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TRAJECTORIES CONNECTING TWO SUBMANIFOLDS ON A NON-COMPLETE LORENTZIAN MANIFOLD

ROSSELLA BARTOLO, ANNA GERMINARIO, & MIGUEL SÁNCHEZ

Abstract. This article presents existence and multiplicity results for orthogonal trajectories joining two submanifolds Σ_1 and Σ_2 of a static space-time manifold M under the action of gravitational and electromagnetic vector potential. The main technical difficulties are because M may not be complete and Σ_1, Σ_2 may not be compact. Hence, a suitable convexity assumption and hypotheses at infinity are needed. These assumptions are widely discussed in terms of the electric and magnetic vector fields naturally associated. Then, these vector fields become relevant from both their physical interpretation and the mathematical gauge invariance of the equation of the trajectories.

1. Introduction

The pair (S, g) is called *Lorentzian manifold* if S is a connected finite dimensional smooth manifold with $\dim S \geq 2$ and g is a *Lorentzian metric* on S, that is g is a smooth, symmetric, two covariant tensor field such that, for any $z \in S$, the bilinear form $g(z)[\cdot, \cdot]$ induced on T_zS is non-degenerate and of index one. A vector $\zeta \in T_zS$ is said timelike (respectively lightlike; spacelike) if $g(z)[\zeta, \zeta] < 0$ (respectively $g(z)[\zeta, \zeta] = 0$, $\zeta \neq 0$; $g(z)[\zeta, \zeta] > 0$ or $\zeta = 0$). The points of S are called *events*. A Lorentzian manifold (S, g) is called *(standard) static* if S is a product manifold

$$S = M \times \mathbb{R},$$

where M is a C^3 connected manifold and g can be written as

$$\langle \zeta, \zeta' \rangle_L = \langle \xi, \xi' \rangle - \beta(x)\tau\tau' \tag{1.1}$$

for any $z = (x, t) \in S$, $\zeta = (\xi, \tau)$, $\zeta' = (\xi', \tau') \in T_zS = T_xM \times \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ and β are respectively a Riemannian metric and a smooth scalar field on M. The smooth function $\mathcal{T}(x, t) = t$ is a time-function, that is the Lorentzian gradient $\nabla^L \mathcal{T}$ is a timelike vector field, where

$$\nabla^L \mathcal{T}(x, t) = \big(\mathbf{0}, -\frac{1}{\beta(x)}\big).$$

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The vector field $\nabla^L \mathcal{T}$ yields a time-orientation on S: a vector $\zeta \in T_zS$, $z \in S$, is said *future-pointing* (respectively *past-pointing*) if $\langle \nabla^L \mathcal{T}(z), \zeta \rangle_L < 0$ (respectively $\langle \nabla^L \mathcal{T}(z), \zeta \rangle_L > 0$).

We refer the reader to [25, 26, 22] for the background material assumed in this paper. Let us consider a smooth stationary vector field A on S, that is

$$A(z) = A(x,t) = A(x) = (A_1(x), A_2(x)) \quad \forall z = (x,t) \in S$$

where $A_1(x)$ can be regarded as a vector field on M and $A_2(x)$ as a function on M.

In previous papers the existence and the multiplicity of trajectories (under the action of A) joining two events in S have been studied. Namely, fixed two events $z, w \in S$, the trajectories joining them satisfy the Euler-Lagrange equation associated to the functional introduced in [9]

$$F(\gamma) = \frac{1}{2}\int_0^1 \langle \dot\gamma, \dot\gamma \rangle_L ds + \int_0^1 \langle A\ (\gamma), \dot\gamma \rangle_L ds \tag{1.2}$$

on

$$\Omega(z, w; S) = \left\{ \gamma \in H^1([0,1], S) : \gamma(0) = z, \gamma(1) = w \right\}$$

(see Section 2 for details), that is

$$D_s \dot\gamma = ((A'(\gamma))^* - A'(\gamma))\,[\dot\gamma] \tag{1.3}$$

where $D_s\dot\gamma$ is the covariant derivative of $\dot\gamma$ along γ, A' denotes the covariant derivative of A (that is $A'(\gamma)[\dot\gamma] = \nabla_{\dot\gamma} A(\gamma)$) and $(A'(z))^*$ denotes for any $z \in S$ the adjoint operator of $A'(z)$ on T_zS with respect to $\langle \cdot, \cdot \rangle_L$.

Recall that equation (1.3) is *gauge invariant*, that is, it remains equal if one adds the gradient of any function to A. In fact, the right hand side of (1.3) is the skew symmetric part of A' or *rotational* of A, and admits a natural decomposition in the "electric" and "magnetic" vector fields (see Section 4). Nevertheless, it is natural to assume that these vector fields are independent of t, and the simpler way to ensure this is to "choose a gauge" such that A is independent of t. At any case, one must bear in mind that A can always be replaced by $A + (\nabla V, c)$, where V is any function on M, ∇ denotes its gradient with respect to $\langle \cdot, \cdot \rangle$ and $c \in \mathbb{R}$.

Remark that equation (1.3) has a prime integral, in fact:

$$\frac{d}{ds}\langle \dot\gamma, \dot\gamma \rangle_L = 2\langle D_s\dot\gamma, \dot\gamma \rangle_L = \langle (A'(\gamma))^*[\dot\gamma] - A'(\gamma)[\dot\gamma], \dot\gamma \rangle_L = 0,$$

hence if $\gamma : [0,1] \to S$ is a trajectory, there exists a constant of the motion $E_\gamma \in \mathbb{R}$ such that

$$\langle \dot\gamma, \dot\gamma \rangle_L = E_\gamma \quad \text{on} \quad [0,1]. \tag{1.4}$$

Therefore a trajectory γ is said to be *timelike, lightlike* or *spacelike* according to the causal character of $\dot\gamma$.

Trajectories joining two given events have been studied in [2], [16] on complete *stationary* Lorentzian manifolds, in [3], [17] on open subsets of stationary Lorentzian manifolds and in [1] in a different setting. It is clear that this problem generalizes the geodesic connectedness one (see e.g. [10, 19]).

We point out that these results have a physical interpretation. Indeed, the Lorentz world-force law which determines the motion of relativistic particles γ submitted to an electromagnetic field is the Euler-Lagrange equation related to the action functional

$$\mathcal{S}(\gamma) = -m_0 c \frac{1}{2}\int_{s_0}^{s_1} \sqrt{-\langle \dot\gamma, \dot\gamma \rangle_L} ds + q\int_{s_0}^{s_1} \langle A\ (\gamma), \dot\gamma \rangle_L ds$$

where m_0 is the rest mass of the particle, q is its charge, c is the speed of light (see [21]). In [9] it is proved that for timelike trajectories the search of critical points of $\mathcal{S}$ is equivalent to that of the critical points of F. In particular, when $E_\gamma < 0$, this constant of the motion turns to be, up to a dimensional factor, the inertial mass (necessarily equal to the gravitational mass), which is determined by the initial conditions (see [9]).

Here we shall look for orthogonal trajectories under the action of a gravitational and electromagnetic field joining two given submanifolds of a static Lorentzian manifold S.

Definition 1.1. Let Σ_1, Σ_2 be two submanifolds of S. A curve $\gamma : [0,1] \to S$ is called *orthogonal trajectory (under the action of A) joining* Σ_1 *to* Σ_2 if

- (i) γ satisfies (1.3)
- (ii) $\gamma(0) \in \Sigma_1, \gamma(1) \in \Sigma_2$ and $\dot\gamma(0) \in T_{\gamma(0)}\Sigma_1^\perp, \dot\gamma(1) \in T_{\gamma(1)}\Sigma_2^\perp$.

This problem has been studied in the case when $A \equiv 0$ in [23, 14] on static Lorentzian manifolds and on orthogonal splitting Lorentzian manifolds (see also [13]).

Let P and Q be two submanifolds of M and let us set

$$\Sigma_1 = P \times \{0\} \quad \Sigma_2 = Q \times \{T\} \text{ for some } T \in \mathbb{R}. \tag{1.5}$$

Of course, we could consider $\Sigma_1 = P \times \{t_0\}, t_0 \in \mathbb{R}$, however, as the metric is static, there is not loss of generality if we assume $t_0 = 0$.

