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PYRAMIDAL CENTRAL CONFIGURATIONS AND PERVERSE SOLUTIONS

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Abstract. For n-body problems, a central configuration (CC) plays an important role. In this paper, we establish the relation between the spatial pyramidal central configuration (PCC) and the planar central configuration. We prove that the base of PCC is also a CC and we also prove that for some given conditions a planar CC can be extended to a PCC. In particular, if the pyramidal central configuration has a regular polygon base, then the masses of base are equal and the distance between the top vertex and the base is fixed and the mass of the top vertex is selective. Furthermore, the pyramidal central configuration gives rise to an example of a perverse solution in $\mathbb{R}^3$.

1. Introduction and Main Results

In this paper, we investigate the quantitative relationship between the spatial pyramidal central configuration and its base. We also investigate perverse solution in $\mathbb{R}^3$. The Newtonian n-body problem concerns the motion of n point particles with masses $m_j \in \mathbb{R}^+$ and positions $\bar{q}_j \in \mathbb{R}^3$ $(j = 1, \dots, n)$. This motion is governed by the Newton's law

$$m_j \ddot{\bar{q}}_j = \frac{\partial U(\bar{q})}{\partial \bar{q}_j}, \tag{1.1}$$

where $\bar{q} = (\bar{q}_1, \dots, \bar{q}_n)$ and the Newtonian potential is

$$U(\bar{q}) = \sum_{1 \le k < j \le n} \frac{m_k m_j}{|\bar{q}_k - \bar{q}_j|}. \tag{1.2}$$

Consider the space

$$X = \big\{ \bar{q} = (\bar{q}_1, \dots, \bar{q}_n) \in \mathbb{R}^{3n} : \sum_{k=1}^{n} m_k \bar{q}_k = 0 \big\}$$

i.e. suppose that the center of mass is fixed at the origin of the space. Because the potential is singular when two particles have the same position, it is natural to assume that the configuration avoids the set $\triangle = \{\bar{q} : \bar{q}_k = \bar{q}_j \text{ for some } k \neq j\}$. The set $X \backslash \triangle$ is called the configuration space.

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Definition A configuration $\bar{q} = (\bar{q}_1, \dots, \bar{q}_n) \in X\backslash\triangle$ is called a central configuration (CC) if there exists a constant λ such that

$$\sum_{j=1, j\neq k}^{n} \frac{m_j m_k}{|\bar{q}_j - \bar{q}_k|^3}(\bar{q}_j - \bar{q}_k) = -\lambda m_k \bar{q}_k, \quad 1 \leq k \leq n. \tag{1.3}$$

The value of the constant λ in (1.3) is uniquely determined by

$$\lambda = \frac{U}{I}, \tag{1.4}$$

where $I = \sum_{k=1}^{n} m_k |\bar{q}_k|^2$.

Definition A central configuration of $N+1$ bodies, N of which are coplanar, the $(N+1)th$ being off the plane, is called a pyramidal central configuration (PCC). Equivalently, we will say that the CC has the shape of a pyramid, where the N bodies or the N positions are called the base of the corresponding pyramidal central configuration.

Central configurations give rise to simple, explicit solutions of the N-body problem [9]. If the bodies are placed in a central configuration and released with zero initial velocity, they will collapse homothetically to a collision at center of mass. If the central configuration is planar, one can also choose initial velocities which lead to a periodic solution for which the configuration rigidly rotates around center of mass with angular velocity $\sqrt{\lambda}$.

A complete understanding of the nature of the central configurations is of fundamental importance to the n-body problem of celestial mechanics as these configurations play an essential role in the global structures of the solutions of the n-body problem.

Although three centuries have passed since Euler, Lagrange, etc. studied these problems, the classification of the central configuration is still unknown even for 4 bodies. It continues to attract much attention and some marvellous results have been obtained [4, 6]. In the celebrated work [1] of 1996, Albouy was able to establish a symmetry and prove that there are exactly three central configurations for the planar 4-body problem with equal masses. In 2002, Yiming Long and Sanzhong Sun studied the central configuration for the 4-body problem under the weaker condition that only the opposite masses are equal.

