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GENERIC SIMPLICITY OF EIGENVALUES FOR A DIRICHLET PROBLEM OF THE BILAPLACIAN OPERATOR

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Abstract. In this work we show that the eigenvalues of the Dirichlet problem for the Bilaplacian are generically simple in the set of C^4-regular regions.

1. Introduction

Boundary perturbations have been studied by several authors through different perspectives, since the pioneering works of Rayleigh [17] and Hadamard [6]. Among others, we mention the works of Micheletti [11] and Uhlenbeck [20] in which generic properties of eigenvalues and eigenfunctions of second order eliptical operators with respect to variation of the domain were obtained.

Many problems of the same type were considered by Henry in [8] where the author developed a general theory on perturbation of domains and proved several results on boundary perturbations for second order eliptic operators. Following his approach Pereira [15] obtained results on the eigenvalues of the Dirichlet's problem for the Laplace operator on regions satisfying symmetry properties. Some results correlating boundary perturbation to the Laplace operator and to reaction-diffusion problems can be found in [16] and [12].

There are also many works on perturbation of the boundary in the literature using the concept of shape differentiation (see e.g. [18, 19]). In particular, Ortega and Zuazua used this concept in [13] to study the eigenvalue problem

$$\begin{aligned} (\Delta^2 + \lambda)u = 0 \quad \text{in } \Omega \\ u = \frac{\partial u}{\partial N} = 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1.1}$$

In this interesting work, the authors, among others results, presented a proof of the generic simplicity of the eigenvalues of (1.1). Unfortunately, there is a problem in their proof. The authors assumed that the eigenvalues and eigenfunctions are analytic with respect to the diffeomorphism giving the variation of the region, which is not true in general.

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Our goal here is to give a somewhat different approach to obtain the simplicity of the eigenvalues of (1.1) using as our main tool a general form of the Transversality Theorem to overcome the problem in their proof.

This paper is organized as follows. In section 2 we state some background results needed in the sequel. In section 3, we prove the continuous dependence of a part of spectrum of (1.1) consisting of a finite system of eigenvalues with respect to variation of domain. In section 4, we prove analytic dependence of the simple eigenvalues with respect to the perturbation of the boundary. In section 5, we prove the main results of the paper, namely, the generic simplicity of the eigenvalues of (1.1) in the set of open, connected, bounded, C^4-regular regions $\Omega \subset \mathbb{R}^n$, $n \geq 2$.

2. Preliminaries

The results in this section were taken from the monograph by Henry [8], where complete proofs can be found.

2.1. **Some notation and geometrical preliminaries.** Given a real function f defined in a neighborhood of $x \in \mathbb{R}^n$, its m-derivative at x can be considered as a m-linear symmetric form $h \mapsto D^m f(x) h^m$ in R^n, with norm

$$|D^m f(x)| = \max_{|h|\leq 1} |D^m f(x) h^m|.$$

If Ω is an open subset of $\mathbb{R}^n$ and E is a normed vector space, $C^m(\Omega, E)$ is the space of m-times continuously and bounded differentiable functions on Ω whose derivatives extend continuously to the closure $\bar{\Omega}$, with the usual norm

$$\|f\|_{C^m(\Omega,E)} = \max_{0\leq j\leq m} sup_{x\in\Omega} |D^j f(x)|.$$

If $E = \mathbb{R}$, we write simply $C^m(\Omega)$.

Let $C^m_{unif}(\Omega, E)$ be the closed subspace of $C^m(\Omega, E)$ consisting of functions whose m^{th} derivative is uniformly continuous.

We say that an open set $\Omega \subset \mathbb{R}^n$ is C^m-regular if there exists $\phi \in C^m(\mathbb{R}^n, \mathbb{R})$, which is at least in $C^1_{\mathrm{unif}}(\mathbb{R}^n, \mathbb{R})$, such that

$$\Omega = \{x \in \mathbb{R}^n; \phi(x) > 0\}$$

and $\phi(x) = 0$ implies $|\nabla \phi| \geq 1$.

Let m be a non negative integer and $p \geq 1$ a real number. We define the Sobolev spaces $W^{m,p}(\Omega)$ and $W^{m,p}_0(\Omega)$, as the completion of $C^m(\Omega)$ and $C^m_0(\Omega)$ respectively under the norm

$$\|u\| = \Big(\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u|^p dx \Big)^{1/p}$$

where $C^m_0(\Omega)$ is the subspace of functions on $C^m(\Omega)$ with compact support (when $p = 2$ we usually write $H^m(\Omega) = W^{m,2}(\Omega)$ and $H^m_0(\Omega) = W^{m,2}_0(\Omega)$). We sometimes need to use differential operators (gradient, divergence and Laplacian) in a hypersurface $S \subset \mathbb{R}^n$.