Here we shall present existence and multiplicity results for timelike orthogonal trajectories joining Σ_1 to Σ_2. We shall use variational methods since it can be easily proved (see Proposition 2.1) that if A is orthogonal to Σ_1 and Σ_2, that is

$$\langle A(z), \zeta\rangle_L = 0 \quad \forall z \in \Sigma_i \quad \forall \zeta \in T_z\Sigma_i \quad i = 1, 2 \tag{1.6}$$

then the orthogonal trajectories joining Σ_1 to Σ_2 are the critical points of F (see (1.2)) on a suitable Hilbert manifold (see Section 2).

We allow both P and Q to be non compact and M to be not complete. Then three problems arise:

- (a) Due to the indefiniteness of the metric (see (1.1)) F is strongly indefinite
- (b) Since P and Q may not be compact, Palais-Smale sequences (see Section 2) could exist which are not bounded
- (c) Due to the possible lack of completeness of M, bounded Palais-Smale sequences may not converge.

In the following L will denote a domain (i.e. an open connected subset) of a static Lorentzian manifold $(S, \langle\cdot,\cdot\rangle_L)$, ∂L its topological boundary and $\overline{L} = L \cup \partial L$.

Let us assume that ∂L is differentiable. Then there exists a differentiable function $\Phi : \overline{L} \to \mathbb{R}$ such that

$$\begin{aligned} \Phi^{-1}(0) &= \partial L \\ \Phi > 0 &\quad \text{on } L \\ \nabla^L\Phi(w) \neq 0 &\quad \forall w \in \partial L \end{aligned} \tag{1.7}$$

where $\nabla^L\Phi$ denotes the Lorentzian gradient of Φ.

We shall use the following definition.

Definition 1.2. A manifold $(L, \langle\cdot,\cdot\rangle_L)$, with $L = D \times \mathbb{R}$, is said to be a static Lorentzian manifold with differentiable boundary $\partial L = \partial D \times \mathbb{R}$ if a static Lorentzian manifold (S, g), with $S = M \times \mathbb{R}$, exists such that D is a domain of M, g restricted to L is $\langle \cdot, \cdot \rangle_L$ and $\overline{D} = D \cup \partial D$ is a complete Riemannian manifold with differentiable boundary.

We remark that if L is a static Lorentzian manifold with differentiable boundary, as ∂D is differentiable, there exists a smooth function $\phi : \overline{D} \to \mathbb{R}$ such that

$$\begin{gathered} \phi^{-1}(0) = \partial D \\ \phi > 0 \quad \text{on } D \\ \nabla\phi(q) \neq 0 \quad \forall q \in \partial D. \end{gathered} \tag{1.8}$$

Moreover Φ in (1.7) can be chosen such that, for any $z = (x, t) \in S$:

$$\Phi(z) = \Phi(x, t) = \phi(x). \tag{1.9}$$

Then

$$\nabla^L \Phi(z) = (\nabla\phi(x), 0). \tag{1.10}$$

Since the metric is stationary, we can overcome the problem in (a) by a slight variant (Proposition 2.3) of the variational principle in [2] (see also [10]) which reduces the study of the orthogonal trajectories joining Σ_1 to Σ_2 to the search of the critical points of a suitable functional J depending only on the "spatial" component.

As in previous papers on this topic, we shall assume that there exist $\eta, b \in \mathbb{R}$ such that

$$0 < \eta \le \beta(x) \le b \quad \forall x \in D \tag{1.11}$$

and that there exist $a_1, a_2 \in \mathbb{R}$ such that

$$\sup_{x \in D} |A_1(x)| = a_1, \quad \sup_{x \in D} |A_2(x)| = a_2. \tag{1.12}$$

Under these two assumptions, J is bounded from below (Remark 2.4). Note that a condition such as (1.12) is not gauge invariant, but combined with other conditions as (1.14) and (1.15) below, it admits more intrinsic interpretations (see Section 4).

Problem (b) above arises on non compact manifolds also in the study of periodic solutions of (1.3) which have been studied in [7, 8] (see [11] for the case $A \equiv 0$). Moreover, this problem appears first in the Riemannian case to ensure the existence of closed geodesics; a detailed discussion of hypothesis at infinity in this case is carried out in [6]. We choose here the simplest hypothesis, concerning the existence of a certain function U. More precisely, we shall assume that

> for some $x_0 \in D$, there exist $U \in C^2(D, \mathbb{R})$ and two positive real constants r, σ such that for any $x \in D$ with $d(x, x_0) \ge r$,
>
> $$H_U(x)[\xi, \xi] \ge \sigma\langle \xi, \xi \rangle \quad \forall \xi \in T_x D, \tag{1.13}$$
>
> where $H_U(x)[\xi, \xi]$ denotes the Riemannian Hessian of U at x in the direction of ξ.

Nevertheless, we need to ensure now the compatibility between the role of U (at infinity) and the other elements of our problem, as well as the compatibility between β, A and the submanifolds P, Q. This compatibility holds under the following technical assumptions: for a suitable $\nu > 0$, which will be defined in (3.8),

$$\begin{aligned} \lim_{d(x,x_0)\to\infty,\ d(x,P)<\nu} \beta(x) &= b, \\ \lim_{d(x,x_0)\to\infty,\ d(x,P)<\nu} |A_1(x)| &= 0 \\ \lim_{d(x,x_0)\to\infty,\ d(x,P)<\nu} A_2(x) &= a_2 \end{aligned} \tag{1.14}$$

$$\begin{aligned} \lim_{d(x,x_0)\to\infty,\ d(x,P)<\nu} |A_1'(x)|_*|\nabla U(x)| &= 0 \\ \lim_{d(x,x_0)\to\infty,\ d(x,P)<\nu} |\nabla A_2(x)||\nabla U(x)| &= 0, \end{aligned} \tag{1.15}$$

where $|\cdot|_*$ denotes the norm for endomorphisms on T_xD induced by the Riemannian metric on D at any $x \in D$, and

$$\lim_{d(x,x_0)\to\infty,\ d(x,P)<\nu} |\nabla\beta(x)||\nabla U(x)| = 0. \tag{1.16}$$

Moreover, we need to ensure the compatibility between the role of U and the boundary conditions of orthogonal trajectories. This will hold if, for trajectories $\gamma(s) = (x(s), t(s))$ as in Definition 1.1 with "$x(0), x(1)$ going to infinity",

$$\langle \nabla U(x(0)), \dot{x}(0)\rangle \geq \langle \nabla U(x(1)), \dot{x}(1)\rangle. \tag{1.17}$$

A simple condition which ensures (1.17), is:

$$\begin{aligned} \nabla U(x) \in T_xP \quad \forall x \in P \text{ with } d(x,x_0) \geq r\,, \\ \nabla U(x) \in T_xQ \quad \forall x \in Q \text{ with } d(x,x_0) \geq r\,. \end{aligned} \tag{1.18}$$

But this is not the unique possibility: If either P or Q are compact then we can assume that this condition is automatically satisfied (as in [4]); we explore another possibility in Remark 1.4.

We remark that only for the sake of simplicity we deal with static (instead of stationary) manifolds, and we follow [11] (in our assumption (1.13) about the existence of function U) instead of using a more intrinsic approach introduced in [12] in the study of periodic geodesics on Riemannian manifolds.

To bypass problem (c) above, we shall deal with static Lorentzian manifolds whose (differentiable) boundaries satisfy suitable convexity assumptions. We recall that ∂L is *convex* if and only if

$$H^L_\Phi(z)[\zeta,\zeta] \leq 0 \quad \forall z \in \partial L, \zeta \in T_z\partial L \tag{1.19}$$

where Φ is as in (1.7) and $H^L_\Phi(z)[\zeta,\zeta]$ denotes the Lorentzian Hessian of Φ at z in the direction of ζ, or equivalently (see [5]) if for any $z, w \in L$ the range of any geodesic $\gamma : [0,1] \to \overline{L}$ such that $\gamma(0) = z, \gamma(1) = w$ satisfies

$$\gamma([0,1]) \subset L. \tag{1.20}$$

Moreover ∂L is *time-convex* (respectively *light-convex, space-convex*) if and only if (1.19) holds on timelike (respectively lightlike, spacelike) vectors or equivalently (see [5]) (1.20) holds for any timelike (respectively lightlike, spacelike) geodesic.