In 1996, Nelly Faycal established a classification of all PCC of the five-body problem with its base admitting a plane of reflexive symmetry. She studied the four cases which corresponds to the base of the pyramid of five bodies that admits one axis of symmetry, two axes of symmetry, or more axes of symmetry. The four cases are: pyramid with a square base, pyramid with a rectangular base, pyramid with a kite-shaped base and pyramid with a trapezoid base. She also generalized some of the results in the case of five masses to N+1 masses. She proved that the coplanar masses are concyclic (i.e. all lie on the same circle), and that the mass off the plane is equidistant from them [5, Theorem 6.1.1]. She also proved that in a pyramidal central configuration the mass off the plane is arbitrary [5, Theorem 6.2.2]. She also investigated the relation between the pyramidal central configuration and its base [5, Corollary 6.2.1].

This paper is distributed as follows. In section 2, we collect some basic properties of PCC that will be useful in the proof of the main theorem in section 3 and section 4. In section 3 we show the relation between spatial pyramidal central configuration and its base and also find the quantitative formulas of masses and distance for a

PCC with regular polygon base. In section 4 we construct an example which gives rise to a perverse solution in $\mathbb{R}^3$. Although some results in section 2 and 3 follow straight ahead from the main theorems of Nelly Faycal [5], we have decided to include our proofs here so that our paper will be completely self-contained.

2. Some General Lemmas

The proof to the following Lemmas can be found in Nelly Faycal [4, 5].

Lemma 2.1 ([5, Theorem 6.1.1]). *If $\bar q = (\bar q_1, \dots, \bar q_{N+1})$ is a PCC such that $\bar q_{N+1}$ is at the top vertex which is off the plane containing $m_1, \dots, m_N$, then m_{N+1} is equidistant from $m_1, \dots, m_N$.*

Proof. Since $\bar q = (\bar q_1, \dots, \bar q_{N+1})$ forms a CC, then there exists a scalar λ such that

$$\sum_{j=1, j\neq i}^{N+1} \frac{m_j m_i}{|\bar q_j - \bar q_i|^3}(\bar q_j - \bar q_i) = -\lambda m_i \bar q_i, 1 \le i \le N+1. \tag{2.1}$$

Writing $\bar q_i = (\bar x_i, \bar y_i, \bar z_i) \in \mathbb{R}^3$ in terms of its coordinate $G\bar x\bar y\bar z$, and $D_{j,i} = |\bar q_j - \bar q_i|$ for $1 \le i, j \le N+1$. Since the masses $m_1, \dots, m_N$ lie on a common plane, we may assume then, without loss of generality, that this plane is parallel to $G\bar x\bar z$. Hence $\bar y_1 = \bar y_2 = \dots = \bar y_N$. Multiplying (2.1) by $\bar y$ which is the unit vector of $\bar y-$direction. We obtain

$$\sum_{j=1, j\neq i}^{N+1} \frac{m_j m_i}{D_{j,i}^3}(\bar q_j - \bar q_i)\bar y = -\lambda m_i \bar q_i \bar y, 1 \le i \le N+1. \tag{2.2}$$

From (2.2), for $i = 1, 2$, we obtain

$$\frac{m_{N+1} m_1}{D_{N+1,1}^3}(\bar y_{N+1} - \bar y_1) = -\lambda m_1 \bar y_1. \tag{2.3}$$

$$\frac{m_{N+1} m_2}{D_{N+1,2}^3}(\bar y_{N+1} - \bar y_2) = -\lambda m_2 \bar y_2. \tag{2.4}$$

Hence (2.3), (2.4) give

$$m_{N+1}\Big(\frac{1}{D_{N+1,1}^3} - \frac{1}{D_{N+1,2}^3}\Big)(\bar y_{N+1} - \bar y_1) = 0. \tag{2.5}$$

