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# BLOW-UP RATE FOR PARABOLIC PROBLEMS WITH NONLOCAL SOURCE AND BOUNDARY FLUX

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Abstract. We determine the blow-up rate and the blow-up set for a class of one-dimensional nonlocal parabolic problems with opposite source term and boundary flux. As a consequence, it is shown that the solutions approach negative infinity in the interior of the domain and positive infinity at one boundary point.

## 1. Introduction

In some nonlinear evolution equations, solutions develop singularities in finite time and can not be continued beyond that time. Such phenomenon is called blow-up and the time at which blow-up occurs is called the blow-up time [8]. The blow-up set is the set of all points $x$ such that the solution blows up at the place $x$ and at time $T$. By the blow-up rate we mean any approximation of the solution near the blow-up time by unbounded quantities. The solution can be estimated for instance pointwise or with respect to the $L^\infty$ norm.

In this paper, we study of the blow-up rate of the following one-dimensional nonlocal problem,

$$\begin{gathered}
u_t - u_{xx} = -a\Big(\int_\Omega u(x,t)dx\Big) \quad \text{in } (0,\ell)\times(0,T),\\
u(0,t) = 0 \quad \text{on } (0,T),\\
u_x(\ell,t) = a\Big(\int_\Omega u(x,t)dx\Big) \quad \text{on } (0,T),\\
\int_\Omega u(x,t)dx \geq 0 \quad \text{on } (0,T),\\
u(\cdot,0) = u_0(\cdot) \quad \text{in } (0,\ell),
\end{gathered} \tag{1.1}$$

where $\ell, T$ are positive numbers, $\Omega = (0,\ell)$ and $a(\cdot)$ is a numerical function defined on $[0,\infty)$. The function $a(\cdot)$ could be for instance a power function.

Nonlocal problems can arise in applications either by assuming from the start that there exists some global interaction between the variables involved or as the result of some simplification of standard local models (see [6]). In the first case, there are, for instance, the examples of heat conduction (see [4] for the above

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problem and [7]) or population models in biology, where it can be assumed that there exists some global mechanism which is important in the process of evolution described (see [2], [3], [9]). We refer to [12] for an example where the second situation occurs.

We would like to point out some structural difficulties appearing in Problem (1.1) namely the presence of two nonlinear terms and the mixed boundary conditions. Moreover some usual and useful tools do not apply here: there is no global Liapunov's functional and the maximum principle does not hold. More precisely, there exit sign changing solutions having positive initial conditions (see Remark 5.2). This phenomenon is essentially due to the fact that the nonlinear terms are nonlocal and have different signs.

The study of blow-up rate for parabolic equations has produced a huge literature which can be divided into three parts depending on the presence of nonlinear terms in the equation or/and in the boundary conditions. We briefly state some results for these three types of problems. First consider problems with a source term.

If $\Omega$ is a bounded domain of $\mathbb{R}^n$ and $p > 1$ is a number satisfying $(n-2)p < n+2$ then the blow-up rate for the problem

$$\begin{gathered} u_t - \Delta u = u^p \quad \text{in } \Omega \times (0,T), \\ u(\cdot,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0(\cdot) \geq 0 \quad \text{in } \Omega, \end{gathered}$$

is given by the estimates

$$\frac{c}{(T-t)^{\frac{1}{p-1}}} \leq \sup_{\Omega} u(\cdot,t) \leq \frac{C}{(T-t)^{\frac{1}{p-1}}},$$

where $c$ and $C$ are positive constants [10]. Nonlocal equations of the form

$$\begin{gathered} u_t - \Delta u = \Big( \int_{\Omega} | u(x,t) |^r \, dx \Big)^{p/r} \quad \text{in } \Omega \times (0,T), \\ u(\cdot,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0(\cdot) \quad \text{in } \Omega, \end{gathered}$$

where $1 \leq r < \infty$, $p > 1$ have the same blow-up rate [16].

For problems with nonlinear boundary conditions such as

$$\begin{gathered} u_t - \Delta u = 0 \quad \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = u^q \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0(\cdot) \geq 0 \quad \text{in } \Omega, \end{gathered}$$

the blow-up rate is

$$\frac{c}{(T-t)^{\frac{1}{2(q-1)}}} \leq \sup_{\Omega} u(\cdot,t) \leq \frac{C}{(T-t)^{\frac{1}{2(q-1)}}}$$

provided, for instance, that $\Omega$ is a smooth bounded domain of $\mathbb{R}^n$, $q > 1$, $(n-2)q < n$ and $\Delta u_0 \geq 0$ (see [11]). The case where $\Omega = \mathbb{R}^n_+$ is addressed in [5].

An example of problems with both source term and boundary flux is

$$\begin{gathered} u_t - u_{xx} = -\lambda u^p \quad \text{in } (0,1)\times(0,T), \\ u_x(0,t) = 0 \quad \text{on } (0,T), \\ u_x(1,t) = u(1,t)^q \quad \text{on } (0,T), \\ u(\cdot,0) = u_0(\cdot) \geq 0 \quad \text{in } \Omega. \end{gathered}$$

If $u_0$ is positive, increasing, verifies a compatibility condition, and $u_{0xx} - \lambda u_0^p \geq \alpha > 0$, then the blow-up rate of any blowing-up solution is given by

$$\frac{c}{(T-t)^\alpha} \leq \sup_\Omega u(\cdot,t) \leq \frac{C}{(T-t)^\alpha},$$

where

$$\alpha = \min\Big(\frac{1}{p-1}, \frac{1}{2(q-1)}\Big)$$

(see [13]). We remark that $1/(p-1)$ (resp. $1/2(q-1)$) is the blow-up rate exponent coming from the source term (resp. the boundary flux) and that the blow-up rate is independent of the sign of the real parameter $\lambda$.

Let us state now our main result in the simple situation where $a$ is a power function.

**Theorem 1.1.** *Let $p > 1$, $a(s) = s^p$ for all $s \geq 0$, $\ell \in (0, \frac{3\pi}{10})$, $u_0 \geq 0$ in $\Omega$ and $u$ be the maximal solution to Problem (1.1) defined on some finite time interval $[0,T)$. Then $u$ blows up at time $T$, $\int_0^\ell u(x,t)dx \to +\infty$ as $t \to T$ and the blowup rate at the point $x = \ell$ is given, for $t$ close to $T$, by*

$$\frac{c}{(T-t)^{\frac{p+1}{2(p-1)}}} \leq u(\ell,t) \leq \frac{C}{(T-t)^{\frac{p+1}{2(p-1)}}}, \tag{1.2}$$

*where $c$ and $C$ are positive numbers independent of time. In addition, for any compact subset $K$ and for $t$ close to $T$,*

$$\frac{-C}{(T-t)^{\frac{1}{p-1}}} \leq u(x,t) \leq \frac{-c}{(T-t)^{\frac{1}{p-1}}} \quad \forall x \in K, \tag{1.3}$$

*where $c, C$ are positive numbers independent of $K$ and $t$.*

The plan for this paper is as follows: In section 2, we recall some results about solutions which blow up in finite time. The blow-up rate for the integral of the solution is computed in section 3. In section 4, the main result, namely Theorem 4.3, is stated and proved. It gives the blow-up rate of the solution to Problem (1.1). The section 5 is devoted to some remarks and applications of the main result.