We shall look for future-pointing orthogonal trajectories joining Σ_1 to Σ_2, thus we assume $T > 0$ in (1.5). Our main result which will be proved in Section 3 is the following theorem.

Theorem 1.3. *Let $L = D \times \mathbb{R}$ be a static Lorentzian manifold with differentiable boundary $\partial D \times \mathbb{R}$ and assume that* (1.6), (1.11), (1.12), (1.13), (1.18), (1.14), (1.15), (1.16) *hold and:*

(i) *∂L is time-convex;*
(ii) *∂D is compact;*
(iii) *Σ_1 and Σ_2 are submanifolds of L as in (1.5) with P, Q closed submanifolds of D;*
(iv) *out of a ball the distance between P and Q is greater than zero, that is there exists $\sigma_1 > 0$ such that $d(P', Q') \geq \sigma_1$ where $P' = P \setminus B_r(x_0)$, $Q' = Q \setminus B_r(x_0)$ (where x_0, r are as in (1.13) and $B_r(x_0) = \{x \in D : d(x, x_0) < r\}$);*
(v) *for any $z \in \partial L$, for any $\zeta \in \partial L$ timelike and future-pointing*
$$\langle((A'(z))^* - A'(z))\,[\zeta], \nabla^L \Phi(z)\rangle_L \leq 0 \tag{1.21}$$
where Φ is as in (1.7).

Then there exists $\overline{T} > 0$ such that for any $T \in \mathbb{R}$ with $T > \overline{T}$ there exists an orthogonal timelike future-pointing trajectory joining Σ_1 to Σ_2.

Remark 1.4. Let us assume that P and Q denote the closures of two open domains with smooth boundaries of M. Let N_1 and N_2 denote respectively the inner normals to ∂P and ∂Q. Any orthogonal trajectory $\gamma(s) = (x(s), t(s))$ with $x(0)$ (resp. $x(1)$) in the interior of P (resp. Q) must have, according to Definition 1.1, $\dot{x}(0) = 0$ (resp. $\dot{x}(1) = 0$); thus, (1.17) will be satisfied. If $x(0) \in \partial P, x(1) \in \partial Q$ and the trajectory does not come into the interior of P and Q then necessarily $\dot{x}(0)$ (resp. $\dot{x}(1)$) is parallel to N_1 (resp. N_2) and points out in the opposite (resp. same) direction. Thus, if
$$\begin{aligned} &\langle \nabla U(x), N_1(x)\rangle \leq 0 \quad \forall x \in \partial P \text{ with } d(x, x_0) \geq r \\ &\langle \nabla U(x), N_2(x)\rangle \geq 0 \quad \forall x \in \partial Q \text{ with } d(x, x_0) \geq r \end{aligned} \tag{1.22}$$
then, essentially, (1.17) will hold.

We refer the reader to Section 4 for further discussions on the hypothesis in relation to references [17], [8], [28]. The following theorem concerns the multiplicity of orthogonal trajectories and will be proved in Section 3.

Theorem 1.5. *Let the assumptions of Theorem 1.3 hold. If D is not contractible in itself and P, Q are both contractible in D, then denoted by $N(T, \Sigma_1, \Sigma_2)$ the number of the timelike future-pointing orthogonal trajectories joining Σ_1 to Σ_2 it results*
$$\lim_{T \to \infty} N(T, \Sigma_1, \Sigma_2) = \infty.$$

We point out that our results hold also for past-pointing timelike trajectories if (1.21) holds for past-pointing timelike vectors tangent to ∂L.

Remark 1.6. Essentially, if P and Q reduce respectively to $\{p\}$ and $\{q\}$, then we reobtain the results in [3] for timelike trajectories joining two fixed events in L, and if either P or Q are compact, the results in [4] are reobtained (in these cases, assumptions at infinity are not needed). See [27] for analogous results on Riemannian manifolds.

2. Functional setting

Let S be a static Lorentzian manifold, with $S = M \times \mathbb{R}$ and let Σ_1, Σ_2 be two submanifolds of S as in (1.5). Hereafter we shall assume that M is a submanifold of $\mathbb{R}^N$ for N sufficiently large (see [24]), thus

$$H^1([0,1],M) = \{x \in H^1([0,1],\mathbb{R}^N) : x([0,1]) \subset M\}$$

where

$$\begin{aligned} H^1([0,1],\mathbb{R}^N) &\equiv H^{1,2}([0,1],\mathbb{R}^N) \\ &= \{y \in L^2([0,1],\mathbb{R}^N) : y \text{ is absolutely cont., } \dot{y} \in L^2([0,1],\mathbb{R}^N)\}. \end{aligned}$$

We shall denote by $\|\cdot\|$ the usual norm on $H^1([0,1],\mathbb{R}^N)$ and by $\|\cdot\|_2$ the usual norm on $L^2([0,1],\mathbb{R}^N)$. Let us introduce the manifold

$$\Gamma(\Sigma_1,\Sigma_2;S) = \{z \in H^1([0,1],S) : z(0) \in \Sigma_1, z(1) \in \Sigma_2\}.$$

It is well known that for any $z \in \Gamma(\Sigma_1,\Sigma_2;S)$

$$T_z\Gamma(\Sigma_1,\Sigma_2;S) = \{\zeta \in T_zH^1([0,1],S) : \zeta(0) \in T_{z(0)}\Sigma_1, \zeta(1) \in T_{z(1)}\Sigma_2\}.$$

By using standard arguments [20, 9] we can prove the following statement.

Proposition 2.1. *Let $\gamma \in \Gamma(\Sigma_1,\Sigma_2;S)$ and assume that (1.6) holds. Then γ is a critical point of F at (1.2) if and only if it is an orthogonal trajectory joining Σ_1 to Σ_2.*

By this proposition, the orthogonal trajectories joining Σ_1 to Σ_2 are the critical points of F on

$$Z_T := \Gamma(\Sigma_1,\Sigma_2;S) = \Omega(P,Q;M) \times H^1(0,T)$$

where

$$\Omega(P,Q;M) = \{x \in H^1([0,1],M) : x(0) \in P, x(1) \in Q\}$$

is a smooth submanifold of $H^1([0,1],M)$ (see [20]) and

$$H^1(0,T) = \{t \in H^1([0,1],\mathbb{R}) : t(0) = 0, t(1) = T\}.$$

For any $z = (x,t) \in Z_T$ it results that

$$T_zZ_T = T_x\Omega(P,Q;M) \times H_0^1([0,1],\mathbb{R})$$

where

$$T_x\Omega(P,Q;M) = \{\xi \in T_xH^1([0,1],M) : \xi(0) \in T_{x(0)}P, \xi(1) \in T_{x(1)}Q\}$$

and

$$H_0^1([0,1],\mathbb{R}) = \{\tau \in H^1([0,1],\mathbb{R}) : \tau(0) = 0 = \tau(1)\}.$$

Remark 2.2. If $\gamma = (x,t)$ is a trajectory joining Σ_1 to Σ_2, (ii) of Definition 1.1 and (1.6) can be respectively written as

$$\begin{gathered} x(0) \in P, t(0) = 0 \quad x(1) \in Q, t(1) = T \\ \dot{x}(0) \in T_{x(0)}P^\perp \quad \dot{x}(1) \in T_{x(1)}Q^\perp \\ \langle A_1(x), \xi\rangle = 0 \quad \forall x \in P \cup Q, \xi \in T_x(P \cup Q). \end{gathered}$$

By Proposition 2.1, orthogonal trajectories joining Σ_1 to Σ_2 are the critical points of $F_T := F$ on Z_T. We have already observed that, as for the geodesic problem on Lorentzian manifolds (see e.g. [10]), the functional F_T is strongly indefinite; nevertheless, as announced in Section 1, the following variational principle can be proved.