Since $\bar y_{N+1} - \bar y_1 \neq 0$ otherwise $m_1, \dots, m_{N+1}$ are coplanar which contradicts to the definition of yramidal central configuration, then

$$D_{N+1,1} = D_{N+1,2}.$$

Similarly, we readily obtain

$$D_{N+1,i} = D_{N+1,j} 1 \le i, j \le N.$$

So m_{N+1} is equidistant from $m_1, \dots, m_N$. $\square$

Remark 2.2. The position $\bar q_1, \dots, \bar q_N$ are concyclic. In fact, they lie on the intersection of a plane with a sphere, since they are coplanar by assumption and they belong to a sphere centered at m_{N+1} by Lemma 2.1.

Remark 2.3. For $N = 3, \bar q_1, \dots, \bar q_4$ form a PCC in addition to the symmetry of the positions then $\bar q_1, \dots, \bar q_4$ are at the vertices of regular tetrahedron.

Lemma 2.4 ([5, Theorem 6.2.1]). *If $\bar q = (\bar q_1, \dots, \bar q_{N+1})$ is a PCC then $\lambda = M_{N+1} g$, where $M_{N+1} = m_1 + \dots + m_{N+1}$ is the total masses and $g = \frac{1}{D_{N+1,i}^3} 1 \le i \le N$.*

Proof. Denote by Oxyz, the coordinate system obtained from G$\bar{x}\bar{y}\bar{z}$ by parallel translation to a new origin $O \in P$, where O belongs to the plane P containing $m_1, \dots, m_N$. Let $q_1, \dots, q_{N+1}$ be the position vectors of $m_1, \dots, m_{N+1}$ in Oxyz. Obviously

$$\overline{OG} = \frac{1}{m} \sum_{j=1}^{N+1} m_j q_j. \tag{2.6}$$

Since $\bar{q} = (\bar{q}_1, \dots, \bar{q}_{N+1})$ is a CC, there exists a λ such that

$$\sum_{j=1, j\neq i}^{N+1} \frac{m_j m_i}{|\bar{q}_j - \bar{q}_i|^3} (\bar{q}_j - \bar{q}_i) = -\lambda m_i \bar{q}_i, 1 \leq i \leq N+1. \tag{2.7}$$

Take for the scalar multiple of equation (2.7) with $\bar{y}$ a unit vector in $\bar{y}$-direction. For $i = 1, \dots, N+1$, we use $\bar{q}_i = q_i - \overline{OG}$ to get

$$\sum_{j=1 j\neq i}^{N+1} \frac{m_j m_i}{|q_j - q_i|^3} (q_j - q_i) = -\lambda m_i (q_i - \overline{OG}), \tag{2.8}$$

that is

$$\sum_{j=1, j\neq i}^{N+1} \frac{m_j m_i}{|q_j - q_i|^3} (q_j - q_i) = -\lambda m_i \Big(\frac{1}{M_{N+1}} \sum_{j=1}^{N+1} m_j q_i - \frac{1}{M_{N+1}} \sum_{j=1}^{N+1} m_j q_j \Big),$$

or

$$\sum_{j=1, j\neq i}^{N+1} m_j m_i \Big(\frac{1}{D_{j,i}^3} - \frac{\lambda}{M_{N+1}} \Big) (q_j - q_i) = 0, \tag{2.9}$$

then

$$\sum_{j=1, j\neq i}^{N+1} m_j m_i \Big(\frac{1}{D_{j,i}^3} - \frac{\lambda}{M_{N+1}} \Big) (q_j - q_i) \bar{y} = 0. \tag{2.10}$$

But $\bar{y}$ is perpendicular to the plane P containing the vectors $q_1, \dots, q_N$ then

$$m_{N+1} m_i \Big(\frac{1}{D_{N+1,i}^3} - \frac{\lambda}{M_{N+1}} \Big) (q_{N+1} - q_i) \bar{y} = 0$$

Hence

$$\lambda = \frac{M_{N+1}}{D_{N+1,i}^3} \quad 1 \leq i \leq N.$$

Note (2.7) holds if and only if (2.9) holds. □

3. Relation Between Pyramidal Central Configuration and Its Base

The following theorem is an extension to arbitrary masses of the Faycal five-body results [5, Corollary 2.3.1 and Theorem 2.3.1].