Let us explain briefly the ideas of the proof of Theorem 4.3. For let us denote by $u$ the solution to Problem (1.1). We compute the blow-up rate of $u_1$, the first coordinate of $u$ in some spectral basis of $L^2(\Omega)$ and prove, roughly speaking, the equivalence between $u_1$ and $\int_\Omega u dx$ near the blow-up time. Estimates for the integral then follow. Using these estimates and assuming the monotonicity of the function $a$, we deduce the blow-up rate of the solution $u$.

Time independent constants will be denoted by $c, C, c_1, \dots$. Different constants may be indicated by the same symbol if no confusion can occur.

## 2. Existence of blowing-up solutions

Existence and uniqueness of a maximal solution to problem (1.1) follow from the classical theory of parabolic equations. We would like to address here the issue of the existence of blowing-up solutions. Since $a$ is defined only on $[0, \infty)$, it could happen that $\int_0^\ell u(x,T)dx$ vanishes at the end point $T$ of the maximal existence time interval. Moreover, as explain above, $u$ can change its sign hence it is not clear that there exist solutions which blow up in finite time. Nevertheless this is true for small $\ell$. Indeed, let us first assume that

$$\text{the function } a \text{ is defined and locally Lipschitz continuous on } [0,\infty) \tag{2.1}$$

and there exist two constants $p \geq 1$ and $C_0 > 0$ such that

$$|a(s)| \leq C_0(1 + |s|^p) \quad \forall s \in [0,\infty). \tag{2.2}$$

Then the function

$$\overline{a}(s) = \begin{cases} a(s) & \text{if } s \geq 0, \\ a(0) & \text{otherwise,} \end{cases} \tag{2.3}$$

is clearly locally Lipschitz continuous on $\mathbb{R}$ and satisfies the growth condition (2.2) on $\mathbb{R}$. Hence according to [14, Theorem 1.1], we have the following theorem.

**Theorem 2.1.** *Under the assumptions and notation (2.1)-(2.3), let us assume in addition that the initial condition $u_0$ belongs to $L^2(\Omega)$. Then the problem*

$$\begin{gathered} u_t - u_{xx} = -\overline{a}\Big(\int_\Omega u(x,t)dx\Big) \quad \text{in } (0,\ell)\times(0,T), \\ u(0,t) = 0 \quad \text{on } (0,T), \\ u_x(\ell,t) = \overline{a}\Big(\int_\Omega u(x,t)dx\Big) \quad \text{on } (0,T), \\ u(\cdot,0) = u_0(\cdot) \quad \text{in } (0,\ell), \end{gathered} \tag{2.4}$$

*has a unique maximal weak solution $u$ defined on $\Omega \times [0,T)$. Moreover, if $T$ is finite then*

$$\lim_{t\to T} |u(\cdot,t)|_{L^2(\Omega)} = +\infty.$$

We refer the reader to [4] for the definition of the maximal weak solution to the above problem.

Theorem 4.2 in [4] gives sufficient conditions ensuring that the integral of the solution to (2.4) remains positive hence under the assumptions of this theorem, the solutions to Problems (1.1) and (2.4) coincide. Using also [14, Theorems 2.1 and 2.2], we obtain the existence of blowing-up solutions for Problem (1.1). More precisely, we have the

**Theorem 2.2.** *There exists $\ell_1 \in [\frac{3\pi}{10}, \infty]$ such that if*

(i) *$\Omega = (0,\ell)$ with $\ell \in (0,\ell_1)$,*
(ii) *the function $a$ satisfies (2.1)-(2.2), is non-negative, non-decreasing on the interval $(0,+\infty)$ and $\int^{+\infty} \frac{ds}{a(s)} < +\infty$,*
(iii) *the initial condition $u_0$ is equal to $\beta\phi$ where $\beta \in \mathbb{R}$ and $\phi$ is a function of $L^2(\Omega)$ satisfying one of the two following conditions:*
    (1) *There exists $\alpha > 0$ such that $\phi \geq \alpha$ a.e. in $\Omega$; or*

(2) *$\phi$ is continuous on $\overline{\Omega}$, positive on $(0,\ell]$, differentiable from the right at 0 and satisfies $\phi(0)=0$, $\phi'(0^+)>0$,*

*then there exists a positive number $\beta_1$ such that for all $\beta \ge \beta_1$, the weak solution to Problem (1.1) blows up in finite time in $L^2$ norm.*

Some blow-up results for large $\ell$ are stated in [15].

## 3. Blowup rate for the integral

In this section, we will assume that

$$a \text{ satisfies (2.1), is positive, non-decreasing on } (0,\infty) \tag{3.1}$$

and

$$\int^{+\infty} \frac{ds}{a(s)} < +\infty. \tag{3.2}$$

These assumptions allow us to introduce the function $A$ defined by

$$A(u) = \int_u^{\infty} \frac{ds}{a(s)} \quad \forall u \in (0,\infty). \tag{3.3}$$

**Remark 3.1.** The function $A$ maps $(0,\infty)$ onto $(0, A(0))$, is continuous and decreasing on $(0,\infty)$. Thus the inverse function $A^{-1}$ of $A$ is well defined on $(0, A(0))$ and its limit at 0 is $+\infty$. If $A(0)$ is finite then we extend $A^{-1}$ by zero outside of $(0, A(0)]$, hence $(A^{-1})'$ is a non-negative piecewise continuous function on $(0,\infty)$.

**Theorem 3.2.** *Let us assume the following:*

(i) $\Omega = (0,\ell)$ *with* $\ell < 3\pi/10$.
(ii) *The function $a$ satisfies the assumptions (2.2), (3.1) and (3.2).*
(iii) *The solution $u$ to Problem (1.1) blows up in finite time $T$ in $L^2$ norm.*

*Then there exist positive constants $c$ and $C$ such that for $t$ close to $T$, it holds that*

$$cA^{-1}\big(C(T-t)\big) \le \int_\Omega u(x,t)dx \le CA^{-1}\big(c(T-t)\big). \tag{3.4}$$

To prove this result, we introduce some auxiliary functions and notation and then state three lemmas.