Proposition 2.3. *Let $\gamma = (x,t) \in Z_T$. The following statements are equivalent:*

(a) *γ is a critical point of F_T*

(b) (i) *$x \in \Omega(P,Q;M)$ is a critical point of the C^2 functional J_T defined on $\Omega(P,Q;M)$ by*

$$\begin{aligned} J_T(x) = &\frac{1}{2}\int_0^1 \langle \dot{x},\dot{x}\rangle\,ds + \int_0^1 \langle A_1(x),\dot{x}\rangle\,ds \\ &+ \frac{1}{2}\int_0^1 \beta(x)A_2^2(x)\,ds - \frac{1}{2}H^2(x)\int_0^1 \frac{1}{\beta(x)}ds \end{aligned} \tag{2.1}$$

where

$$H(x) = \frac{T + \int_0^1 A_2(x)\,ds}{\int_0^1 \frac{1}{\beta(x)}\,ds} \tag{2.2}$$

(ii) *$t \in H^1(0,T)$ is the solution of the Cauchy problem*

$$\begin{gathered} \dot{t} = \frac{H(x)}{\beta(x)} - A_2(x) \\ t(0) = 0. \end{gathered} \tag{2.3}$$

Moreover, if (a) or (b) is true, then $F_T(\gamma) = J_T(x)$.

Remark 2.4. By (1.11), (1.12) and the Hölder inequality for any $x \in \Omega(P,Q;M)$ we get

$$J_T(x) \geq \frac{1}{2}\|\dot{x}\|_2^2 - a_1\|\dot{x}\|_2 - b\Big(\frac{T^2}{2} + \frac{a_2^2}{2} + Ta_2\Big) \tag{2.4}$$

hence J_T is bounded from below.

In the remaining of this section we shall denote by X a C^2 Hilbert manifold endowed with a Riemannian metric. Let us recall some definitions and results to be used in the next section.

A function f in $C^1(X,\mathbb{R})$ satisfies the *Palais-Smale condition* if every sequence $\{y_m\}$ such that

$$\{f(y_m)\} \text{ is bounded} \tag{2.5}$$

and

$$\lim_{m\to\infty} \|f'(y_m)\|_* = 0 \tag{2.6}$$

contains a converging subsequence (where $\|\cdot\|_*$ is the norm induced on the cotangent bundle by the Riemannian metric on X). A sequence satisfying (2.5) and (2.6) is said a *Palais-Smale* sequence.

Let A be a subspace of X. The *category* of A in X, denoted by $\operatorname{cat}_X A$, is the minimum number of closed and contractible subsets of X covering A (possibly ∞). We shall write $\operatorname{cat} X = \operatorname{cat}_X X$.

We shall obtain multiplicity results thanks to the following theorem [15, 18].

Theorem 2.5. *Let D be a noncontractible in itself C^3 Riemannian manifold. Let P and Q be two submanifolds of D both contractible in D. Then there exists a sequence $\{K_m\}$ of compact subsets of $\Omega(P,Q;D)$ such that*

$$\lim_{m\to\infty} \operatorname{cat}_{\,\Omega(P,Q;D)} K_m = \infty.$$

3. Proof of Theorems 1.3 and 1.5

Let us consider

$$\Omega(P,Q;D)=\big\{x\in H^1([0,1],D): x(0)\in P, x(1)\in Q\big\}$$

which is an open submanifold of $\Omega(P,Q;M)$. Following [11], we penalize the functional F_T in a suitable way. For any $\epsilon\in]0,1]$, we consider a non-negative increasing function $\psi_\epsilon\in C^2(\mathbb{R},\mathbb{R})$ defined by

$$\psi_\epsilon(s)=\begin{cases}0 & \text{if } s\le 1/\epsilon\\ \sum_{m=3}^{\infty}\frac{1}{m!}\sigma^m\big(s-\frac{1}{\epsilon}\big)^m & \text{if } s>1/\epsilon\end{cases} \tag{3.1}$$

where σ is as in (1.13). Set, for any $\epsilon\in]0,1]$, $\gamma=(x,t)\in Z_T=\Omega(P,Q;D)\times H^1(0,T)$

$$F_{T,\epsilon}(\gamma)=F_T(\gamma)+\int_0^1\psi_\epsilon(U(x))\,ds+\int_0^1\psi_\epsilon\big(\frac{1}{\Phi^2(\gamma)}\big)\,ds$$

and for any $\epsilon\in]0,1]$, $x\in\Omega(P,Q;D)$

$$J_{T,\epsilon}(x)=J_T(x)+\int_0^1\psi_\epsilon(U(x))ds+\int_0^1\psi_\epsilon\big(\frac{1}{\phi^2(x)}\big)\,ds \tag{3.2}$$

where the function U is as in (1.13) and Φ,ϕ are respectively as in (1.7), (1.8). It is clear that the first penalization term takes into account the lack of boundedness of the submanifolds P,Q and the second one the presence of the boundary ∂D.

Remark 3.1. Since the penalization terms do not depend on t, Proposition 2.3 still holds when F_T and J_T are respectively replaced by $F_{T,\epsilon}$, $J_{T,\epsilon}$.

For the proof of the following proposition we refer the reader to [11, 3].

Proposition 3.2. *For any $\epsilon\in]0,1]$ and $c\in\mathbf{R}$, the sublevels*

$$J^c_{T,\epsilon}=\{x\in\Omega(P,Q;D): J_{T,\epsilon}(x)\le c\}$$

are complete metric subspaces of $\Omega(P,Q;D)$ and $J_{T,\epsilon}$ satisfies the Palais-Smale condition.

The following lemma can be found in [11, Lemma 2.2].

Lemma 3.3. *Let U,r,σ,x_0 be as in (1.13). Then there exist c_1, c_2, $c_3>0$ such that for any $x\in D$:*

$$\begin{gathered}\langle\nabla U(x),\nabla U(x)\rangle^{1/2}\ge\sigma d(x,x_0)-c_1\\ U(x)\ge\frac{\sigma}{2}d^2(x,x_0)-c_2d(x,x_0)-c_3.\end{gathered}$$

Remark 3.4. Since $J_{T,\epsilon}(x)\ge J_T(x)$ for any $\epsilon\in]0,1]$, $x\in\Omega(P,Q;D)$, by Remark 2.4 $J_{T,\epsilon}$ is bounded from below. Then, by Proposition 3.2, $J_{T,\epsilon}$ attains its infimum at a point $x_\epsilon\in\Omega(P,Q;D)$. We set

$$K=\min_{x\in\Omega(P,Q;D)}J_{T,1}(x). \tag{3.3}$$

By the form of the penalization, it results $J_{T,\epsilon}(x_\epsilon)\le J_{T,\epsilon}(x_1)\le K$.

The following lemma will be crucial in the proof of Theorem 1.3.

Lemma 3.5. *For any $\epsilon \in]0,1]$ let $x_\epsilon \in \Omega(P,Q;D)$ be a critical point of the functional $J_{T,\epsilon}$ satisfying*

$$-b\big(\frac{T^2}{2} + Ta_2\big) + \delta \leq J_{T,\epsilon}(x_\epsilon) \leq K \tag{3.4}$$

where δ is a suitable real constant independent of ϵ, K is as in (3.3), b is as in (1.11) and a_2 is as in (1.12). Then there exist $\epsilon_0 \in]0,1]$ and $\overline{T} > 0$ such that, for any $\epsilon \in]0,\epsilon_0]$ and for any $T \in \mathbb{R}$ with $T > \overline{T}$, x_ϵ is necessarily a critical point of J_T.

Proof. By the form of the penalization, it suffices to prove the existence of a $\epsilon_1 \in]0,1]$ such that for any $\epsilon \in]0,\epsilon_1]$ it results

$$\sup_{s\in[0,1]} d(x_\epsilon(s), x_0) \leq M_1 \tag{3.5}$$

for a suitable $M_1 > 0$, the existence of $\epsilon_2 \in]0,1]$ such that for any $\epsilon \in]0,\epsilon_2]$ it results

$$\phi(x_\epsilon(s)) \geq \sqrt{\epsilon} \quad \forall s \in [0,1] \tag{3.6}$$

and set $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$.
Step 1: Let us prove (3.5). Assume by contradiction that there exist an infinitesimal and decreasing sequence $\{\epsilon_m\}$ of numbers in $]0,1]$ and a sequence of critical points $\{x_m\}$ of $J_{T,m} \equiv J_{T,\epsilon_m}$ satisfying (3.4) and such that

$$\sup\{d(x_m(s), x_0) | s \in [0,1], m \in \mathbf{N}\} = \infty. \tag{3.7}$$

By *(ii)* of Theorem 1.3, there exists $\mu > 0$ such that for m large

$$\phi(x_m(s)) \geq \mu > 0 \quad \forall s \in [0,1].$$

Therefore, from (3.1), for m large enough we get

$$\psi_{\epsilon_m}\big(\frac{1}{\phi^2(x_m(s))}\big) = 0 \quad \forall s \in [0,1].$$