Theorem 3.1. *If $\bar{q} = (\bar{q}_1, \dots, \bar{q}_{N+1}) (N \geq 3)$ is a PCC, such that $\bar{q}_{N+1}$ is at the top vertex which is off the plane containing $m_1, \dots, m_N$, then the particles of the base $m_1, \dots, m_N$ also form a CC.*

Conversely, if $m_1, m_2, \dots, m_N$ with position $q_1, q_2, \dots, q_N$, are coplanar and form a CC with multiplier λ and if there exists a position q_{N+1} such that

$$\frac{1}{|q_{N+1} - q_i|^3} = \frac{\lambda}{M_N}, \quad 1 \leq i \leq N,$$

where $M_N = \sum_{i=1}^N m_i$, *then for any mass* m_{N+1} *with position* q_{N+1}, $m_1, m_2, \dots, m_{N+1}$ *form a PCC*

Proof. If $\bar{q} = (\bar{q}_1, \dots, \bar{q}_{N+1})$ is a PCC, similar to the proof of Lemma 2.4 and according to the results (2.9) of Lemma 2.4, we have

$$\sum_{j=1, j\neq i}^{N} m_j \Big(\frac{1}{D_{j,i}^3} - \frac{1}{D_{N+1,1}^3}\Big)(q_j - q_i) = 0\,. \tag{3.1}$$

Furthermore, we choose the new origin O in Lemma 2.4 as the center of masses $m_1, \dots, m_N$, (i.e. $\sum_{j=1}^N m_j q_j = 0$). Then we have

$$\begin{aligned}
\sum_{j=1, j\neq i}^{N} \frac{m_j}{D_{j,i}^3}(q_j - q_i) &= \sum_{j=1, j\neq i}^{N} \frac{m_j}{D_{N+1,1}^3}(q_j - q_i) \\
&= \frac{1}{D_{N+1,1}^3} \sum_{j=1, j\neq i}^{N} m_j (q_j - q_i) \\
&= \frac{1}{D_{N+1,1}^3} \sum_{j=1}^{N} m_j (q_j - q_i) \\
&= \frac{1}{D_{N+1,1}^3} \sum_{j=1}^{N} m_j q_j - \frac{1}{D_{N+1,1}^3} \sum_{j=1}^{N} m_j q_i \\
&= -\frac{\sum_{j=1}^N m_j}{D_{N+1,1}^3} q_i = -\frac{M_N}{D_{N+1,1}^3} q_i .
\end{aligned}$$

Let $\lambda = (\sum_{j=1}^N m_j)/D_{N+1,1}^3$. Then $q_1, \dots, q_N$ form a central configuration. Conversely because $m_1, m_2, \dots, m_N$ with positions $q_1, q_2, \dots, q_N$, form a CC then

$$\sum_{j=1, j\neq i}^{N} \frac{m_j m_i}{|q_j - q_i|^3}(q_j - q_i) = -\lambda m_i q_i, \quad 1 \le i \le N. \tag{3.2}$$

Let

$$z_0 = \frac{1}{M_{N+1}} \sum_{j=1}^{N+1} m_j q_j, \quad \bar{q}_j = q_j - z_0. \tag{3.3}$$

Then $\sum_{j=1}^{N} m_j\bar{q}_j = -m_{N+1}\bar{q}_{N+1}$. For $i \neq N+1$, we obtain

