

which is a closed convex subset of the Banach space $\mathcal{H}(T)$.

We now want to show that there is a $T \le T_0$ such that, if $v \in \mathcal{F}(T, C_f)$, then the integral

$$w(t,x) := \int_0^t d\tau \int_{-\infty}^{\infty} G(t-\tau, x-\xi) f(\tau, \xi, \partial_2^p v(\mu(\tau)\tau, \xi), \partial_2^q v(\tau, \nu(\tau)\xi)) d\xi \tag{4.2}$$

is well-defined for all $(t,x) \in [0,T) \times \mathbb{R}$ and the function $w : [0,T) \times \mathbb{R} \to \mathbb{R}$ defined by (4.2) is again in $\mathcal{F}(T, C_f)$.

In order for the integral on the right-hand side of (4.2) to be well-defined it is necessary that the values $\partial_2^p v(\mu(\tau)\tau, \xi)$ and $\partial_2^q v(\tau, \nu(\tau)\xi)$ can be substituted for y and z in $f(\tau, \xi, y, z)$, respectively. It is enough for this that the inequality

$$\max\{|\partial_2^p v(\mu(\tau)\tau, \xi)|, |\partial_2^q v(\tau, \nu(\tau)\xi)|\} < R \tag{4.3}$$

holds. Now suppose that v is in $\mathcal{F}(T, C_f)$. Then we have

$$|\partial_2 v(t,\cdot)|_{h(t)} \le 2^{-3} h(t) \|v(t,\cdot)\|_{h(t)} \le 2^{-3} h(t) C_f t, \quad 0 \le t < T. \tag{4.4}$$

Therefore, for $(t,x) \in [0,T) \times \mathbb{R}$,

$$\begin{aligned} |\partial_2^p v(\mu(t)t, x)| &= |\partial_2^{p-1}(\partial_2 v)(\mu(t)t, x)| \\ &\le |\partial_2 v(\mu(t)t, \cdot)|_{h(\mu(t)t)} \cdot h(\mu(t)t)^{p-1} \cdot \Gamma_\lambda(p-1) \\ &\le 2^{-3} \|\partial_2 v(\mu(t)t, \cdot)\|_{h(\mu(t)t)} \cdot h(\mu(t)t)^p \cdot \Gamma_\lambda(p-1) \\ &\le 2^{-3} \mathbb{N}_{\mathcal{H}(T)}[v] \mu(t)t \cdot h(\mu(t)t)^p \cdot \Gamma_\lambda(p-1) \\ &\le 2^{-3} C_f \cdot \mu(t)t \cdot h(\mu(t)t)^p \cdot \Gamma_\lambda(p-1) \\ &< 2^{-3} C_f T \cdot h(T)^p \cdot \Gamma_\lambda(p-1) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} |\partial_2^q v(t, \nu(t)x)| &\le \sup_{\xi \in \mathbb{R}} |\partial_2^q v(t,\xi)| = \sup_{\xi \in \mathbb{R}} |\partial_2^{q-1}(\partial_2 v)(t,\xi)| \\ &\le |\partial_2 v(t,\cdot)|_{h(t)} \cdot h(t)^{q-1} \cdot \Gamma_\lambda(q-1) \\ &\le 2^{-3} C_f \cdot t \cdot h(t)^q \cdot \Gamma_\lambda(q-1) \\ &< 2^{-3} C_f \cdot T \cdot h(T)^q \cdot \Gamma_\lambda(q-1). \end{aligned}$$

Therefore, if $T \le T_0$ satisfies the inequality

$$2^{-3} C_f \cdot T \cdot \max\{h(T)^p \Gamma_\lambda(p-1),\ h(T)^q \Gamma_\lambda(q-1)\} \le R \tag{4.6}$$

and v is in $\mathcal{F}(T, C_f)$, then the inequality (4.3) holds for $(t,x) \in [0,T) \times \mathbb{R}$ and the expression $f(t, x, \partial_2^p v(\mu(t)t, x), \partial_2^q v(t, \nu(t)x))$ makes sense. For this reason we define

$$T_1 = \max\{T : T \le T_0 \text{ and (4.6) holds}\}.$$

Now suppose $0 < T \le T_1$ and take an element v of $\mathcal{F}(T, C_f)$. Then we can put

$$\varphi(t,x) = f(t, x, \partial_2^p v(\mu(t)t, x), \partial_2^q v(t, \nu(t)x)). \tag{4.7}$$