From (2.4) and (3.4) for any $m \in \mathbb{N}$

$$\|\dot{x}_m\|_2 \leq \nu = a_1 + \sqrt{a_1^2 + 2b\big(\frac{T^2}{2} + \frac{a_2^2}{2} + Ta_2\big) + 2K}. \tag{3.8}$$

From (3.7) and (3.8) it follows

$$\lim_{m\to\infty} \inf_{[0,1]} d(x_m(s), x_0) = \infty. \tag{3.9}$$

If t_m is the solution of (2.3) corresponding to x_m, by Proposition 2.3 and Remark 3.1 it follows that $\gamma_m = (x_m, t_m)$ is a critical point of $F_{T,m} \equiv F_{T,\epsilon_m}$. Therefore, for any $\xi \in C_0^\infty([0,1], \mathbb{R}^N)$,

$$\begin{aligned}
&F'_{T,m}(x_m, t_m)[\xi, 0] \\
&= -\int_0^1 \langle D_s \dot{x}_m, \xi\rangle ds - \frac{1}{2}\int_0^1 \langle \nabla\beta(x_m), \xi\rangle \dot{t}_m^2 ds \\
&\quad + \int_0^1 \langle ((\nabla A_1(x_m))^* - \nabla A_1(x_m))[\dot{x}_m], \xi\rangle ds - \int_0^1 [\langle \nabla\beta(x_m), \xi\rangle A_2(x_m) \\
&\quad + \beta(x_m)\langle \nabla A_2(x_m), \xi\rangle]\dot{t}_m ds + \int_0^1 \psi'_{\epsilon_m}(U(x_m))\langle \nabla U(x_m), \xi\rangle ds = 0.
\end{aligned}$$

Then from (2.3) we get

$$\begin{aligned} D_s\dot x_m = &-\frac{1}{2}H(x_m)\frac{\nabla\beta(x_m)}{\beta(x_m)}\dot t_m - \frac{1}{2}\nabla\beta(x_m)A_2(x_m)\dot t_m \\ &+ \big((\nabla A_1(x_m))^* - \nabla A_1(x_m)\big)[\dot x_m] - \beta(x_m)\nabla A_2(x_m)\dot t_m \\ &+ \psi'_{\epsilon_m}(U(x_m))\nabla U(x_m). \end{aligned} \tag{3.10}$$

Now set, for any $m \in \mathbf{N}$, $s \in [0,1]$, $u_m(s) = U(x_m(s))$. Then, as x_m is a critical point of $J_{T,m}$, by (3.9), (3.10) and (1.13) for m large enough, it results that

$$\begin{aligned} \int_0^1 \ddot u_m\,ds = &\int_0^1 H_U(x_m)[\dot x_m,\dot x_m]ds + \int_0^1 \langle\nabla U(x_m), D_s\dot x_m\rangle ds \\ \ge\ &\sigma\int_0^1 \langle \dot x_m,\dot x_m\rangle ds - \frac{1}{2}H(x_m)\int_0^1 \langle\nabla U(x_m), \frac{\nabla\beta(x_m)}{\beta(x_m)}\rangle \dot t_m ds \\ &- \frac{1}{2}\int_0^1 \langle\nabla U(x_m),\nabla\beta(x_m)\rangle A_2(x_m)\dot t_m\,ds \\ &+ \int_0^1 \langle\nabla U(x_m), \big((\nabla A_1(x_m))^* - \nabla A_1(x_m)\big)[\dot x_m]\rangle ds \\ &- \int_0^1 \langle\nabla U(x_m),\nabla A_2(x_m)\rangle\beta(x_m)\dot t_m ds \\ &+ \int_0^1 \psi'_{\epsilon_m}(U(x_m))\langle\nabla U(x_m),\nabla U(x_m)\rangle ds. \end{aligned} \tag{3.11}$$

Again from (1.11) and (1.12) it follows that $\{H(x_m)\}, \{\dot t_m\}$ are bounded too (see (2.2) and (2.3)) and then, from (1.16), it follows that

$$H(x_m)\int_0^1 \langle\nabla U(x_m), \frac{\nabla\beta(x_m)}{\beta(x_m)}\rangle \dot t_m ds = o(1). \tag{3.12}$$

Indeed it results

$$\begin{aligned} H(x_m)\int_0^1 \langle\nabla U(x_m), \frac{\nabla\beta(x_m)}{\beta(x_m)}\rangle \dot t_m ds &\le K \max_{s\in[0,1]} |\nabla U(x_m(s))|\,|\nabla\beta(x_m(s))| \\ &= K|\nabla U(x_m(\bar s))|\,|\nabla\beta(x_m(\bar s))| \end{aligned}$$

for suitable $K > 0$, $\bar s \in [0,1]$. Since $x_m(0) \in P$,

$$d(x_m(\bar s), P) \le d(x_m(\bar s), x_m(0)) \le \|\dot x_m\|_2,$$

thus (3.12) follows from (3.8), (3.9) and (1.16). By a similar argument we also obtain

$$\int_0^1 \langle\nabla U(x_m),\nabla\beta(x_m)\rangle A_2(x_m)\dot t_m ds = o(1)$$

and, from (1.15) and (3.8)

$$\begin{gathered} \int_0^1 \langle\nabla U(x_m), \big((\nabla A_1(x_m))^* - \nabla A_1(x_m)\big)[\dot x_m]\rangle ds = o(1), \\ \int_0^1 \langle\nabla U(x_m),\nabla A_2(x_m)\rangle\beta(x_m)\dot t_m ds = o(1). \end{gathered}$$

From (1.18) it is $\dot{u}_m(0) = 0 = \dot{u}_m(1)$, hence from (3.11) we have:

$$\begin{aligned} 0 &= \int_0^1 \ddot{u}_m(s)ds \\ &\geq \sigma \int_0^1 \langle \dot{x}_m, \dot{x}_m \rangle ds + \int_0^1 \psi'_{\epsilon_m}(U(x_m))\langle \nabla U(x_m), \nabla U(x_m)\rangle ds + o(1). \end{aligned} \tag{3.13}$$

From (3.4) and (3.2) we get

$$\begin{aligned} &\frac{1}{2}\int_0^1 \langle \dot{x}_m, \dot{x}_m \rangle ds \\ &\geq -b\big(\frac{T^2}{2} + Ta_2\big) + \delta - \int_0^1 \langle A_1(x_m), \dot{x}_m\rangle ds - \frac{1}{2}\int_0^1 \beta(x_m)A_2^2(x_m)\, ds \\ &+ \frac{1}{2}H^2(x_m)\int_0^1 \frac{1}{\beta(x_m)}\, ds - \int_0^1 \psi_{\epsilon_m}(U(x_m))\, ds. \end{aligned} \tag{3.14}$$

Moreover from (3.9) and (1.14) for any positive real number α and for m large enough,

$$a_2 + \alpha > A_2(x_m) > a_2 - \alpha, \tag{3.15}$$

$$b + \alpha > \beta(x_m) > b - \alpha. \tag{3.16}$$

Then from (3.9), (3.14), (3.15), (3.16) and (1.14), it follows that, for m large enough,

$$\begin{aligned} &\frac{1}{2}\int_0^1 \langle \dot{x}_m, \dot{x}_m \rangle ds \\ &\geq \delta - \alpha\big(a_2^2 + \alpha^2 + 2a_2 b + bT + \frac{T}{2} + Ta_2 - T\alpha\big) - \int_0^1 \psi_{\epsilon_m}(U(x_m))ds + o(1). \end{aligned} \tag{3.17}$$

Hence, chosen α small enough such that

$$\frac{\delta}{2} > \alpha\big(a_2^2 + \alpha^2 + 2a_2 b + bT + \frac{T}{2} + Ta_2 - T\alpha\big),$$

(3.13) implies

$$\begin{aligned} 0 = \int_0^1 \ddot{u}_m\, ds &\geq \sigma\,(\delta + o(1)) - 2\sigma \int_0^1 \psi_{\epsilon_m}(U(x_m))ds \\ &\quad + \int_0^1 \psi'_{\epsilon_m}(U(x_m))\langle \nabla U(x_m), \nabla U(x_m)\rangle ds + o(1). \end{aligned} \tag{3.18}$$

By Lemma 3.3 and (3.9), for m large, $\langle \nabla U(x_m), \nabla U(x_m)\rangle \geq 2$. Hence, by (3.18) and the form of the penalization, we obtain

$$\begin{aligned} 0 &= \int_0^1 \ddot{u}_m\, ds \\ &\geq \sigma\delta + 2\int_0^1 \big(\psi'_{\epsilon_m}(U(x_m)) - \sigma\psi_{\epsilon_m}(U(x_m))\big)\, ds + o(1) \\ &\geq \sigma\delta + o(1), \end{aligned}$$

which is a contradiction.