$$\begin{aligned}
&\sum_{j=1,j\neq i}^{N+1} \frac{m_j m_i}{|\bar{q}_j-\bar{q}_i|^3}(\bar{q}_j-\bar{q}_i) + \frac{\lambda M_{N+1}}{M_N} m_i\bar{q}_i \\
&= \sum_{j=1,j\neq i}^{N+1} \frac{m_j m_i}{|\bar{q}_j-\bar{q}_i|^3}(\bar{q}_j-\bar{q}_i) + \frac{\lambda}{M_N} m_i \sum_{j=1,j\neq i}^{N+1} m_j(\bar{q}_i-\bar{q}_j) \\
&= \sum_{j=1,j\neq i}^{N+1} m_j m_i\Big(\frac{1}{|\bar{q}_j-\bar{q}_i|^3} - \frac{\lambda}{M_N}\Big)(\bar{q}_j-\bar{q}_i) \\
&= \sum_{j=1,j\neq i}^{N} m_j m_i\Big(\frac{1}{D_{j,i}^3} - \frac{\lambda}{M_N}\Big)(\bar{q}_j-\bar{q}_i) \\
&= \sum_{j=1,j\neq i}^{N} m_j m_i\Big(\frac{1}{D_{j,i}^3} - \frac{\lambda}{M_N}\Big)(q_j-q_i) = 0.
\end{aligned}$$

That is

$$\sum_{j=1,j\neq i}^{N+1} \frac{m_j m_i}{|\bar{q}_j-\bar{q}_i|^3}(\bar{q}_j-\bar{q}_i) = -\frac{\lambda M_{N+1}}{M_N} m_i\bar{q}_i = -\lambda' m_i\bar{q}_i,$$

where

$$\lambda' = \frac{M_{N+1}}{|q_j-q_{N+1}|^3}.$$

And for $i = N+1$,

$$\begin{aligned}
\sum_{j=1}^{N} \frac{m_j m_{N+1}}{|\bar{q}_j-\bar{q}_{N+1}|^3}(\bar{q}_j-\bar{q}_{N+1}) &= \frac{m_{N+1}}{D_{N+1,j}^3}\sum_{j=1}^{N} m_j(\bar{q}_j-\bar{q}_{N+1}) \\
&= \frac{m_{N+1}}{D_{N+1,j}^3}(-m_{N+1}\bar{q}_{N+1} - M_N\bar{q}_{N+1}) \\
&= -\frac{M_{N+1}}{D_{N+1,j}^3} m_{N+1}\bar{q}_{N+1} \\
&= -\lambda' m_{N+1}\bar{q}_{N+1}.
\end{aligned}$$

The proof is complete. □

The following theorem is an extension to the case of arbitrary masses of the Faycal five-body result [5, Theorem 3.1.1].

Theorem 3.2. *For $N \geq 3$ the $N+1$ body problem with masses $m_1, m_2, \dots, m_{N+1}$ in $\mathbb{R}^+$, and positions $\bar{q}_1, \dots, \bar{q}_{N+1} \in \mathbb{R}^3$, assume $\bar{q}_1, \dots, \bar{q}_N$ are coplanar and lie at the vertices of a regular polygon inscribed on a unit circle, and the $(N+1)$th is off the plane. Then the $N+1$ bodies form a PCC if and only if the distance between top vertex and the vertices of the base satisfies*

$$\frac{1}{D_{N+1,k}^3} = \frac{1}{4N}\sum_{j=1}^{N-1} \csc\big(\frac{\pi j}{N}\big) < 1, \quad 1 \leq k \leq N, \tag{3.4}$$

where $D_{k,j} = |\bar{q}_k - \bar{q}_j|$, and the masses in the base are equal $m_1 = m_2 = \dots = m_N$, the masse m_{N+1} in the top vertex is arbitrary.

We remark that there is no loss of generality in assuming that the regular polygon is inscribed on the unit circle since the CC (1.3) is invariant under the transformation $\bar{q}_k \to \bar{q}_k/a, \lambda \to a^2\lambda$. In addition, the distance between top vertex and the vertices of the base doesn't depend on the masses and is completely determined by the base. The central configurations of N bodies cannot be extended to any pyramidal central configuration for $N \geq 473$ because for $N \geq 473$, $\frac{1}{4N}\sum_{j=1}^{N-1}\csc(\frac{\pi j}{N}) > 1$. So a planar central configuration can not always be extended to a pyramidal central configuration.You can find more comments in Moeckel [9].