Let us seek a condition under which this function φ enters the family $\widetilde{\mathcal{G}}_h([0,T), \mathbb{R})$. For this purpose set $m = \sup_{0 \le t < T_0} \max\{\mu(t), \nu(t)\}$. Then, since v is in $\mathcal{F}(T, C_f)$, we have (4.4) and, in virtue of Proposition 3.4,

$$
\begin{aligned}
&|\partial_2^{p+1} v(\mu(t)t,\cdot)|_{h(t)} \\
&\le h(t)^{(1+\lambda)p}\Big(\frac{\lambda p}{h(t)-h(\mu(t)t)}\Big)^{\lambda p}|\partial_2 v(\mu(t)t,\cdot)|_{h(\mu(t)t)} \\
&\le h(t)^{2\lambda p}M^{-\lambda p}\Big(\frac{\lambda p}{(1-m^s)t^s}\Big)^{\lambda p}2^{-3}h(\mu(t)t)\|v(\mu(t)t,\cdot)\|_{h(\mu(t)t)} \\
&\le \Big(\frac{h(t)^2\lambda p}{M(1-m^s)t^s}\Big)^{\lambda p}2^{-3}h(\mu(t)t)C_f\mu(t)t \\
&\le \Big(\frac{h(T)^2\lambda p}{M(1-m^s)}\Big)^{\lambda p}2^{-3}h(t)C_f t^{1-\lambda ps} \\
&\le 2^{-3}C_f\cdot\Big(\frac{h(T)^2\lambda p}{M(1-m^s)}\Big)^{\lambda p}T^{1/2}h(t).
\end{aligned} \tag{4.8}
$$

Therefore, if T satisfies the inequality

$$
2^{-3}C_f\cdot\Big(\frac{h(T)^2\lambda p}{M(1-m^s)}\Big)^{\lambda p}T^{1/2}\le \frac{1}{L}-\frac{1}{M}, \tag{4.9}
$$

then

$$
|\partial_2(\partial_2^p v)(\mu(t)t,\cdot)|_{h(t)} = |\partial_2^{p+1}v(\mu(t)t,\cdot)|_{h(t)} \le \big(\frac{1}{L}-\frac{1}{M}\big)h(t)\le \frac{h(t)}{L}-1. \tag{4.10}
$$

Next we have to estimate the Gevrey norm of the function $x\mapsto \partial_2^q v(t,\nu(t)x)$. For this purpose define $\psi(t,x)=\partial_2^q v(t,\nu(t)x)$. Then we have

$$
\partial_2^{k+1}\psi(t,x)=\nu(t)^{k+1}\partial_2^{q+k+1}v(t,\nu(t)x),\quad k=0,1,2,\dots,
$$

and

$$
\begin{aligned}
|\partial_2\psi(t,\cdot)|_{h(t)} &= \sup_{k,x}\frac{|\partial_2^{k+1}\psi(t,x)|}{h(t)^k\Gamma_\lambda(k)} = \sup_{k,x}\frac{|\partial_2^{q+k+1}v(t,\nu(t)x)|}{h(t)^k\Gamma_\lambda(k)}\nu(t)^{k+1} \\
&= \nu(t)\sup_{k,\xi}\frac{|\partial_2^{q+k+1}v(t,\xi)|}{(h(t)/\nu(t))^k\Gamma_\lambda(k)} = \nu(t)|\partial_2^{q+1}v(t,\cdot)|_{h(t)/\nu(t)} \\
&\le m|\partial_2^{q+1}v(t,\cdot)|_{h(t)/m}.
\end{aligned} \tag{4.11}
$$