The weak formulation of (1.1) reads

$$\frac{d}{dt}\int_\Omega u\varphi dx + \int_\Omega u_x\varphi_x dx = a\Big(\int_\Omega u(x,t)dx\Big)\big(\varphi(\ell) - \int_\Omega \varphi dx\big),$$

for all test function $\varphi$ belonging to $\{\varphi \in H^1(\Omega) : \varphi(0) = 0\}$. Taking $\varphi = \varphi_k$ the $k^{th}$ element of the normalized spectral basis (with non-negative integral) associated with the Laplacian operator with homogeneous mixed boundary conditions, we get

$$u_k'(t) + \lambda_k u_k(t) = a\big(\int_\Omega u(x,t)dx\big)D(\varphi_k), \tag{3.5}$$

where $u_k(t)$ denotes the $k^{th}$ coordinate of $u(t)$ in the basis $(\varphi_k)_{k\ge1}$ and

$$D(\varphi_k) = \varphi_k(\ell) - \int_\Omega \varphi_k dx = \sqrt{\frac{2}{\ell}}\big((-1)^{k+1} - \frac{1}{\sqrt{\lambda_k}}\big). \tag{3.6}$$

Note that, in our one-dimensional setting, we have clearly

$$\lambda_k = \frac{\pi^2}{4\ell^2}(2k-1)^2, \quad \varphi_k(x) = \sqrt{\frac{2}{\ell}}\sin\big(\sqrt{\lambda_k}x\big) = \sqrt{\frac{2}{\ell}}\sin\big(\frac{\pi}{2\ell}(2k-1)x\big) \tag{3.7}$$

and therefore

$$\varphi_k(\ell) = \sqrt{\frac{2}{\ell}}(-1)^{k+1}, \quad \int_\Omega \varphi_k dx = \sqrt{\frac{2}{\ell}}\frac{1}{\sqrt{\lambda_k}}. \tag{3.8}$$

**Lemma 3.3.** *Let $\ell \in (0, 3\pi/10)$ and $u$ be the solution to Problem (1.1). Then, for some positive constants $c$ and $C$ independent of time,*

$$cu_1(t) \le \int_\Omega u(x,t)dx + C \quad \forall t \in [0,T). \tag{3.9}$$

*Proof.* Let $t \in [0,T)$. From (3.5), we deduce the following representation of $u_k$,

$$u_k(t) = e^{-\lambda_k t}u_{0_k} + \int_0^t e^{-\lambda_k(t-s)}a\Big(\int_\Omega u(x,s)dx\Big)dsD(\varphi_k), \tag{3.10}$$

where $u_{0_k}$ is the $k^{th}$ coordinate of the initial condition i.e. $u_{0_k} = \int_\Omega u_0(x)\varphi_k(x)dx$. For any integer $n \ge 1$, let us consider the sum

$$S_n(t) = \sum_{k=1}^n u_k(t)\int_\Omega \varphi_k dx.$$

With (3.10), denoting

$$E_k = D(\varphi_k)\int_\Omega \varphi_k dx = \frac{2}{\ell}\Big(\frac{(-1)^{k+1}}{\sqrt{\lambda_k}} - \frac{1}{\lambda_k}\Big) \tag{3.11}$$

and

$$I_k = \int_0^t e^{-\lambda_k(t-s)}a\Big(\int_\Omega u(x,s)dx\Big)ds,$$

we write $S_n(t)$ in the form

$$S_n(t) = \sum_{k=1}^n e^{-\lambda_k t}u_{0_k}\int_\Omega \varphi_k dx + \sum_{k=1}^n I_kE_k = S_n^1(t) + S_n^2(t), \tag{3.12}$$

where $S_n^1(t)$ and $S_n^2(t)$ are defined in an obvious way.

Considering first the sum $S_n^2(t)$, a direct computation leads to (see (3.11) and (3.7)),

$$E_k + E_{k+1} = \frac{\pi^2}{2\ell^4\lambda_k\lambda_{k+1}}\left(\pi(4k^2-1) - 2(4k^2+1)\ell\right),$$

for $k \ge 1$ odd. Thus

$$\ell < \frac{3\pi}{10} \Longrightarrow E_k + E_{k+1} > 0 \quad \forall k \ge 1, \text{ odd.} \tag{3.13}$$

In particular, for $k = 1$, there exists a positive constant $\epsilon$ depending only on $\ell$ such that

$$E_1 + E_2 \ge \epsilon E_1. \tag{3.14}$$

Furthermore, $k \mapsto I_k$ is non-increasing. It follows that for all $k \ge 1$ odd,

$$I_{k+1}E_{k+1} \ge I_kE_{k+1}, \tag{3.15}$$

since $E_{k+1} \le 0$. Hence, for every even integer $n \ge 4$, writing $S_n^2(t)$ under the form

$$S_n^2(t) = I_1E_1 + I_2E_2 + \sum_{k=3,odd}^{n-1} I_kE_k + I_{k+1}E_{k+1}$$

and using (3.15), it appears

$$S_n^2(t) \geq I_1(E_1+E_2) + \sum_{k=3,odd}^{n-1} I_k(E_k+E_{k+1}).$$

Now $I_k \geq 0$ thus with (3.14) and (3.13), we obtain

$$S_n^2(t) \geq \epsilon I_1 E_1. \tag{3.16}$$

Going back to (3.12) we get with (3.16) and next (3.10),

$$\begin{aligned} S_n(t) \geq & S_n^1(t) + \epsilon I_1 E_1 \\ \geq & S_n^1(t) - \epsilon e^{-\lambda_1 t} u_{0_1} \int_\Omega \varphi_1 dx + \epsilon\Big(e^{-\lambda_1 t} u_{0_1} \int_\Omega \varphi_1 dx + I_1 E_1\Big) \\ \geq & S_n^1(t) - \epsilon e^{-\lambda_1 t} u_{0_1} \int_\Omega \varphi_1 dx + \epsilon \int_\Omega \varphi_1 dx\, u_1(t). \end{aligned}$$

Now, $S_n^1(t) - \epsilon e^{-\lambda_1 t} u_{0_1} \int_\Omega \varphi_1 dx$ is the integral over $\Omega$ of the solution $w^n$ to the linear problem

$$\begin{gathered} w_t^n - w_{xx}^n = 0 \quad \text{in } (0,\ell)\times(0,T), \\ w^n(0,t) = 0, \quad w_x^n(\ell,t) = 0 \quad \text{on } (0,T), \\ w^n(x,0) = (1-\epsilon)u_{0_1}\varphi_1(x) + \sum_{k=2}^n u_{0_k}\varphi_k(x) \quad \text{in } (0,\ell). \end{gathered} \tag{3.17}$$