Let S be a C^1 hypersurface in $\mathbb{R}^n$ and let $\phi : S \to \mathbb{R}$ be C^1 (so it can be extended to be C^1 on a neighborhood of S), then $\nabla_S \phi$ is the tangent vector field in S such that, for each C^1 curve $t \mapsto x(t) \subset S$, we have

$$\frac{d}{dt}\phi(x(t)) = \nabla_S \phi(x(t)) \cdot \dot{x}(t).$$

Let S be a C^2 hypersurface in $\mathbb{R}^n$ and $\vec{a} : S \to \mathbb{R}^n$ a C^1 vector field tangent to S. Then $\operatorname{div}_S \vec{a} : S \to \mathbb{R}^n$ is the continuous function such that, for every C^1 $\phi : S \to \mathbb{R}$ with compact support in S,

$$\int_S (\operatorname{div}_S \vec{a})\phi = -\int_S \vec{a} \cdot \nabla_S \phi.$$

Also, if $u : S \to \mathbb{R}$ is C^2, then $\Delta_S u = \operatorname{div}_S(\nabla_S u)$ or, equivalently, for all C^1 $\phi : S \to \mathbb{R}$ with compact support

$$\int_S \phi \Delta_S u = -\int_S \nabla_S \phi \cdot \nabla_S u.$$

Theorem 2.1. (1) *If S is a C^1 hypersurface and $\phi : \mathbb{R}^n \to \mathbb{R}$ is C^1 in a neighborhood of S, then, on S, $\nabla_S \phi(x)$ is the component of $\nabla \phi(x)$ tangent S at x, that is*

$$\nabla_S \phi(x) = \nabla \phi(x) - \frac{\partial \phi}{\partial N}(x) N(x)$$

where N is an unit normal field on S.

(2) *If S is a C^2 hypersurface in $\mathbb{R}^n$, $\vec{a} : S \to \mathbb{R}^n$ is C^1 in a neighborhood of S, $N : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 unit normal field in a neighborhood of S and $H = \operatorname{div} N$ is the mean curvature of S, then*

$$\operatorname{div}_S \vec{a} = \operatorname{div} \vec{a} - H \vec{a} \cdot N - \frac{\partial}{\partial N}(a \cdot N)$$

on S.

(3) *If S is a C^2 hypersurface, $u : \mathbb{R}^n \to \mathbb{R}$ is C^2 in a neighborhood of S and N is a normal vector field for S, then*

$$\Delta_S u = \Delta u - H \frac{\partial u}{\partial N} - \frac{\partial^2 u}{\partial N^2} + \nabla_S u \cdot \frac{\partial N}{\partial N}$$

on S. We may choose N so that $\frac{\partial N}{\partial N} = 0$ on S and then the final term vanishes.

Remark 2.2. *If $u \in H^4 \cap H_0^2(\Omega)$, we have $\Delta u = \frac{\partial^2 u}{\partial N^2}$ on $\partial\Omega$. In fact, by Theorem 2.1*

$$\begin{aligned} 0 &= \Delta_{\partial\Omega} u \\ &= \Delta u - \operatorname{div} N \frac{\partial u}{\partial N} - \frac{\partial^2 u}{\partial N^2} + \nabla_{\partial\Omega} u \cdot \frac{\partial N}{\partial N} \\ &= \Delta u - \frac{\partial^2 u}{\partial N^2} \quad \text{on } \partial\Omega. \end{aligned}$$

We often need the following uniqueness theorem for the Bilaplacian.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded, C^4-regular region and B a ball which meets $\partial\Omega$ in a C^4 hypersurface $B \cap \partial\Omega$. Assume $u \in H^4(\Omega)$ and for some constant K*

$$\begin{gathered} |\Delta^2 u| \le K\Big(|\Delta u| + |\nabla u| + |u|\Big) \quad \text{a.e. } \Omega \text{ with} \\ u = \frac{\partial u}{\partial N} = \frac{\partial^2 u}{\partial N^2} = \frac{\partial^3 u}{\partial N^3} = 0 \quad \text{on } B \cap \partial\Omega. \end{gathered} \tag{2.1}$$

Then $u \equiv 0$ in Ω.

This theorem follows from [10, Theorem 8.9.1].