On the other hand, by Proposition 3.3,

$$
\begin{aligned}
|\partial_2^{q+1}v(t,\cdot)|_{h(t)/m} &\le (h(t)/m)^q\Big(\frac{\lambda q}{\log m^{-1}}\Big)^{\lambda q}|\partial_2 v(t,\cdot)|_{h(t)} \\
&\le (h(t)/m)^q\Big(\frac{\lambda q}{\log m^{-1}}\Big)^{\lambda q}2^{-3}h(t)\|v(t,\cdot)\|_{h(t)} \\
&\le 2^{-3}m^{-q}\Big(\frac{\lambda q}{\log m^{-1}}\Big)^{\lambda q}C_f\cdot h(t)^{1+q}t \\
&\le 2^{-3}C_f m^{-q}\Big(\frac{\lambda q}{\log m^{-1}}\Big)^{\lambda q}h(T)^qTh(t).
\end{aligned} \tag{4.12}
$$

From (4.11) and (4.12) we obtain the inequality

$$
|\partial_2\psi(t,\cdot)|_{h(t)}\le 2^{-3}C_f m^{1-q}\Big(\frac{\lambda q}{\log m^{-1}}\Big)^{\lambda q}h(T)^qTh(t),\quad 0\le t<T. \tag{4.13}
$$

By (4.13) that, if T satisfies the condition

$$2^{-3}C_f m^{1-q}\Big(\frac{\lambda q}{\log m^{-1}}\Big)^{\lambda q} h(T)^q T \le \frac{1}{L} - \frac{1}{M}, \tag{4.14}$$

then

$$|\partial_2\psi(t,\cdot)|_{h(t)} \le \Big(\frac{1}{L} - \frac{1}{M}\Big)h(t) \le \frac{h(t)}{L} - 1\,. \tag{4.15}$$

Since (4.9) implies (4.10) and (4.14) implies (4.15), we know that, if $T \in (0, T_1]$ satisfies both (4.9) and (4.14), then

$$\max\{|\partial_2(\partial_2^p v)(\mu(t)t,\cdot)|_{h(t)}, |\partial_2\psi(t,\cdot)|_{h(t)}\} \le \frac{h(t)}{L} - 1$$

and

$$L(1 + \max\{|\partial_2(\partial_2^p v)(\mu(t)t,\cdot)|_{h(t)}, |\partial_2\psi(t,\cdot)|_{h(t)}\}) \le h(t).$$

This inequality is of the same type as (3.3) in Proposition 3.7. Therefore, by Proposition 3.7, if $T \in (0, T_1]$ satisfies the inequalities (4.9) and (4.14), then the function φ defined by (4.7) belongs to the family $\widetilde{\mathcal{G}}_h([0,T),\mathbb{R})$ and satisfies the inequality

$$\|\varphi(t,\cdot)\|_{h(t)} \le \|f(t,\cdot,\cdot,\cdot)\|_L \le C_f\,. \tag{4.16}$$

For this reason we define

$$T_2 = \max\{T \in (0, T_1] : (4.9) \text{ and } (4.14) \text{ holds}\}.$$

We now know that, if $0 < T \le T_2$ and $v \in \mathcal{F}(T, C_f)$, then the function $\mathbb{R} \ni x \mapsto \varphi(t,x)$ is in $\mathcal{G}_{h(t)}(\mathbb{R})$. Further we see that the function φ is in $C^{(0,\infty)}([0,T),\mathbb{R})$, since f is in $C^{(0,\infty)}([0,T_0),U)$ and v is in $C^{(0,\infty)}([0,T),\mathbb{R})$. It follows that φ is in $\widetilde{\mathcal{G}}_h([0,T),\mathbb{R})$. Since φ is in $\widetilde{\mathcal{G}}_h([0,T),\mathbb{R})$ and satisfies the inequality (4.16), we can now use Proposition 3.8 and see that the function w defined by (4.2) belongs to $\widetilde{\mathcal{G}}_h([0,T),\mathbb{R})$ and satisfies the inequality

$$\|w(t,\cdot)\|_{h(t)} \le C_f t$$

for $0 \le t < T$. This means that w is in $\mathcal{F}(T, C_f)$. For T with $0 < T \le T_2$ we denote by Φ_T the mapping which maps each $v \in \mathcal{F}(T, C_f)$ to $w \in \mathcal{F}(T, C_f)$ given by (4.2).