Classically, the $L^2$-norm $|w^n(\cdot,t)|_{L^2(\Omega)}$ of $w^n(\cdot,t)$ can be estimated by $|w^n(\cdot,0)|_{L^2(\Omega)}$ which is bounded up to a constant by $|u_0|_{2,\Omega}$. Thus for some positive constant $C>0$ independent of time,

$$\Big|\int_\Omega w^n(x,t)dx\Big| \leq C \quad \forall n \geq 1. \tag{3.18}$$

Hence, $S_n(t) \geq -C + cu_1(t)$ for all $t \in [0,T)$ and letting $n \to \infty$, we arrive at (3.9). □

**Lemma 3.4.** *Let $\ell \in (0,\infty)$ and $u$ be the solution to Problem (1.1). Then, for a positive constant $C$,*

$$\int_\Omega u(x,t)dx \leq Cu_1(t) + C \quad \forall t \in [0,T). \tag{3.19}$$

*Proof.* Using the notation of the previous lemma, we have

$$S_n(t) = \int_\Omega \varphi_1 dx\, u_1(t) + \sum_{k=2}^n e^{-\lambda_k t} u_{0_k} \int_\Omega \varphi_k dx + \sum_{k=2}^n I_k E_k.$$

Let us show that this last sum is non-positive if $n$ is odd. Indeed

$$\sum_{k=2}^n I_k E_k = \sum_{k=2,even}^{n-1} I_k E_k + I_{k+1}E_{k+1} \leq \sum_{k=2,even}^{n-1} I_{k+1}(E_k + E_{k+1}) \leq 0,$$

since $E_k \leq 0$ and $E_k + E_{k+1} \leq 0$ for all $k$ even and all $\ell > 0$. Using also (3.17), (3.18) with $\epsilon = 1$, we obtain $S_n(t) \leq \int_\Omega \varphi_1 dx\, u_1(t) + C$ and the result follows letting $n \to \infty$. □

**Lemma 3.5.** *Under the assumptions of Theorem 3.2, it holds that, as* $t \to T$*,*

$$u_1(t) = \int_\Omega u(x,t)\varphi_1(x)dx \to \infty \quad \textit{and} \quad \int_\Omega u(x,t)dx \to \infty. \tag{3.20}$$

*Moreover, there exist* $c$ *and* $C$ *positive, such that for* $t$ *close to* $T$*,*

$$0 \le cu_1(t) \le \int_\Omega u(x,t)dx \le Cu_1(t). \tag{3.21}$$

*Proof.* Putting $k = 1$ in (3.5) we get

$$u_1'(t) + \lambda_1 u_1(t) = a\Big(\int_\Omega u(x,t)dx\Big)D(\varphi_1). \tag{3.22}$$

By (3.9), (3.1) and $D(\varphi_1) > 0$,

$$u_1'(t) + \lambda_1 u_1(t) \ge a\big((cu_1(t) - C)^+\big)D(\varphi_1) \quad \forall t \in [0,T),$$

where $(\cdot)^+$ denotes the positive part of the argument. Setting, for $s \in \mathbb{R}$, $f(s) := D(\varphi_1)a\big((cs - C)^+\big) - \lambda_1 s$, the above inequality reads

$$u_1'(t) \ge f(u_1(t)) \quad \forall t \in [0,T). \tag{3.23}$$

Since $u$ blows up in $L^2$ norm at time $T$ and $\int_\Omega u(x,t)dx \ge 0$, it follows in a standard way that

$$\limsup_{t \to T} \int_\Omega u(x,t)dx = +\infty. \tag{3.24}$$

Thus by (3.19) and (3.2), there exists a time $t_1 \in [0,T)$ such that $u_1(t_1)$ is larger than any zero of $f$ and $f(u_1(t_1)) > 0$. We then deduce with (3.23) that $u_1$ is increasing on $[t_1,T)$. Thus $u_1$ admits a limit in $T$ which must be $+\infty$ due to (3.19) and (3.24). Now with (3.9) we have then $\int_\Omega u(x,t)dx \to \infty$. (3.21) follows easily from (3.20), (3.9) and (3.19) which completes the proof of the lemma. □

*Proof of Theorem 3.2.* Due to (3.20)-(3.22), (3.1) and (3.2), we deduce that, for some time $t_1 \in [0,T)$, it holds that $u_1'(t) \ge \frac{D(\varphi_1)}{2}a(cu_1(t))$ on $[t_1,T)$; we recall that $D(\varphi_1)$ defined by (3.6) is positive since $\ell < \frac{3\pi}{10}$. Integrating this inequality on $[t,T_1]$ for any $t, T_1 \in [t_1,T)$ with $t < T_1$, we have

$$A\big(cu_1(t)\big) - A\big(cu_1(T_1)\big) \ge \frac{cD(\varphi_1)}{2}(T_1 - t),$$

where $A$ is defined by (3.3). When $T_1 \to T$, we get by (3.20)

$$A\big(cu_1(t)\big) \ge \frac{cD(\varphi_1)}{2}(T - t) \quad \forall t \in [t_1,T).$$

Since $A^{-1}$ is decreasing,

$$u_1(t) \le CA^{-1}\big(c(T-t)\big) \quad \forall t \in [t_1,T),$$

which together with (3.21) provides the upper bound of $\int_\Omega u(x,t)dx$ claimed in (3.4) for a new constant $C$. The lower bound is obtained in the same manner. Indeed, from (3.21) and (3.1), we deduce that $a(\int_\Omega u(x,t)dx) \le a(Cu_1(t))$ on $[t_1,T)$. Combining this with (3.22) leads to

$$u_1'(t) \le D(\varphi_1)a(Cu_1(t)) \quad \forall t \in [t_1,T).$$

Hence, in view of Remark 3.1, $\int_\Omega u(x,t) \ge c'A^{-1}\big(C'(T-t)\big)$ thanks to (3.21). Moreover we may assume $c = c'$ and $C = C'$ in (3.4) since $A^{-1}$ is non-increasing. □

## 4. Blow-up rate for the solution

Taking advantage of the semi-linear structure of Problem (1.1) and using the results of the previous section, we are led to study, by the superposition principle, three simpler linear problems. Each has only one non-trivial term coming respectively from the contribution of the source term or the boundary flux or the initial condition. The third problem gives rise, of course, to bounded quantities and the first one can be treated using results of [16]. Indeed, let $f$ be a numerical function defined on $[0,T)$ and let us consider the problem

$$\begin{gathered} u_t - u_{xx} = -f(t) \quad \text{in } (0,\ell)\times(0,T), \\ u(0,t) = 0 \quad \text{on } (0,T), \\ u_x(\ell,t) = 0 \quad \text{on } (0,T), \\ u(\cdot,0) = 0 \quad \text{in } (0,\ell). \end{gathered} \tag{4.1}$$

Then we have the following statement.