2.2. **Differential calculus of boundary perturbation.** Consider a formal nonlinear differential operator $u \mapsto v$

$$v(y) = f\Big(y, u(y), \frac{\partial u}{\partial y_1}(y), \dots, \frac{\partial u}{\partial y_n}(y), \frac{\partial^2 u}{\partial y_1^2}(x), \frac{\partial^2 u}{\partial y_1 \partial y_2}(y), \dots\Big) \quad y \in \mathbb{R}^n$$

To simplify the notation, we define a constant matrix coefficient differential operator L

$$Lu(y) = \Big(u(y), \frac{\partial u}{\partial y_1}(y), \dots, \frac{\partial u}{\partial y_n}(y), \frac{\partial^2 u}{\partial y_1^2}(y), \frac{\partial^2 u}{\partial y_1 \partial y_2}(y), \dots\Big) \quad y \in \mathbb{R}^n$$

with as many terms as needed, so our nonlinear operator becomes

$$u \mapsto v(\cdot) = f(\cdot, Lu(\cdot))$$

More precisely, suppose $Lu(\cdot)$ has values in $\mathbb{R}^p$ and $f(y, \lambda)$ is defined for (y, λ) in some open set $O \subset \mathbb{R}^n \times \mathbb{R}^p$. For subsets $\Omega \subset \mathbb{R}^n$ define F_Ω by

$$F_\Omega(u)(y) = f(y, Lu(y)),\ y \in \Omega$$

for sufficiently smooth functions u in Ω such that $(y, Lu(y)) \in O$ for any $y \in \bar{\Omega}$. For example, if f is continuous, Ω is bounded and L involves derivatives of order $\leq m$, the domain of F_Ω is an open subset (perhaps empty) of $C^m(\Omega)$, and the values of F_Ω are in $C^0(\Omega)$. (Other function spaces could be used with obvious modifications).

If $h : \Omega \mapsto \mathbb{R}^n$ is a C^k imbedding, we can also consider $F_{h(\Omega)} : C^m(h(\Omega)) \mapsto C^0(h(\Omega))$. But then the problem will be posed in different spaces. To bring it back to the original spaces we consider the 'pull-back' of h

$$h^\star : C^k(h(\Omega)) \mapsto C^k(\Omega) \quad (0 \leq k \leq m)$$

defined by $h^\star(\varphi) = \varphi \circ h$ (which is a diffeomorphism) and then $h^\star F_{h(\Omega)} h^{\star -1}$ is again a map from $C^m(\Omega)$ into $C_0(\Omega)$. In this sense, we may express problems of perturbation of the boundary of a boundary value problem as problems of differential calculus in Banach spaces. This is more convenient to apply tools like the Implicit Function or Transversality theorems. On the other hand, a new variable h is introduced. We then need to study the differentiability properties of the function $(h, u) \mapsto h^\star F_{h(\Omega)} h^{\star -1} u$. This has been done in [8] where it is shown that, if $(y, \lambda) \mapsto f(y, \lambda)$ is C^k or analytic then so is the map above, considered as a map from $\mathrm{Diff}^m(\Omega) \times C^m(\Omega)$ to $C^0(\Omega)$ (other function spaces can be used instead of C^m) where

$$\mathrm{Diff}^{\,m}(\Omega) = \Big\{h \in C^m(\Omega, \mathbb{R}^n) : h \text{ is injective and } \frac{1}{|\det h'(x)|} \text{ is bounded in } \Omega\Big\}$$

is an open subset of $C^m(\Omega, \mathbb{R}^n)$ (given an open, bounded, C^m region $\Omega_0 \subset \mathbb{R}^n$). To compute the derivative we then need only compute the Gateaux derivative that is, the t-derivative along a smooth curve $t \mapsto (h(t, .), u(t, .)) \in \mathrm{Diff}^m(\Omega) \times C^m(\Omega)$.

Suppose we want to compute

$$\frac{\partial}{\partial t} F_{\Omega(t)}(v)(y) = \frac{\partial}{\partial t} f(y, Lv(y))$$

with $y = h(t, x)$ fixed in $\Omega(t) = h(t, \Omega)$. To keep y fixed we must take $x = x(t)$, $y = h(t, x(t))$ with

$$0 = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} x'(t) \Longrightarrow x'(t) = -(\frac{\partial h}{\partial x})^{-1} \frac{\partial h}{\partial t},$$

that is, $x(t)$ is the solution of the differential equation $\frac{dx}{dt} = -U(x,t)$ where $U(x,t) = (\frac{\partial h}{\partial x})^{-1}\frac{\partial h}{\partial t}$. The differential operator

$$D_t = \frac{\partial}{\partial t} - U(x,t)\frac{\partial}{\partial x}, \quad U(x,t) = \big(\frac{\partial h}{\partial x}\big)^{-1}\frac{\partial h}{\partial t}$$

is called the *anti-convective derivative*. This derivative at a fixed point x corresponds to the t-derivative at $y = h(t,x)$ fixed. The results (theorems 2.4, 2.7) below are the main tools used to compute derivatives.