We need next to estimate the difference $\Phi_T(v_1) - \Phi_T(v_0)$, where v_1 and v_0 are arbitrary two elements of $\mathcal{F}(T, C_f)$. For this purpose let us first estimate the difference

$$\begin{aligned}&\delta_{(v_1,v_0)}(\tau,\xi)\\ &:= f(\tau,\xi,\partial_2^p v_1(\mu(\tau)\tau,\xi),\partial_2^q v_1(\tau,\nu(\tau)\xi)) - f(\tau,\xi,\partial_2^p v_0(\mu(\tau)\tau,\xi),\partial_2^q v_0(\tau,\nu(\tau)\xi)).\end{aligned} \tag{4.17}$$

In order to do so it is convenient to write

$$v_\theta(t,x) = \theta v_1(t,x) + (1-\theta)v_0(t,x) \tag{4.18}$$

for $0 \le \theta \le 1$. Since v_0 and v_1 are in the convex set $\mathcal{F}(T, C_f)$, so is v_θ. Note that

(i) The values $\partial_i f(t,x,\partial_2^p v_\theta(\mu(t)t,x),\partial_2^q v_\theta(t,\nu(t)x))$, $i = 3,4$, are well-defined for $(t,x) \in [0,T) \times \mathbb{R}$

(ii) The functions $\partial_i f(t,x,\partial_2^p v_\theta(\mu(t)t,x),\partial_2^q v_\theta(t,\nu(t)x))$, $i = 3,4$, of (t,x), belong to the family $\widetilde{\mathcal{G}}_h([0,T),\mathbb{R})$

(iii) The inequalities

$$\|\partial_i f(t,\cdot,\partial_2^p v_\theta(\mu(t)t,\cdot),\partial_2^q v_\theta(t,\nu(t)\cdot))\|_{h(t)} \le \|\partial_i f(t,\cdot,\cdot,\cdot)\|_L \le C_f, \tag{4.19}$$

hold for $i = 3, 4$.

These facts are proved by almost the same reasoning as in the proof of similar facts about the function $\varphi(t,x) = f(t,x,\partial_2^p v(\mu(t)t,x),\partial_2^q v(t,\nu(t)x))$. In virtue of the facts (i), (ii) and (iii) above we can change the expression (4.17) of $\delta_{(v_1,v_0)}(\tau,\xi)$ as follows.

$$\begin{aligned}
&\delta_{(v_1,v_0)}(\tau,\xi)\\
&= \int_0^1 \partial_3 f(\tau,\xi,\partial_2^p v_\theta(\mu(\tau)\tau,\xi),\partial_2^q v_\theta(\tau,\nu(\tau)\xi))(\partial_2^p v_1(\mu(\tau)\tau,\xi)-\partial_2^p v_0(\mu(\tau)\tau,\xi))d\theta\\
&\quad + \int_0^1 \partial_4 f(\tau,\xi,\partial_2^p v_\theta(\mu(\tau)\tau,\xi),\partial_2^q v_\theta(\tau,\nu(\tau)\xi))(\partial_2^q v_1(\tau,\nu(\tau)\xi)-\partial_2^q v_0(\tau,\nu(\tau)\xi))d\theta.
\end{aligned}$$

By (4.19) and Proposition 3.2, $\delta_{(v_1,v_0)}$ satisfies the inequality

$$\begin{aligned}
\|\delta_{(v_1,v_0)}(\tau,\cdot)\|_{h(\tau)} &\le \int_0^1 \|\partial_3 f(\tau,\cdot,\partial_2^p v_\theta(\mu(\tau)\tau,\cdot),\partial_2^q v_\theta(\tau,\nu(\tau)\cdot))\|_{h(\tau)}\\
&\quad \times \|\partial_2^p v_1(\mu(\tau)\tau,\cdot)-\partial_2^p v_0(\mu(\tau)\tau,\cdot)\|_{h(\tau)}d\theta\\
&\quad + \int_0^1 \|\partial_4 f(\tau,\cdot,\partial_2^p v_\theta(\mu(\tau)\tau,\cdot),\partial_2^q v_\theta(\tau,\nu(\tau)\cdot))\|_{h(\tau)}\\
&\quad \times \|\partial_2^q v_1(\tau,\nu(\tau)\cdot)-\partial_2^q v_0(\tau,\nu(\tau)\cdot)\|_{h(\tau)}d\theta\\
&\le C_f\Big\{\|\partial_2^p v_1(\mu(\tau)\tau,\cdot)-\partial_2^p v_0(\mu(\tau)\tau,\cdot)\|_{h(\tau)}\\
&\quad + \|\partial_2^q v_1(\tau,\nu(\tau)\cdot)-\partial_2^q v_0(\tau,\nu(\tau)\cdot)\|_{h(\tau)}\Big\}.
\end{aligned} \tag{4.20}$$