Step 2: In order to prove (3.6) assume by contradiction that there exist an infinitesimal and decreasing sequence $\{\epsilon_m\}$ of numbers in $]0,1]$ and a sequence of critical points $\{x_m\}$ of $J_{T,m} \equiv J_{T,\epsilon_m}$ satisfying (3.4) and such that

$$\phi(x_m(s_m)) < \sqrt{\epsilon_m} \tag{3.19}$$

where for any $m \in \mathbb{N}$ s_m is a minimum point for $h_m(s) = \phi(x_m(s))$ on $[0,1]$. From (3.5) it follows that $\{\|x_m\|_\infty\}$ is bounded. Therefore from (3.8) we get that $\{x_m\}$ is bounded in $\Omega(P,Q;D)$ and, up to a subsequence,

$$x_m \to x \quad \text{uniformly}. \tag{3.20}$$

Remark that, up to a subsequence, there exists $s_0 \in [0,1]$ such that

$$\lim_{m\to\infty} s_m = s_0. \tag{3.21}$$

Since

$$|x_m(s_m) - x(s_0)| \le \|x_m - x\|_\infty + |x(s_m) - x(s_0)|$$

from (3.20), (3.21), and the continuity of x we get that $\{x_m(s_m)\}$ converges to $x(s_0) \in \partial D$. It results

$$\phi(x_m(s_m)) \to \phi(x(s_0))$$

thus from (3.19) $\phi(x(s_0)) = 0$, that is $x(s_0) \in \partial D$ (see (1.8)). Since the set

$$\{x_m(0), x_m(1) : m \in \mathbb{N}\}$$

is relatively compact in D, there exists $\delta_1 > 0$ such that $\phi(x_m(0)) \ge \delta_1, \phi(x_m(1)) \ge \delta_1$ for any $m \in \mathbb{N}$, thus $s_0 \in]0,1[$. In order to obtain a contradiction, we shall exploit the convexity assumption on the boundary. From (3.4), reasoning as in [3, Lemma 4.5] we get the existence of a curve $\gamma = (x,t) \in \Omega(P,Q;\overline{D}) \times H^1(0,T)$ such that, up to a subsequence,

$$\gamma_m \to \gamma \quad \text{in } H^1([0,1], \mathbb{R}^{N+1}). \tag{3.22}$$

Moreover, $\gamma \in H^2([0,1], \mathbb{R}^{N+1})$ and it solves the equation

$$D_s\dot\gamma = ((A'(\gamma))^* - A'(\gamma))\,[\dot\gamma] - \mu(s)\nabla^L\Phi(\gamma) \tag{3.23}$$

where $\mu \in L^2([0,1], \mathbb{R})$ is positive almost everywhere in $[0,1]$ and vanishes if $\gamma(s) \in L$. From (3.23) we easily get that

$$\langle D_s\dot\gamma, \dot\gamma\rangle_L + \mu(s)\langle \nabla^L\Phi(\gamma), \dot\gamma\rangle_L = 0$$

and standard arguments show that $\langle D_s\dot\gamma, \dot\gamma\rangle_L = 0$ a.e. on $[0,1]$ (see e.g. [19, Theorem 5.1]); therefore, there exists $E_\gamma \in \mathbb{R}$ such that $E_\gamma = \langle \dot\gamma, \dot\gamma\rangle_L$. We claim that for T large enough E_γ is negative. By Remark 3.4 and (1.11)

$$c_{T,m} := J_{T,m}(x_m) \le c_1 - \frac{1}{2}\eta T^2 \tag{3.24}$$

for a suitable $c_1 > 0$. From (3.22) we get

$$\frac{1}{2}E_\gamma = \lim_{m\to\infty}\Big[c_{T,m} - \int_0^1 \psi_\epsilon\big(\frac{1}{\Phi^2(\gamma_m)}\big)ds - \int_0^1 \langle A(\gamma_m), \dot\gamma\rangle_L ds\Big]. \tag{3.25}$$

Standard calculations, (1.11) and (1.12) show that

$$\Big|\int_0^1 \langle A(\gamma_m), \dot\gamma_m\rangle_L ds\Big| \le a_1 \int_0^1 |\dot x_m|\, ds + c_2T + c_3$$

for suitable $c_2, c_3 > 0$, hence from (3.24) and (3.25) we get

$$\frac{1}{2}E_\gamma \leq c_1 - \frac{1}{2}\eta T^2 + a_1 \lim_{m\to\infty} \int_0^1 |\dot{x}_m|\, ds + c_2 T + c_3.$$

By the Young inequality

$$a_1\|\dot{x}\|_2 \leq \frac{1}{4}\|\dot{x}\|_2^2 + 4a_1^2\,, \tag{3.26}$$

(2.4), (3.24) and the Hölder inequality, we get

$$\frac{1}{2}E_\gamma \leq c_4 + c_2 T - \frac{1}{2}\eta T^2 + a_1\sqrt{K_1 + K_2 T + K_3 T^2}$$

for suitable $c_4, K_1, K_2, K_3 > 0$. Therefore, for T large enough γ is a timelike curve. From (2.3) it follows that for T large enough γ is also future-pointing.

We have already shown that there exists a s_0 as in (3.21). Then, set $h(s) = \Phi(\gamma(s))$ we get

$$H^L_\Phi(\gamma(s_0))[\dot{\gamma}(s_0), \dot{\gamma}(s_0)] + \langle((A'(\gamma(s_0)))^* - A'(\gamma(s_0)))\,[\dot{\gamma}(s_0)], \nabla^L\Phi(\gamma(s_0))\rangle_L \\ - \mu(s_0)\langle\nabla^L\Phi(\gamma(s_0)), \nabla^L\Phi(\gamma(s_0))\rangle_L \geq 0. \tag{3.27}$$

Thus by (1.21) and (i) of Theorem 1.3 (remark that as $\langle\nabla^L\Phi(\gamma(s_0)), \dot{\gamma}(s_0)\rangle_L = 0$, $\dot{\gamma}(s_0) \in T_{\gamma(s_0)}\partial L$),

$$\mu(s_0)\langle\nabla^L\Phi(\gamma(s_0)), \nabla^L\Phi(\gamma(s_0))\rangle_L \leq 0$$

and this implies $\mu(s_0) = 0$ since from (1.9), (1.10) and (1.8)

$$\langle\nabla^L\Phi(\gamma(s_0)), \nabla^L\Phi(\gamma(s_0))\rangle_L = \langle\nabla\phi(x(s_0)), \nabla\phi(x(s_0))\rangle > 0. \tag{3.28}$$

Moreover, it can be proved that if $\bar{s} \in [0,1]$ is such that $\gamma(\bar{s}) \in L$, there exists a neighborhood $\mathcal{I}$ of $\bar{s}$ such that $\mu(s) = 0$ for every $s \in \mathcal{I}$. Thus from (3.23) γ is a orthogonal (timelike, future-pointing) trajectory joining Σ_1 to Σ_2.