Theorem 2.4. *Suppose $f(t,y,\lambda)$ is C^1 in an open set in $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^p$, L is a constant-coefficient differential operator of order $\le m$ with $Lv(y) \in \mathbb{R}^p$ (where defined). For open sets $Q \subset \mathbb{R}^n$ and C^m functions v on Q, let $F_Q(t)v$ be the function*

$$y \to f(t,y,Lv(y)),\ y \in Q.$$

where defined. Suppose $t \to h(t,\cdot)$ is a curve of imbeddings of an open set $\Omega \subset \mathbb{R}^n$, $\Omega(t) = h(t,\Omega)$ and for $|j| \le m$, $|k| \le m+1$ $(t,x) \to \partial_t\partial_x^j h(t,x)$, $\partial_x^k h(t,x)$, $\partial_x^k u(t,x)$ are continuous on $\mathbb{R}\times\Omega$ near $t=0$, and $h(t,\cdot)^{*^{-1}}u(t,\cdot)$ is in the domain of $F_{\Omega(t)}$. Then, at points of Ω

$$D_t(h^*F_{\Omega(t)}(t)h^{*^{-1}})(u) = (h^*\dot{F}_{\Omega(t)}(t)h^{*^{-1}})(u) + (h^*F'_{\Omega(t)}(t)h^{*^{-1}})(u)\cdot D_t u$$

where D_t is the anti-convective derivative defined above,

$$\dot{F}_Q(t)v(y) = \frac{\partial f}{\partial t}(t,y,Lv(y))$$

and

$$F'_Q(t)v\cdot w(y) = \frac{\partial f}{\partial\lambda}(t,y,Lv(y))\cdot Lw(y), y \in Q$$

is the linearization of $v \to F_Q(t)v$.

Remark 2.5. Suppose we deal with a linear operator $A = \sum_{|\alpha|\le m} a_\alpha(y)\big(\frac{\partial}{\partial y}\big)^\alpha$ not explicitly dependent on t, and $h(t,x) = x + tV(x) + o(t)$ as $t \to 0$ and $x \in \Omega$. Then at $t = 0$

$$\begin{aligned}
\frac{\partial}{\partial t}(h^*Ah^{*^{-1}}u)\big|_{t=0} &= D_t(h^*Ah^{*^{-1}}u)\big|_{t=0} + h_x^{-1}h_t\nabla(h^*Ah^{*^{-1}}u)\Big|_{t=0}\\
&= A(\frac{\partial u}{\partial t} - V\cdot\nabla u) + V\cdot\nabla(Au)\\
&= A\frac{\partial u}{\partial t} + [V\cdot\nabla, A]u
\end{aligned}$$

since $\frac{\partial}{\partial t}A = 0$. Note that the commutator $[V\cdot\nabla, A](\cdot)$ is still an operator of order m.

Consider now a boundary condition of the form

$$b(t,y,Lv(y),MN_{\Omega(t)}(y)) = 0 \quad \text{for } y \in \partial\Omega(t),$$

where L, M are constant-coefficient differential operators and $N_{\Omega(t)}(y)$ is the outward unit normal for $y \in \partial\Omega(t)$, extended smoothly as a unit vector field on a neighborhood of $\partial\Omega(t)$. Choose some extension of N_Ω in the reference region and then define $N_{\Omega(t)} = N_{h(t,\Omega)}$ by

$$h^*N_{h(t,\Omega)}(x) = N_{h(t,\Omega)}(h(x)) = \frac{(h_x^{-1})^T N_\Omega(x)}{\|(h_x^{-1})^T N_\Omega(x)\|} \tag{2.2}$$

for x near $\partial\Omega$, where $(h_x^{-1})^T$ is the inverse-transpose of the Jacobian matrix h_x and $\|\cdot\|$ is the Euclidean norm. This is the extension understood in the above boundary condition: $b(t, y, Lv(y), MN_{\Omega(t)}(y))$ is defined for $y \in \Omega$ near $\partial\Omega$ and has limit zero (in some sense, depending on the functional space employed) as $y \to \partial\Omega$.