So we need now to estimate the Gevrey norms of the two differences

$$\partial_2^p v_1(\mu(\tau)\tau,\cdot)-\partial_2^p v_0(\mu(\tau)\tau,\cdot) \quad\text{and}\quad \partial_2^q v_1(\tau,\nu(\tau)\cdot)-\partial_2^q v_0(\tau,\nu(\tau)\cdot). \tag{4.21}$$

As for the first of these we can perform, almost in the same way as in (4.8), the following estimation.

$$\begin{aligned}
&|\partial_2^{p+1} v_1(\mu(\tau)\tau,\cdot)-\partial_2^{p+1} v_0(\mu(\tau)\tau,\cdot)|_{h(\tau)}\\
&\le \Big(\frac{h(\tau)^2\lambda p}{M(1-m^s)\tau^s}\Big)^{\lambda p} 2^{-3} h(\mu(\tau)\tau)\|v_1(\mu(\tau)\tau,\cdot)-v_0(\mu(\tau)\tau,\cdot)\|_{h(\mu(\tau)\tau)}\\
&\le \Big(\frac{h(\tau)^2\lambda p}{M(1-m^s)\tau^s}\Big)^{\lambda p} 2^{-3} h(\mu(\tau)\tau)\cdot \mathbb{N}_{\mathcal{H}(T)}[v_1-v_0]\cdot\tau\\
&\le \Big(\frac{h(\tau)^2\lambda p}{M(1-m^s)}\Big)^{\lambda p} 2^{-3} h(\mu(\tau)\tau)\cdot \mathbb{N}_{\mathcal{H}(T)}[v_1-v_0]\cdot\tau^{1-\lambda ps}\\
&\le 2^{-3}\Big(\frac{h(T)^2\lambda p}{M(1-m^s)}\Big)^{\lambda p} T^{1/2} h(\tau)\cdot \mathbb{N}_{\mathcal{H}(T)}[v_1-v_0]
\end{aligned}$$

and

$$2^3 h(\tau)^{-1}|\partial_2(\partial_2^p(v_1-v_0))(\mu(\tau)\tau,\cdot)|_{h(\tau)} \le \Big(\frac{h(T)^2\lambda p}{M(1-m^s)}\Big)^{\lambda p} T^{1/2}\cdot \mathbb{N}_{\mathcal{H}(T)}[v_1-v_0]. \tag{4.22}$$

Further, as in (4.5), we see that

$$\begin{aligned} &2^6|\partial_2^p(v_1-v_0)(\mu(\tau)\tau,\cdot)| \\ &= 2^6\sup_\xi|\partial_2^{p-1}(\partial_2(v_1-v_0))(\mu(\tau)\tau,\xi)| \\ &\le 2^6|\partial_2(v_1-v_0)(\mu(\tau)\tau,\cdot)|_{h(\mu(\tau)\tau)}h(\mu(\tau)\tau)^{p-1}\Gamma_\lambda(p-1) \\ &\le 2^3h(\mu(\tau)\tau)^p\Gamma_\lambda(p-1)\cdot\|(v_1-v_0)(\mu(\tau)\tau,\cdot)\|_{h(\mu(\tau)\tau)} \\ &\le 2^3h(T)^p\Gamma_\lambda(p-1)\cdot\mathbb{N}_{\mathcal{H}(T)}[v_1-v_0]\cdot\mu(\tau)\tau \\ &\le 2^3h(T)^p\Gamma_\lambda(p-1)T\cdot\mathbb{N}_{\mathcal{H}(T)}[v_1-v_0]. \end{aligned} \tag{4.23}$$