**Theorem 4.1.** *Let $f$ be a positive function, continuous on $[0,T)$ and locally Hölder continuous on $(0,T)$. Assume that the solution $u$ to (4.1) satisfies*

$$\lim_{t\to T} |u(\cdot,t)|_{L^\infty(\Omega)} = \infty.$$

*Then*

$$\lim_{t\to T} \frac{u(x,t)}{\int_0^t f(s)ds} = -1,$$

*uniformly on compact subsets of $(0,\ell]$.*

*Proof.* It is well know that the problem

$$\begin{gathered} \xi_t - \xi_{xx} = -f(t) \quad \text{in } (0,2\ell)\times(0,T), \\ \xi(0,t) = 0 \quad \text{on } (0,T), \\ \xi(2\ell,t) = 0 \quad \text{on } (0,T), \\ \xi(\cdot,0) = 0 \quad \text{in } (0,2\ell), \end{gathered}$$

has a unique classical solution. Moreover it is symmetric with respect to $\ell$ thus $\xi = u$ on $(0,\ell)\times(0,T)$. We conclude using [16, Theorem 4.1]. □

Hence it remains to examine the second problem. For this end, we consider the problem

$$\begin{gathered} u_t - u_{xx} = 0 \quad \text{in } (0,\ell)\times(0,T), \\ u(0,t) = 0 \quad \text{on } (0,T), \\ u_x(\ell,t) = g(t) \quad \text{on } (0,T), \\ u(\cdot,0) = 0 \quad \text{in } (0,\ell), \end{gathered} \tag{4.2}$$

where $g$ is a numerical function defined on $[0,T)$.

**Theorem 4.2.** *Let us assume that $g$ is a continuous function defined from $[0,T)$ into $[0,\infty)$ and for $t$ close to $T$,*

$$g(t) \le C_1 \exp\Big(\frac{\ell^2}{8(T-t)}\Big). \tag{4.3}$$

*Then the solution $u$ to (4.2) satisfies, for all $t \in [0,T)$,*

$$-C + \int_0^t \frac{g(s)}{\sqrt{\pi(t-s)}} ds \le u(\ell,t) \le C + C\int_0^t \frac{\sup_{[0,s]} g}{\sqrt{t-s}} ds. \tag{4.4}$$

*If, instead of (4.3), we suppose that for $t$ close to $T$,*

$$g(t) \le \frac{C_1}{(T-t)^\beta}, \tag{4.5}$$

*where $\beta > 0$, then for each compact subset $K$ of $\Omega$,*

$$\sup_{K\times[0,T)} u < \infty. \tag{4.6}$$

*Proof.* Let

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\big(-\frac{x^2}{4t}\big).$$

According to [1, Theorem 7.1.1], the solution to (4.2) has the form

$$u(x,t) = -2\int_0^t \partial_x K(x,t-s)\varphi_1(s)ds + 2\int_0^t K(x-\ell,t-s)\varphi_2(s)ds, \tag{4.7}$$

where $\varphi_1$, $\varphi_2$ are piecewise-continuous solutions to

$$\begin{gathered} \varphi_1(t) = -2\int_0^t K(-\ell,t-s)\varphi_2(s)ds, \\ \varphi_2(t) = 2\int_0^t \partial_{xx}K(\ell,t-s)\varphi_1(s)ds + g(t). \end{gathered} \tag{4.8}$$

To solve (4.8), we put

$$\vec\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \quad H(t) = \begin{pmatrix} 0 & -2K(-\ell,t) \\ 2\partial_{xx}K(\ell,t) & 0\end{pmatrix}, \quad \vec g = \begin{pmatrix}0\\ g\end{pmatrix}.$$

Then (4.8) is equivalent to

$$\vec\varphi(t) = \int_0^t H(t-s)\vec\varphi(s)ds + \vec g(t). \tag{4.9}$$

Since the kernel $H$ is smooth on $[0,\infty)$ and $g$ is continuous on $[0,T']$ for all $T' \in (0,T)$, this equation has a unique continuous solution $\vec\varphi$ on $[0,T']$. Moreover $\vec\varphi$ can be clearly extended into a solution on $[0,T)$. Also

$$\|\vec\varphi(t)\| \le C \sup_{[0,t]} g \quad \forall t \in (0,T), \tag{4.10}$$

where $\|\vec\varphi\|^2 := \varphi_1^2 + \varphi_2^2$. Indeed since $H$ is bounded, we deduce from (4.9) that

$$\|\vec\varphi(t)\| \le C\int_0^t \|\vec\varphi(s)\|ds + \sup_{[0,t]}\|\vec g\|.$$

By Gronwall's Lemma, $\|\vec\varphi(t)\| \le \sup_{[0,t]}\|\vec g\| \exp(CT)$, for $t < T$ and (4.10) follows.

Now we can give the lower bound for $u(\ell,\cdot)$. The choice $x = \ell$ in (4.7) implies

$$u(\ell,t) = -2\int_0^t \partial_x K(\ell,t-s)\varphi_1(s)ds + \int_0^t \frac{\varphi_2(s)}{\sqrt{\pi(t-s)}} ds. \tag{4.11}$$

Let us estimate the first integral above. According to (4.3), for $s < t < T$,

$$g(s) \leq C_1 \exp\big(\frac{\ell^2}{8(t-s)}\big). \tag{4.12}$$

Hence, by (4.10),

$$\|\vec{\varphi}(s)\| \leq C_1 \exp\big(\frac{\ell^2}{8(t-s)}\big). \tag{4.13}$$

Thus

$$\big|\int_0^t \partial_x K(\ell, t-s)\varphi_1(s)ds\big| \leq C \int_0^t \frac{1}{(t-s)^{3/2}} \exp\big(-\frac{\ell^2}{8(t-s)}\big)ds$$

which is bounded on $[0, T]$. Let us consider the second integral in (4.11) denoted by $I_2$. Going back to the second equation of (4.8), we obtain with (4.13),

$$\varphi_2(t) \geq -C + g(t).$$

Thus

$$I_2 \geq -C \int_0^t \frac{ds}{\sqrt{\pi(t-s)}} + \int_0^t \frac{g(s)\,ds}{\sqrt{\pi(t-s)}}$$

and the left hand side of (4.4) follows. To get the upper bound, we go back to (4.11) and recall that its first integral is bounded. Combining this with (4.10), we arrive at

$$u(\ell, t) \leq C + C \int_0^t \frac{\sup_{[0,s]} g\, ds}{\sqrt{t-s}},$$

for a new constant $C$ and (4.4) is proved.

Let us prove (4.6). From the integral representation (4.7) and (4.10), (4.5), we deduce, arguing as above, that $u$ is bounded on $K \times [0, T)$. This completes the proof of the theorem. $\square$

Let us now state our main result.

**Theorem 4.3.** *Assume the following:*

(i) $\Omega = (0, \ell)$ *with* $\ell < 3\pi/10$.