Proof of Theorem 3.2. By lemma 2.1, for $1 \leq k, j \leq N$.

$$\frac{1}{D^3_{N+1,k}} = \frac{1}{D^3_{N+1,j}}\,. \tag{3.5}$$

By theorem 3.1, $\bar{q}_1, \bar{q}_2, \dots, \bar{q}_N$ form a planar central configuration. Then these particles can rotate about the center of masses by theorem (Perko-Walter [10] and Xie-Zhang [12]).

$$\lambda = \frac{M_{N+1}\gamma}{N} = \frac{M_{N+1}}{D^3_{N+1,i}}, \tag{3.6}$$

where $\gamma = \frac{1}{4N}\sum_{j=1}^{N-1}\csc(\frac{\pi j}{N})$. Then

$$\frac{1}{D^3_{N+1,i}} = \frac{1}{4N}\sum_{j=1}^{N-1}\csc\big(\frac{\pi j}{N}\big). \tag{3.7}$$

By Theorem 3.1, $\bar{q_1}, \dots, \bar{q_N}$ form a planar central configuration. As a result of [10, 12], $m_1 = m_2 = \dots = m_N$. Although the proof in [12] is not complete, the flaw pointed out by Chenciner [2] does not affect the conclusion, $m_1 = m_2 = \dots = m_N$. $\square$

Conversely, by Theorem 3.1, we know that we can put an arbitrary mass body at the top vertex and the $N+1$ bodies form a pyramidal central configuration.

4. Perverse Solutions in $\mathbb{R}^3$

Let $\bar{q}(t) = (\bar{q}_1(t), \bar{q}_2(t), \dots, \bar{q}_n(t))$ be a solution of the n-body problem with Newtonian potential and masses $m_1, m_2, \dots, m_n$. Chenciner [2] proposed the following two questions:

(1) Does there exist another system of masses, $(m_1', m_2', \dots, m_n')$, for which $\bar{q}(t)$ is still a solution?
(2) The same as question 1 but insisting that the sum $M = \sum_{i=1}^n m_i$ of the masses and the center of mass $C = \frac{1}{M}\sum_{i=1}^n m_i\bar{q}_i$ do not change.

Definition. If the answer to the first (resp. second) question is yes, we shall say $\bar{q}(t)$ is a perverse (resp. really perverse) solution and the allowed systems of masses will be called admissible.

Chenciner investigated the perverse solutions in the planar case. He proved for n=2 that no solution is perverse, and for $n \geq 3$ that perverse solutions do exist by constructing an example of a regular polygon rotating around the body lying in the center of the regular polygon. Now, we construct a perverse solution in $\mathbb{R}^3$. Let $\bar{q}(t) = (\bar{q}_1(t), \bar{q}_2(t), \dots, \bar{q}_N(t), \bar{q}_{N+1}(t), 0)$ be a total collision solution with $N+2$ masses $(m_1, m_2, \dots, m_N, m_{N+1}, m_{N+2})$ and satisfy the following initial conditions:

(1) $(\bar{q}_1(0), \bar{q}_2(0), \dots, \bar{q}_N(0), \bar{q}_{N+1}(0))$ is a pyramidal central configuration such that $\bar{q}_{N+1}(0)$ is at the top vertex which is off the plane containing $\bar{q}_1(0)$, $\bar{q}_2(0), \dots, \bar{q}_N(0)$
(2) The center of mass is at the origin i.e. $m_1\bar{q}_1(0) + \cdots + m_{N+1}\bar{q}_{N+1}(0) + m_{N+2} \cdot 0 = 0$
(3) $|\bar{q}_i| = |\bar{q}_j|, 1 \le i, j \le N+1$
(4) The initial velocity is zero i.e. $\bar{q}'(0) = 0$.