By (4.22) and (4.23) we see that the inequality

$$\|\partial_2^p(v_1-v_0)(\mu(\tau)\tau,\cdot)\|_{h(\tau)} \le E(T)\mathbb{N}_{\mathcal{H}(T)}[v_1-v_0] \tag{4.24}$$

holds, where

$$E(T) = \max\Big\{\big(\frac{h(T)^2\lambda p}{M(1-m^s)}\big)^{\lambda p}T^{1/2},\ 2^3h(T)^p\Gamma_\lambda(p-1)T\Big\}. \tag{4.25}$$

To estimate the second difference in (4.21), we define

$$\varphi(\tau,\xi) = \partial_2^q v_1(\tau,\nu(\tau)\xi) - \partial_2^q v_0(\tau,\nu(\tau)\xi).$$

In almost the same way as in (4.11) and (4.12) the norm $|\partial_2\varphi(\tau,\cdot)|_{h(\tau)}$ is estimated as follows.

$$\begin{aligned} |\partial_2\varphi(\tau,\cdot)|_{h(\tau)} &= \nu(\tau)|\partial_2^{q+1}(v_1-v_0)(\tau,\cdot)|_{h(\tau)/\nu(\tau)} \\ &\le m|\partial_2^{q+1}(v_1-v_0)(\tau,\cdot)|_{h(\tau)/m} \\ &\le m(h(\tau)/m)^q\big(\frac{\lambda q}{\log m^{-1}}\big)^{\lambda q}|\partial_2(v_1-v_0)(\tau,\cdot)|_{h(\tau)} \\ &\le m(h(\tau)/m)^q\big(\frac{\lambda q}{\log m^{-1}}\big)^{\lambda q}2^{-3}h(\tau)\|(v_1-v_0)(\tau,\cdot)\|_{h(\tau)} \\ &\le 2^{-3}m^{1-q}\big(\frac{\lambda q}{\log m^{-1}}\big)^{\lambda q}h(\tau)^{1+q}\mathbb{N}_{\mathcal{H}(T)}[v_1-v_0]\cdot\tau \\ &\le 2^{-3}m^{1-q}\big(\frac{\lambda q}{\log m^{-1}}\big)^{\lambda q}h(T)^qTh(\tau)\mathbb{N}_{\mathcal{H}(T)}[v_1-v_0]. \end{aligned} \tag{4.26}$$

Further, as (4.23), we have

$$\begin{aligned} |\varphi(\tau,\cdot)| &\le \sup_x|\partial_2^q(v_1-v_0)(\tau,x)| \\ &\le |\partial_2(v_1-v_0)(\tau,\cdot)|_{h(\tau)}h(\tau)^{q-1}\Gamma_\lambda(q-1) \\ &\le 2^{-3}h(T)^q\Gamma_\lambda(q-1)\|(v_1-v_0)(\tau,\cdot)\|_{h(\tau)} \\ &\le 2^{-3}h(T)^q\Gamma_\lambda(q-1)\mathbb{N}_{\mathcal{H}(T)}[v_1-v_0]\cdot\tau \\ &\le 2^{-3}h(T)^q\Gamma_\lambda(q-1)T\mathbb{N}_{\mathcal{H}(T)}[v_1-v_0]. \end{aligned} \tag{4.27}$$

From (4.26) and (4.27) we obtain

$$\begin{aligned} \|\partial_2^q(v_1-v_0)(\tau,\nu(\tau)\cdot)\|_{h(\tau)} &= \|\varphi(\tau,\cdot)\|_{h(\tau)} \\ &= \max\{2^6|\varphi(\tau,\cdot)|, 2^3h(\tau)^{-1}|\partial_2\varphi(\tau,\cdot)|_{h(\tau)}\} \\ &\le F(T)\mathbb{N}_{\mathcal{H}(T)}[v_1-v_0], \end{aligned} \tag{4.28}$$

where

$$F(T) = \max\Big\{2^3\Gamma_\lambda(q-1), m^{1-q}\big(\frac{\lambda q}{\log m^{-1}}\big)^{\lambda q}\Big\}h(T)^q \cdot T. \tag{4.29}$$