Now it suffices to prove that the range of γ is contained in L. Let $C = \{s \in [0,1] : \gamma(s) \in \partial L\}$. From (3.19) we have shown that there exists $s_0 \in]0,1[$ such that $s_0 \in C$. Clearly C is compact; say $s_M \in]0,1[$ its maximum. Using the Gronwall Lemma we shall prove that there exists $\delta_1 > 0$ such that $[s_M, s_M + \delta_1] \subset C$, getting a contradiction. Indeed, for $\eta_1 > 0$ there exists $\delta_1 > 0$ such that

$$\Phi(\gamma(s)) < \eta_1 \quad \forall s \in [s_M, s_M + \delta_1]$$

and we can consider the projection $\gamma_p = (x_p, t_p) : [s_M, s_M + \delta_1] \to \partial L$ of γ on ∂L obtained by using the flow of the vector field $-\nabla\Phi/|\nabla\Phi|^2$ where

$$t_p(s) = c\int_0^s \frac{1}{\beta(x_p)}\, d\tau,$$

see also [5]. Let us remark that δ_1 can be chosen such that the projected curve γ_p is (future-pointing and) timelike on $[s_M, s_M + \delta_1]$. Indeed, by continuity, it is sufficient to check that $\dot{x}(s_M) = \dot{x}_p(s_M)$. Denote by $\eta(s,x)$ the flow of $-\nabla\Phi/|\nabla\Phi|^2$; then

$$x_p(s) = \eta(h(s), x(s))$$

and

$$\dot{x}_p(s) = \eta_x(h(s), x(s))[\dot{x}(s)] - \frac{\nabla\phi(x_p(s))}{|\nabla\phi(x_p(s))|^2}\dot{h}(s).$$

Since $h(s_M) = 0$ and $\dot{h}(s_M) = 0$, clearly $\dot{x}_p(s_M) = \dot{x}(s_M)$, which implies the required equality.

As the geometric-time convexity is equivalent to the variational one we get

$$H^L_\Phi(\gamma_p(s))[\dot\gamma_p(s),\dot\gamma_p(s)] \le 0 \quad \forall s\in[s_M, s_M+\delta_1].$$

Hence, for any $s\in[s_M,s_M+\delta_1]$ it is

$$\begin{aligned}\ddot h(s) &\le H^L_\Phi(\gamma(s))[\dot\gamma(s),\dot\gamma(s)] - H^L_\Phi(\gamma_p(s))[\dot\gamma_p(s),\dot\gamma_p(s)]\\ &\quad + \langle ((A'(\gamma(s)))^* - A'(\gamma(s)))\,[\dot\gamma(s)], \nabla^L\Phi(\gamma(s))\rangle_L.\end{aligned}$$

Reasoning as in [5, Theorem 4.3] for any $s\in[s_M,s_M+\delta_1]$ it results

$$H^L_\Phi(\gamma(s))[\dot\gamma(s),\dot\gamma(s)] - H^L_\Phi(\gamma_p(s))[\dot\gamma_p(s),\dot\gamma_p(s)] \le M_1 h(s) + M_2\dot h(s) \tag{3.29}$$

for some $M_1, M_2>0$. Moreover from (1.21),

$$\begin{aligned}&\langle \nabla^L\Phi(\gamma(s)), ((A'(\gamma(s)))^* - A'(\gamma(s)))\,[\dot\gamma(s)]\rangle_L\\ &\le \langle \nabla^L\Phi(\gamma(s)), ((A'(\gamma(s)))^* - A'(\gamma(s)))\,[\dot\gamma(s)]\rangle_L\\ &\quad - \langle \nabla^L\Phi(\gamma_p(s)), ((A'(\gamma_p(s)))^* - A'(\gamma_p(s)))\,[\dot\gamma_p(s)]\rangle_L\\ &= \langle \nabla^L\Phi(\gamma(s)), ((A'(\gamma(s)))^* - A'(\gamma(s)))\,[\dot\gamma(s)]\rangle_L\\ &\quad - \langle \nabla^L\Phi(\gamma_p(s)), ((A'(\gamma(s)))^* - A'(\gamma(s)))\,[\dot\gamma(s)]\rangle_L\\ &\quad + \langle \nabla^L\Phi(\gamma_p(s)), ((A'(\gamma(s)))^* - A'(\gamma(s)))\,[\dot\gamma(s)]\rangle_L\\ &\quad - \langle \nabla^L\Phi(\gamma_p(s)), ((A'(\gamma_p(s)))^* - A'(\gamma_p(s)))\,[\dot\gamma_p(s)]\rangle_L\,.\end{aligned}$$

Using arguments similar to those used to prove (3.29), because A and Φ are C^2, there exists $M_3>0$ such that

$$\langle \nabla^L\Phi(\gamma(s)), ((A'(\gamma(s)))^* - A'(\gamma(s)))\,[\dot\gamma(s)]\rangle_L \le M_3|x(s)-x_p(s)| \le M_3 h(s)\,. \tag{3.30}$$

Therefore,

$$\ddot h(s) \le (M_1+M_3)h(s) + M_2\dot h(s) \quad \forall s\in[s_M,s_M+\delta_1].$$

Since $h(s_M)=0$, $\dot h(s_M)=0$ by the Gronwall lemma we obtain $h\equiv 0$ in $[s_M, s_M+\delta_1]$, which is a contradiction. □

Remark 3.6. In the proof of Lemma 3.5 we have proved in particular that to the critical point of J_T corresponds a *timelike future-pointing* orthogonal trajectory.

Lemma 3.7. *Let* (iv) *of Theorem 1.3 hold, and for each $\epsilon\in]0,1]$ let $x_\epsilon\in\Omega(P,Q;D)$ be a critical point of the functional $J_{T,\epsilon}$ satisfying*

$$J_{T,\epsilon}(x_\epsilon)\le K \tag{3.31}$$

where K is as in (3.3). Then there exist $\epsilon_0\in]0,1]$ and $\overline{T}>0$ such that, for any $\epsilon\in]0,\epsilon_0]$ and for any $T\in\mathbb{R}$ with $T>\overline{T}$, x_ϵ is necessarily a critical point of J_T.

Proof. The only difference with the proof of Lemma 3.5 is the following. If by contradiction there exists a subsequence $\{x_m\}$ such that

$$\lim_{m\to\infty} J_T(x_m) = -b\Big(\frac{T^2}{2}+Ta_2\Big) \tag{3.32}$$

(see (3.4)) and (3.9) holds, then from (3.32) and (1.14) it follows that

$$\lim_{m\to\infty}\int_0^1 \langle \dot x_m,\dot x_m\rangle ds = 0$$

and this contradicts assumption (iv). Therefore, by the form of the penalization we get the existence of $\delta > 0$ such that (3.4) holds and we can repeat the proof of Lemma 3.5. □

Proof of Theorem 1.3. The existence of a critical point of J_T for T large follows from Remark 3.4 and Lemma 3.7. Finally, by Propositions 2.3, 2.1 and Remark 3.6 the proof is complete. □

Proof of Theorem 1.5. For any $c \in \mathbb{R}$, set

$$J_{T,c} = \{x \in \Omega(P, Q; D) : J_T(x) \geq c\},$$
$$J_{T,\epsilon,c} = \{x \in \Omega(P, Q; D) : J_{T,\epsilon}(x) \geq c\}.$$

It can be proved that even if J does not satisfy the Palais-Smale condition,

$$\operatorname{cat}_{\Omega(P,Q;D)} J^c < \infty \tag{3.33}$$

(see [11]). By Theorem 2.5 for any $m \in \mathbb{N}$ there exists $m = m(c) \in \mathbb{N}$ such that, for any $A \in \Gamma_m = \{B \subset \Omega(P, Q; D) : \operatorname{cat}_{\Omega(P,Q;D)} B \geq m\}$,

$$A \cap J_{T,c} \neq \emptyset$$

and since $J_{T,c} \subset J_{T,\epsilon,c}$, for any $A \in \Gamma_m$ it also results

$$A \cap J_{T,\epsilon,c} \neq \emptyset \quad \forall \epsilon \in]0, 1].$$

Proposition 3.2 and classical arguments in critical point theory imply that for any $i \in \{1, \dots, m\}$ the values

$$c_{T,\epsilon,i} = \inf_{A \in \Gamma_i} \sup_{x \in A} J_{T,\epsilon}(x)$$

are well defined and are critical values of $J_{T,\epsilon}$. Moreover

$$c \leq c_{T,\epsilon,1} \leq \dots \leq c_{T,\epsilon,m} \quad \forall \epsilon \in]0, 1].$$

Now let K be a compact subset of Γ_m; then for any $i \in \{1, \dots, m\}$ we have

$$c \leq c_{T,\epsilon,1} \leq \dots \leq c_{T,\epsilon,m} \leq \max_{x \in K} J_{T,\epsilon}(x) \quad \forall \epsilon \in]0, 1].$$

Hence there exist at least m critical points of $J_{T,\epsilon}$. As K is compact, for T large enough, we can reason as in Lemma 3.5 (see also [4, Theorem 1.4]) obtaining at least m distinct timelike future-pointing orthogonal trajectories joining Σ_1 to Σ_2. □

4. Discussion: A time-convexity, electric and magnetic vector fields

In this section, firstly we shall discuss the notion of A-timeconvexity and shall show that the proof of Theorem 1.3 holds under other definitions of convexity. Secondly we introduce the electric and magnetic fields E, B associated to A, and discuss the meaning of the boundary condition (1.21) in terms of these fields. Finally, some comments about the translation of the other hypotheses on A to hypotheses on E, B are given.