(ii) *The function $a$ satisfies (3.1) and for all $C \in (1, \infty)$,*

$$\limsup_{s\to+\infty} \frac{a(Cs)}{a(s)} < \infty, \tag{4.14}$$

$$\limsup_{C\to+\infty} \liminf_{s\to+\infty} \frac{a(Cs)}{Ca(s)} = \infty. \tag{4.15}$$

(iii) *The solution $u$ to Problem (1.1) blows up in finite time $T$ in $L^2$ norm.*

*Then there exist positive constants $c$ and $C$ such that for $t$ close to $T$, it holds that*

$$c \int_0^t \frac{-(A^{-1})'\big(C(T-s)\big)}{\sqrt{t-s}} ds \leq u(\ell, t) \leq C \int_0^t \frac{-(A^{-1})'\big(c(T-s)\big)}{\sqrt{t-s}} ds. \tag{4.16}$$

*Additionally, for any compact subset $K$ of $\Omega$, with $t$ close to $T$,*

$$-CA^{-1}\big(c(T-t)\big) \leq u(x, t) \leq -cA^{-1}\big(C(T-t)\big), \tag{4.17}$$

*for all $x \in K$.*

We refer the reader to Remark 3.1 for the definition of $A^{-1}$.

**Lemma 4.4.** *If a satisfies the assumption (ii) of the above theorem then there exist positive constants $p$, $\epsilon$ and $s_0$ such that*

$$s^{1+\epsilon} \le a(s) \le s^p \quad \forall s \ge s_0. \tag{4.18}$$

*Proof.* By (4.14) with $C = 2$ and (3.1), there exists a positive number $M > 1$ such that

$$a(2s) \le Ma(s) \quad \forall s \ge 1. \tag{4.19}$$

For fixed $s \ge 1$, there exits $n \in \mathbb{N} \setminus \{0\}$ satisfying $2^{n-1} \le s \le 2^n$. Since $a$ is non-decreasing, we have with (4.19),

$$a(s) \le a(2^n) \le M^{n-1}a(2). \tag{4.20}$$

Now, from $2^{n-1} \le s$ and $M > 1$, we deduce that

$$a(s) \le a(2)s^{(\log M)/(\log 2)}.$$

Thus the right hand side of (4.18) holds with $p = \frac{\log M}{\log 2} + 1$ and for some $s_0 > 1$.

Let us prove its left hand side. According to (4.15), there exist $C > 1$ and $s_2 > 0$ such that

$$a(Cs) \ge 2Ca(s) \quad \forall s \ge s_2.$$

For fixed $s \ge s_2$, there exits $n \in \mathbb{N}$ satisfying

$$C^n s_2 \le s \le C^{n+1}s_2.$$

Hence

$$a(s) \ge 2^n C^n a(s_2) = \frac{a(s_2)}{2C}2^{n+1}C^{n+1} \ge \frac{a(s_2)}{2Cs_2}2^{n+1}s.$$

Now from $s \le C^{n+1}s_2$ and $C > 1$, we deduce that $\log(s/s_2)/\log C \le n+1$. Therefore,

$$a(s) \ge \frac{a(s_2)}{2Cs_2}s\big(\frac{s}{s_2}\big)^{\log 2/\log C}.$$

The expected estimate of $a(s)$ follows setting $\epsilon = \frac{\log 2}{2\log C}$ and for some $s_0 > s_2$. □

**Lemma 4.5.** *Let us suppose that a satisfies the assumption (ii) of Theorem 4.3. Then, for all $c, C > 0$, there exist $M, \beta > 0$ such that*

$$a(cA^{-1}(t)) \le t^{-\beta}, \tag{4.21}$$

$$A^{-1}(ct) \le MA^{-1}(Ct), \tag{4.22}$$

*for $t > 0$ close to 0.*

*Proof.* In order to obtain (4.21), by Lemma 4.4, it is sufficient to show that

$$c^p \left(A^{-1}(t)\right)^p \le t^{-\beta},$$

or equivalently, since $A$ is decreasing, that

$$A\big(\frac{1}{c}t^{-\beta/p}\big) \le t. \tag{4.23}$$

Now, by Lemma 4.4,

$$A\big(\frac{1}{c}t^{-\beta/p}\big) = \int_{\frac{1}{c}t^{-\beta/p}}^{\infty} \frac{ds}{a(s)} \le \int_{\frac{1}{c}t^{-\beta/p}}^{\infty} \frac{ds}{s^{1+\epsilon}} = Ct^{\frac{\beta}{p}\epsilon}.$$

Thus (4.23) holds if $t^{\frac{\beta}{p}\epsilon - 1} \le \frac{1}{C}$ for $t$ close to 0. Choosing $\beta = \frac{p}{\epsilon} + 1$, this inequality holds true if $t$ is small enough.

Let us show (4.22). By (4.15), there exists $M = M(C/c)$ such that

$$\liminf_{s\to+\infty} \frac{a(Ms)}{Ma(s)} \geq 2C/c.$$

Hence there exists $s_M > 0$ such that

$$\frac{M}{a(Ms)} \leq \frac{c}{Ca(s)} \quad \forall s > s_M.$$

Therefore, for $t$ close to 0, we have

$$A(MA^{-1}(Ct)) = \int_{MA^{-1}(Ct)}^{\infty} \frac{ds}{a(s)} = \int_{A^{-1}(Ct)}^{\infty} \frac{Mds}{a(Ms)} \leq \frac{c}{C} \int_{A^{-1}(Ct)}^{\infty} \frac{ds}{a(s)} = ct$$

and (4.22) follows since $A^{-1}$ is non-increasing. □

*Proof of Theorem 4.3.* Using Theorem 3.2, we choose $t_1$ in $[0, T)$ such that (3.4) holds on $(t_1, T)$ and

$$C(T - t_1) < A(0), \tag{4.24}$$

where $C$ is the constant appearing in (3.4). Denoting by $u$ the solution to (1.1) on $\Omega \times (0, T)$, we define

$$T_1 = T - t_1, \quad v(x, t) = u(x, t + t_1) \quad \forall (x, t) \in \Omega \times (0, T_1).$$

Let us prove the right hand side of (4.16). For for all $t \in [0, T_1)$, we set

$$f(t) = a\left(cA^{-1}\big(C(T_1 - t)\big)\right), \quad g(t) = a\left(CA^{-1}\big(c(T_1 - t)\big)\right), \tag{4.25}$$

where the constants $c, C$ are given by (3.4) and denote by $u^1$ (resp. $u^2$) the solution to (4.1) (resp. (4.2)) on $(0, T_1)$. Let us also define $u^3$ to be the solution to

$$\begin{gathered}
u^3_t - u^3_{xx} = 0 \quad \text{in } (0, \ell) \times (0, T_1), \\
u^3(0, t) = 0 \quad \text{on } (0, T_1), \\
u^3_x(\ell, t) = 0 \quad \text{on } (0, T_1), \\
u^3(\cdot, 0) = u(\cdot, t_1) \quad \text{in } (0, \ell).
\end{gathered}$$