Looking at (4.25) and (4.29), it is clear that there is a $T > 0$ such that $T \le T_2$ and $E(T) + F(T) \le (2C_f)^{-1}$. So we can define

$$T^* = \max\{T : T \le T_2 \ \text{ and } \ E(T) + F(T) \le (2C_f)^{-1}\}. \tag{4.30}$$

By (4.24), (4.28) and (4.30), if v_0 and v_1 are in $\mathcal{F}(T^*, C_f)$, then the inequality

$$\|\partial_2^p(v_1-v_0)(\mu(\tau)\tau,\cdot)\|_{h(\tau)}+\|\partial_2^q(v_1-v_0)(\tau,\nu(\tau)\cdot)\|_{h(\tau)} \le \frac{1}{2C_f}\mathbb{N}_{\mathcal{H}(T^*)}[v_1-v_0] \tag{4.31}$$

holds for $0 \le \tau \le T^*$. By (4.20) and (4.31) we obtain

$$\|\delta_{(v_1,v_0)}(\tau,\cdot)\|_{h(\tau)} \le \frac{1}{2}\mathbb{N}_{\mathcal{H}(T^*)}[v_1 - v_0]. \tag{4.32}$$

Now, our aim is to estimate the difference $\Phi_{T^*}(v_1) - \Phi_{T^*}(v_0)$. Using the notation $\delta_{(v_1,v_0)}$, this difference is expressed as

$$\Phi_{T^*}(v_1)(t,x) - \Phi_{T^*}(v_0)(t,x) = \int_0^t d\tau \int_{-\infty}^{\infty} G(t-\tau, x-\xi)\delta_{(v_1,v_0)}(\tau,\xi)d\xi \tag{4.33}$$

By (4.33), (4.32) and Proposition 3.8, we know that

$$\mathbb{N}_{\mathcal{H}(T^*)}[\Phi_{T^*}(v_1) - \Phi_{T^*}(v_0)] \le \frac{1}{2}\mathbb{N}_{\mathcal{H}(T^*)}[v_1 - v_0].$$

This implies that the mapping Φ_{T^*} from the closed subset $\mathcal{F}(T^*, C_f)$ of the Banach space $\mathcal{H}(T^*)$ into itself is a contraction. Therefore, there is one and only one element v of $\mathcal{F}(T^*, C_f)$ such that

$$v = \Phi_{T^*}(v).$$

This element $v \in \mathcal{F}(T^*, C_f)$ is a solution of the integral equation (1.7) and, accordingly, a solution of the Cauchy problem (1.6)-(1.4).

Since v is in $\mathcal{F}(T^*, C_f)$, it belongs to the family $\widetilde{\mathcal{G}}_h([0, T^*), \mathbb{R})$ and satisfies the inequality (4.1). Further, since v is a solution of the differential equation (1.6), it satisfies the equality

$$\partial_1 v(t,x) = a\partial_2^2 v(t,x) + f(t,x,\partial_2^p v(\mu(t)t,x), \partial_2^q v(t,\nu(t)x)). \tag{4.34}$$

From this equality we see that $\partial_1 v(t,x)$ is infinitely differentiable in x, because so is the right-hand side of (4.34). This completes the proof of the fact that the Cauchy problem (1.6)-(1.4) has a solution $u \in \widetilde{\mathcal{G}}_h([0,T^*),\mathbb{R})\bigcap C^{(1,\infty)}([0,T^*),\mathbb{R})$ such that the inequality (4.1) holds.

The fact that the Cauchy problem has only one such solution is easily confirmed. In fact assume that $v \in \widetilde{\mathcal{G}}_h([0,T^*),\mathbb{R})\bigcap C^{(1,\infty)}([0,T^*),\mathbb{R})$ is a solution of the Cauchy problem (1.6)-(1.4) and the inequality (4.1) holds. Then v is in the set $\mathcal{F}(T^*, C_f)$ and satisfies the integral equation (1.7). This means that v is the unique fixed point of the contraction $\Phi_{T^*} : \mathcal{F}(T^*, C_f) \to \mathcal{F}(T^*, C_f)$. □

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