Theorem 4.1. *$\bar{q}(t)$ is a perverse solution with a one parameter family of admissible sets of masses.*

Proof. $\bar{q}(t)$ is a solution of the Newton's equation

$$m_j\ddot{\bar{q}}_j = \sum_{k=1,k\neq j}^{N+2} \frac{m_k m_j}{|\bar{q}_k - \bar{q}_j|^3}(\bar{q}_k - \bar{q}_j), \quad 1 \le j \le N+2. \tag{4.1}$$

where $\bar{q}_{N+2}(t) = 0$ for all t. Because $\bar{q}(t)$ satisfies the above initial conditions, $\bar{q}(t)$ will collapse homothetically to a collision at the center of mass at zero while keeping the shape in the whole motion. Therefore, $\bar{q}(t)$ is a perverse solution and $(m_1', m_2', \dots, m_N', m_{N+1}', m_{N+2}')$ is an admissible system of masses if and only if the accelerations $\ddot{\bar{q}}_i(t)$ (for all $1 \le i \le N+2$) do not change with respect to the admissible masses. In fact, for $j \neq N+2$, we have

$$\begin{aligned}
\ddot{\bar{q}}_j &= \sum_{k=1,k\neq j}^{N+2} \frac{m_k}{|\bar{q}_k - \bar{q}_j|^3}(\bar{q}_k - \bar{q}_j) \\
&= \sum_{k=1,k\neq j}^{N+1} \frac{m_k}{|\bar{q}_k - \bar{q}_j|^3}(\bar{q}_k - \bar{q}_j) + \frac{m_{N+2}}{|\bar{q}_{N+2} - \bar{q}_j|^3}(\bar{q}_{N+2} - \bar{q}_j) \\
&= -\frac{M_{N+1}}{D_{N+1,j}^3}\bar{q}_j - \frac{m_{N+2}}{|\bar{q}_j|^3}\bar{q}_j \quad \text{by lemma 2.4} \\
&= -\frac{\beta M_{N+1}}{|\bar{q}_j|^3}\bar{q}_j - \frac{m_{N+2}}{|\bar{q}_j|^3}\bar{q}_j \\
&= -(\beta M_{N+1} + m_{N+2})\frac{\bar{q}_j}{|\bar{q}_j|^3}
\end{aligned}$$

where $\beta = |\bar{q}_j|^3/D_{N+1,j}^3$ is a constant for all $1 \le j \le N+1$ and for all t because $|\bar{q}_j| = |\bar{q}_k|$, $D_{N+1,j} = D_{N+1,k}$ and the motion keeps the same shape. In addition, for $j = N+2$, $\bar{q}_{N+2}$ is fixed at origin. Therefore, $(m_1', m_2', \dots, m_N', m_{N+1}', m_{N+2}')$ is an admissible masses if $\beta M_{N+1}' + m_{N+2}' = \beta M_{N+1} + m_{N+2}$ and the initial conditions are satisfied. For example, we can choose $m_j' = \rho m_j$ for $1 \le j \le N+1$ which leads the initial conditions to be satisfied and choose $m_{N+2}' = \beta M_{N+1} + m_{N+2} - \beta\rho M_{N+1}$. It follows that ρ may be chosen as a parameter of the set of admissible masses. In particular, $\bar{q}(t)$ is perverse but not really perverse since $\beta < 1$. $\square$

Corollary 4.2. *Under the same conditions as theorem 4.1, but inscribing the base $\bar{q}_1(0), \dots, \bar{q}_N(0)$ on the vertex of a unit regular polygon, the function $\bar{q}(t)$ is a perverse solution for $N = 3, 4, 5, 6, 7, 8$.*

Proof. We only need to check the conditions (1) and (2) are satisfied if we choose $m_1 = \cdots = m_N$ and the distance $D_{N+1,k}$ between the $\bar{q}_{N+1}$ and $\bar{q}_k$ satisfying (3.4). For $N = 3, 4, 5, 6, 7, 8$, it could choose masses such that (3) and (4) are satisfied. But for $N \geq 9$, $D_{N+1,i} < 1.394$ then it is impossible to make $|\bar{q}_i| = |\bar{q}_j|$ for $1 \leq i, j \leq N+1$. $\square$

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