According to Theorem 3.2, $v$ satisfies by the maximum principle,

$$v \leq u^1 + u^2 + u^3 \quad \text{on } [0, \ell] \times (0, T_1). \tag{4.26}$$

Since $u^1$ is non-positive and $u^3$ is bounded, we have

$$u^1 + u^3 \leq C' \quad \text{on } [0, \ell] \times (0, T_1). \tag{4.27}$$

Let us estimate $u^2(\ell, t)$. By Lemma 4.5, (4.3) holds with $g$ defined by (4.25). Hence, since $g$ is non-decreasing, it follows from Theorem 4.2 that

$$u^2(\ell, t) \leq C' + C' \int_0^t \frac{g(s)}{\sqrt{t-s}} ds \quad \forall t \in [0, T_1). \tag{4.28}$$

By (4.26), (4.27) and the maximum principle for Problem (4.2), we have

$$\int_\Omega v(x, t)dx \leq \int_\Omega u^2(x, t)dx + |\Omega|C' \leq |\Omega|u^2(\ell, t) + |\Omega|C'.$$

Then $u^2(\ell, t) \to +\infty$ according to Lemma 3.5. Hence with (4.28)

$$\int_0^t \frac{g(s)}{\sqrt{t-s}} ds \to \infty \quad \text{when } t \to T_1.$$

Combining this with (4.26)-(4.28) leads to

$$v(\ell,t) \le 2C' \int_0^t \frac{g(s)}{\sqrt{t-s}} ds \quad \forall t \in [t_2, T_1).$$

Now by (4.14), (4.24) and (3.1), there exists $M$ depending on $C$ such that for every $s \in (0, T_1)$,

$$g(s) = a\big(CA^{-1}\big(c(T_1-s)\big)\big) \le Ma\big(A^{-1}\big(c(T_1-s)\big)\big) = -M(A^{-1})'\big(c(T_1-s)\big).$$

Hence for a new constant $C$,

$$u(\ell,t) = v(\ell, t-t_1) \le C \int_0^{t-t_1} \frac{-(A^{-1})'\big(c(T_1-s)\big)}{\sqrt{t-t_1-s}} ds \quad \forall t \in [t_1+t_2, T).$$

By a change of variable and since $-(A^{-1})'$ is non-negative (see Remark 3.1), we have

$$\begin{aligned} u(\ell,t) &\le C \int_{t_1}^t \frac{-(A^{-1})'\big(c(T-s)\big)}{\sqrt{t-s}} ds \\ &\le C \int_0^t \frac{-(A^{-1})'\big(c(T-s)\big)}{\sqrt{t-s}} ds \quad \forall t \in [t_1+t_2, T), \end{aligned}$$

which gives the right hand side of (4.16). To obtain the lower bound, we exchange $f$ and $g$ in (4.25), impose furthermore, $t_1 \ge t_0$ where $t_0$ will be fixed below and define the auxiliary functions $u^i$ in the same way. We then get $u^1 + u^2 + u^3 \le v$. Arguing as above and using Theorem 4.1, it comes (maybe for a new $T_1$),

$$-2\int_0^t f(s)\, ds + \int_0^t \frac{g(s)}{\sqrt{\pi(t-s)}} ds - C' \le v(\ell,t). \tag{4.29}$$

Let us prove that there exist $M = M(C,c)$ and $t_2 \in [0,T)$ such that

$$a\big(CA^{-1}\big(c(T-s)\big)\big) \le Ma\big(cA^{-1}\big(C(T-s)\big)\big) \quad \forall s \in [t_2, T). \tag{4.30}$$

Indeed, by (4.22), there exists $M_1 > 0$ such that

$$A^{-1}(ct) \le \frac{M_1 c}{C} A^{-1}(Ct),$$

for $t$ close to 0. Hence by (4.14) and (3.1), there exist $M > 0$ and $t_2 \in [0,T)$ such that for all $t \in (0, T-t_2]$,

$$a(CA^{-1}(ct)) \le a(M_1 cA^{-1}(Ct)) \le Ma(cA^{-1}(Ct)). \tag{4.31}$$

As a consequence, (4.30) follows setting $t = T - s$ in (4.31). Defining $t_3$ by $(2\sqrt{\pi(T-t_3)})^{-1} = 2M$, we set $t_0 = max(t_2, t_3)$. Then $f \le Mg$ in $[0, T_1)$ and for $t \in [0, T_1)$, it holds that

$$\int_0^t -2f(s) + \frac{1}{2}\frac{g(s)}{\sqrt{\pi(t-s)}}\, ds \ge \big(-2M + \frac{1}{2\sqrt{\pi T_1}}\big) \int_0^t g(s)\, ds \ge 0.$$

Hence going back to (4.29), we obtain, for $t$ in $[0, T_1)$,

$$\frac{1}{2\sqrt{\pi T_1}} \int_0^t g(s)\, ds - C' \le \int_0^t \frac{g(s)}{2\sqrt{\pi(t-s)}} ds - C' \le v(\ell,t). \tag{4.32}$$

Now, we claim that

$$\int_0^{T_1} g(s)\, ds = +\infty. \tag{4.33}$$

Indeed, by (4.14), (4.24) and (3.1), there exists $M > 0$ such that for every $s \in [0, T_1)$,

$$\begin{aligned} g(s) &= a\left(cA^{-1}\big(C(T_1 - s)\big)\right) \\ &\geq \frac{1}{M} a\left(A^{-1}\big(C(T_1 - s)\big)\right) = -\frac{1}{M}(A^{-1})'\big(C(T_1 - s)\big) \end{aligned} \tag{4.34}$$

and by a change of variable

$$\int_0^t g(s)\,ds \geq \frac{1}{MC}\big\{A^{-1}(C(T_1 - t)) - A^{-1}(CT_1)\big\}. \tag{4.35}$$

Then (4.33) follows since $A^{-1}(0) = +\infty$ . Combining (4.32) and (4.33), we obtain

$$\frac{1}{4}\int_0^t \frac{g(s)}{\sqrt{\pi(t-s)}}ds \leq v(\ell, t) \quad \text{for } t \text{ close to } T_1.$$

Now, the left hand side of (4.16) follows easily due to (4.34). It then remains to prove (4.17) but we will only prove its right hand side since the left hand side can be proved in the same way. Let $K$ be any given compact subset of $\Omega$. Then we go back to (4.26) with $f$ and $g$ defined by (4.25) and $T_1$ satisfying (4.24). It follows from Lemma 4.5, Theorem 4.2 and the boundedness of $u^3$ that

$$u^2 + u^3 \leq C_K \quad \text{on } K \times (0, T_1),$$

where $C_K$ is independent of time. Moreover with (4.35) (recall that $f$ and $g$ have been permuted in (4.35)),

$$-\int_0^t f(s)\,ds \leq \frac{1}{MC}\big(A^{-1}(CT_1) - A^{-1}(C(T_1 - t))\big).$$

Thus, using Theorem 4.1 and $A^{-1}(0) = +\infty$, there exists a time $t_4 \in [t_1, T)$ such that (see (4.26)),

$$v \leq -cA^{-1}(C(T_1 - t)) \quad \text{on } K \times (t_4, T_1).$$

We then go back to $u$ which completes the proof of the theorem. □

## 5. Remarks and applications

As a consequence of Theorem 4.3, we have the following statement.