The following definition of convexity has been introduced in [17].

> ∂L is A-timeconvex in the future if for any future-pointing timelike solution $\gamma : [0, 1] \to L \cup \partial L$ of (1.3) such that $\gamma(0), \gamma(1) \in L$ it results
> $$\gamma([0, 1]) \subset L. \tag{4.1}$$

Then, if $\gamma : [0,1] \to L \cup \partial L$ is a timelike future-pointing orthogonal trajectory joining Σ_1 to Σ_2, (4.1) holds. Moreover, it can be proved that if ∂L is A-timeconvex, then for all $z \in \partial L$ and for any future-pointing timelike $\zeta \in T_z\partial L$

$$H^L_\Phi(z)[\zeta,\zeta] + \langle ((A'(z))^* - A'(z))\,[\zeta], \nabla^L \Phi(z)\rangle_L \leq 0. \tag{4.2}$$

Conversely, if this inequality holds strictly then ∂L is A-timeconvex in the future. Recall that time-convexity of ∂L and formula (1.21) for any future-pointing timelike ζ (that is, hypotheses (i), (v) of Theorem 1.3) also imply (4.2). Moreover, if, additionally, one of these two conditions is strict (that is, either (1.19) or (1.21) holds strictly for any future-pointing timelike ζ) then ∂L is A-timeconvex in the future.

A-timeconvexity in the future can replace assumptions (i), (v) in Theorem 1.3. In fact, we have proved in *Step 2* of Lemma 3.5 the existence of a timelike curve $\gamma = (x,t) \in \Omega(P,Q;\overline{D}) \times H^1(0,T)$ such that, up to a subsequence, (3.22) holds. From (3.27), using (4.2), it follows that γ is a orthogonal trajectory joining Σ_1 to Σ_2. We have also proved that from (3.19) it follows that γ touches the boundary of L, and this is an absurd for the A-timeconvexity of the boundary.

Inequality (1.21), as well as other hypotheses in Theorem 1.3, can be interpreted in terms of the electric and magnetic parts of the electromagnetic field for the natural observers. In fact, assume that the spacetime is 4-dimensional. The electric and magnetic fields associated to A for the observers in ∂_t are defined as follows (see for example [26, p. 75]). The electric field is

$$E = (A')^*(\bar\partial_t) - (A')(\bar\partial_t) \tag{4.3}$$

where $\bar\partial_t = \partial_t/(\beta)^{1/2}$. Explicitly, from (4.3) and the expression of ∇ in a static manifold (see for example [25, Proposition 7.35])

$$E = -(\beta)^{1/2}\nabla A_2 - \nabla\beta. \tag{4.4}$$

For the the magnetic field B, firstly one fixes an orientation on S (it is enough on M, or just in the tangent space to ∂D), and constructs the volume element Ω associated to the metric and the orientation at each point. Then, B is the unique vector tangent to M satisfying

$$\Omega(X,Y,B,\bar\partial_t) = \langle ((A')^* - A')(X), Y\rangle_L, \tag{4.5}$$

for all X, Y tangent to M. Thus, essentially, $B = \operatorname{curl} A_1$ (B is the rotational of the vector field A_1 in the corresponding slice $t =$ constant, up to a sign which depends of the chosen orientations).

Note that the electric and magnetic vector fields are physically measurable quantities, and they remain invariant under the allowed gauge transformation $A \to A + (\nabla V, c)$.

Proposition 4.1. *Let $N = \nabla\phi/|\nabla\phi|$ be the unitary inner normal vector to ∂D at any point x. Inequality (1.21) holds for any timelike future-pointing vector ζ if and only if the electric and magnetic vector fields E, B associated to A satisfy:*

- (i) $\langle E, N\rangle \leq 0$ *(E does not point out inward the boundary)*
- (ii) *The norm of the projection of the magnetic field B on the tangent of ∂D is smaller or equal to $|\langle E, N\rangle|$.*

Proof. Recall first that the necessity of (i) is obvious applying inequality (1.21) to $\bar{\partial}_t$, and using $\nabla^L \Phi \equiv \nabla \phi$ and (4.3). Now, put $\zeta = \bar{\partial}_t + ae$ where e is a unitary vector tangent to M and $|a| < 1$. Using (4.5), inequality (1.21) can be written as

$$a\Omega(e, N, B, \bar{\partial}_t) \leq -\langle E, N \rangle. \tag{4.6}$$

As Ω is the volume element in $(L, \langle \cdot, \cdot \rangle_L)$, then

$$|\Omega(e, N, B, \bar{\partial}_t)| = |\Omega^M(e, N, B)| = |\langle e \times N, B \rangle|$$

where Ω^M and $\times$ are, respectively, the volume element and vectorial product in $(M, \langle \cdot, \cdot \rangle)$. Thus, the result follows applying (4.6) to any direction e and any $a \in (-1, 1)$. □

When the boundary is time-convex and inequalities in (i), (ii) are strict, one obtains A-timeconvexity in the future. From (4.4), condition (i) of Proposition 4.1 can be rewritten as

$$-\langle N, \nabla \beta \rangle \leq (\beta)^{1/2} \langle N, \nabla A_2 \rangle.$$

Remark 4.2. Recall that if $\beta = 1$, $E = -\nabla V$ and $B = 0$ (that is, E is the opposite of the gradient of a potential function $V = A_2$ on M, and $A = A_2 \partial_t$) then we obtain results about connecting geodesics orthogonal to two Riemannian submanifolds (compare with [20], [27]).

Finally, it is worth discussing the other hypotheses on $A = (A_1, A_2)$ of our theorems (see formulas (1.12), (1.14), (1.15)), in terms of E and B. From (4.4), (1.11) and (1.16), the condition on ∇A_2 in (1.15) is equivalent to

$$\lim_{d(x,x_0)\to\infty,\ d(x,P)<\nu} |E(x)||\nabla U(x)| = 0.$$

The conditions on A_2 in (1.12) and (1.14) say that the supremum of A_2 is approximated if $d(x, x_0) \to \infty, d(x, P) < \nu$; this also holds for β from (1.11) and (1.14). Thus, essentially, the gradients of A_2 and β "points out to infinity" when $d(x, x_0) \to \infty, d(x, P) < \nu$; from (4.4), the electric vector field E "does not point out to infinity".

The condition on A_1' in (1.15) applies to both, the skew-symmetric (i.e. the magnetic vector field B) and the symmetric parts of A_1'. Thus, one has

$$\lim_{d(x,x_0)\to\infty,\ d(x,P)<\nu} |B(x)|\,|\nabla U(x)| = 0,$$

and an analogous limit for the symmetric part $\mathrm{Sym}A_1'$ of A_1'. This limit for $\mathrm{Sym}A_1'$ is not gauge invariant: if one takes $\hat{A}_1 = A_1 + V$ then $\mathrm{Sym}\,\hat{A}_1' = \mathrm{Sym}\,A_1' + \mathrm{Hess}\,V$. Nevertheless, recall that hypothesis (1.15) is used in addition to (1.14) (for A_1) and (1.13) (for U), which, essentially, fixes a restricted class of gauges.

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Rossella Bartolo
Dipartimento di Matematica, Politecnico di Bari, Via G. Amendola, 126/B, 70126 Bari Italy
E-mail address: rossella@poliba.it

Anna Germinario
Dipartimento di Matematica, Università degli Studi di Bari, Via E. Orabona, 4, 70125 Bari Italy
E-mail address: germinar@dm.uniba.it

Miguel Sánchez
Departamento de Geometría y Topología, Fac. Ciencias, Univ. Granada, Avenida Fuentenueva s/n, 18071 Granada Spain
E-mail address: sanchezm@ugr.es