**Corollary 5.1.** *Under the assumptions of Theorem 4.3, let $u$ be a solution to Problem (1.1) which blows up in finite time $T$ in $L^2$ norm. Then the blow up set of $u$ is $(0, \ell]$. Moreover, $u(\ell, t) \to +\infty$ and for all $x \in (0, \ell)$,*

$$u(x, t) \to -\infty \quad \textit{when } t \to T.$$

**Remark 5.2.** Combining Theorem 2.2 and Corollary 5.1 leads to the existence of sign changing solutions for Problem (1.1). More precisely, for each blowing up solution $u$ with positive initial condition and for all compact subset $K$ of $\Omega$, there exits a time $t_K$ such that

$$u < 0 \quad \text{on } K \times [t_K, T).$$

**Remark 5.3.** There exist functions satisfying the assumption (ii) of Theorem 4.3. For instance, let us consider two polynomials $P$ and $Q$ with non-negative coefficients and denote their degrees respectively by $m$ and $n$. Let us assume that $m \geq 2$ and $Q \not\equiv 0$. Then the function

$$a(s) = P(s)Q\big(\log(s + 1)\big), \quad s \geq 0$$

satisfies assumption (ii) of Theorem 4.3. Indeed (3.1) clearly holds. Moreover, for $C > 1$, we have when $s \to \infty$

$$\frac{a(Cs)}{a(s)} \simeq \frac{C^m \log^n\big(Cs+1\big)}{\log^n(s+1)} \simeq C^m,$$

which implies (4.14) and (4.15).

Let us consider the particular case where $a$ is a power function.

*Proof of Theorem 1.1.* If $a(s) = s^p$ then we have

$$A(t) = \frac{1}{p-1} t^{1-p}, \quad A^{-1}(t) = (p-1)^{\frac{1}{1-p}} t^{\frac{1}{1-p}}.$$

Therefore, (1.3) follows from (4.17). Let us show the left hand side of (1.2). Setting $\alpha = \frac{p+1}{2(p-1)}$, we have

$$\int_0^t \frac{(T-s)^{\frac{p}{1-p}}}{(t-s)^{1/2}} ds \geq \int_0^t \frac{(T-s)^{\frac{p}{1-p}}}{(T-s)^{1/2}} ds = \frac{1}{\alpha}(T-t)^{-\alpha} - \frac{1}{\alpha} T^{-\alpha}. \tag{5.1}$$

The lower bound of $u(\ell, t)$ in (1.2) then follows from (4.16). It remains to prove the upper bound. After a change of variable, the first integral of (5.1) is equal to

$$\begin{aligned} &\int_0^t s^{-1/2}(T-t+s)^{-\alpha-1/2} ds \\ &= \int_0^{T-t} s^{-1/2}(T-t+s)^{-\alpha-1/2} ds + \int_{T-t}^t s^{-1/2}(T-t+s)^{-\alpha-1/2} ds. \end{aligned}$$

Let us denote by $I_1$ and $I_2$ the first and second integrals of this sum. Then

$$I_1 \leq (T-t)^{-\alpha-1/2} \int_0^{T-t} s^{-1/2} ds = 2(T-t)^{-\alpha}$$

and, for $t > T/2$,

$$I_2 \leq \int_{T-t}^t s^{-\alpha-1} ds \leq \frac{1}{\alpha}(T-t)^{-\alpha}.$$

The expected estimate follows from the above inequalities and (4.16). $\square$

Let us focus now on the case where $\Omega$ is a unbounded domain by considering for instance the problem

$$\begin{gathered} u_t - u_{xx} = 0 \quad \text{in } (0,\infty) \times (0,T), \\ u_x(0,t) = -\Big(\int_0^\infty u(x,t)dx\Big)^p \quad \text{on } (0,T), \\ u(\cdot, 0) = u_0(\cdot) \quad \text{in } (0,\infty). \end{gathered} \tag{5.2}$$

Then using methods similar to those of sections 3 and 4, one can prove the following theorem.

**Theorem 5.4.** *Let us assume the following:*

(i) $u_0 \in C([0,\infty)) \cap L^1(0,\infty)$, $u_0 \geq 0$, $u_0 \not\equiv 0$.
(ii) $u_{0_x} \in L^1(0,\infty) \cap L^\infty(0,\infty)$.
(iii) $p \in (1,\infty)$.

*Then it holds that*

(i) *The solution $u$ to (5.2) blows up in finite time in $L^1$ norm.*

(ii) *The blow-up set of* $u$ *is* $\{0\}$.
(iii) *There exist positive constants* $c$ *and* $C$ *such that for* $t \in [0, T)$,

$$\frac{c}{(T-t)^{\frac{p+1}{2(p-1)}}} \le u(0,t) \le \frac{C}{(T-t)^{\frac{p+1}{2(p-1)}}}.$$

**Remark 5.5.** If $\Omega = \mathbb{R}^N_+ = \{(x_1, \dots, x_N) \mid x_1 > 0\}$ and $N \ge 2$ then the solution (if it exists) to the problem

$$\begin{gathered} u_t - \Delta u = 0 \quad \text{in } \Omega \times (0,T), \\ \partial_n u = -u_{x_1} = \Big(\int_\Omega u(x,t)dx\Big)^p \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot, 0) = u_0(\cdot) \quad \text{in } \Omega, \end{gathered}$$

is not integrable in $\mathbb{R}^N_+$ if $u_0 \ge 0$, $u_0 \not\equiv 0$. Indeed the solution $v$ to

$$\begin{gathered} v_t - \Delta v = 0 \quad \text{in } \Omega \times (0,T), \\ \partial_n v = -v_{x_1} = \Big(\int_\Omega u(x,t)dx\Big)^p \quad \text{on } \partial\Omega \times (0,T), \\ v(\cdot, 0) = 0 \quad \text{in } \Omega, \end{gathered}$$

satisfies $v(x_1, \dots, x_N) = v(x_1)$ and $u \ge v$. Thus $\int_\Omega u \ge \int_\Omega v = \infty$.

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