Lemma 2.6. *Let Ω be a C^2-regular region, $N_{\Omega(\cdot)}$ a C^1 unit-vector field defined on a neighborhood of $\partial\Omega$ which is the outward normal on $\partial\Omega$, and for a C^2 function $h : \Omega \to \mathbb{R}^n$ define $N_{h(\Omega)}$ on a neighborhood of $h(\partial\Omega) = \partial h(\Omega)$ by (2.2) above. Suppose $h(t, \cdot)$ is an imbedding for each t, defined by*

$$\frac{\partial}{\partial t} h(t,x) = V(t, h(t,x)) \text{ for } x \in \Omega, h(0,x) = x,$$

$(t, y) \to V(t, y)$ is C^2 and $\Omega(t) = h(t, \Omega)$, $N_{\Omega(t)} = N_{h(t,\Omega)}$. Then for x near $\partial\Omega$, $y = h(t, x)$ near $\partial\Omega(t)$, we may compute the derivative $(\frac{\partial}{\partial t})_{y=constant}$ and, if $y \in \partial\Omega$,

$$\frac{\partial}{\partial t} N_{\Omega(t)}(y) = D_t(h^* N_{h(t,\Omega)})(x) = -\Big(\nabla_{\partial\Omega(t)}\sigma + \sigma \frac{\partial N_{\Omega(t)}}{\partial N_{\Omega(t)}}(y)\Big)$$

where $\sigma = V \cdot N_{\Omega(t)}$ is the normal velocity and $\nabla_{\partial\Omega(t)}\sigma$ is the component of the gradient tangent to $\partial\Omega$.

Theorem 2.7. *Let $b(t, y, \lambda, \mu)$ be a C^1 function on an open set of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$ and let L, M be constant-coefficient differential operators with order $\leq m$ of appropriate dimensions so $b(t, y, Lv(y), MN_{\Omega(t)}(y))$ makes sense. Assume that Ω is a C^{m+1} region, $N_\Omega(x)$ is a C^m unit-vector field near $\partial\Omega$ which is the outward normal on $\partial\Omega$, and define $N_{h(t,\Omega)}$ by (2.2) when $h : \Omega \to \mathbb{R}^n$ is a C^{m+1} smooth imbedding. Also define $\mathcal{B}_{h(\Omega)}(t)$ by*

$$\mathcal{B}_{h(\Omega)} v(y) = b(t, y, Lv(y), MN_{h(\Omega)}(y))$$

for $y \in h(\Omega)$ near $\partial h(\Omega)$.

If $t \to h(t, \cdot)$ is a curve of C^{m+1} imbeddings of Ω and for $|j| \leq m$, $|k| \leq m+1$, $(t, x) \to (\partial_t\partial_x^j h, \partial_x^k, \partial_t\partial_x^j u, \partial_x^k u)(t, x)$ are continuous on $\mathbb{R} \times \Omega$ near $t = 0$, then at points of Ω near $\partial\Omega$

$$\begin{aligned} D_t(h^*\mathcal{B}_{h(\Omega)}h^{*-1})(u) &= (h^*\dot{\mathcal{B}}_{h(\Omega)}h^{*-1})(u) + (h^*\mathcal{B}'_{h(\Omega)}h^{*-1})(u) \cdot D_t u \\ &\quad + (h^* \frac{\partial \mathcal{B}_{h(\Omega)}}{\partial N} h^{*-1})(u) \cdot D_t(h^* N_{\Omega(t)}) \end{aligned}$$

where $h = h(t, \cdot)$; $\dot{\mathcal{B}}_{h(\Omega)}$ e $\mathcal{B}'_{h(\Omega)}$ are defined as in Theorem 2.2,

$$\frac{\partial \mathcal{B}_{h(\Omega)}}{\partial N}(v) \cdot n(y) = \frac{\partial b}{\partial \mu}(t, y, Lv(y), MN_{h(\Omega)}(y)) \cdot Mn(y)$$

and $D_t(h^* N_{\Omega(t)})\big|_{\partial\Omega}$ is computed in Lemma 2.6.

2.3. **Change of Origin.** In the above, the "origin" or reference region is Ω. But we may easily transfer the origin to any Ω_0 diffeomorphic to Ω. Let $H_0 : \Omega \to \Omega_0$ be the diffeomorphism and for every imbedding $h : \Omega \to \mathbb{R}^n$ define the imbedding $h_0 = h \circ H_0^{-1} : \Omega_0 \to \mathbb{R}^n$. Similarly define

$$x_0 = H_0(x), u_0 = H_0^{*-1} u,$$

$$N_{\Omega_0}(x_0) = N_{H_0(\Omega)}(H_0(x_0)) = \frac{(H_{0,x}^{-1})^T N_\Omega(x)}{\|(H_{0,x}^{-1})^T N_\Omega(x)\|}$$

and then $h(\Omega) = h_0(\Omega_0)$,

$$h^* F_{h(\Omega)} {h^*}^{-1} u(x) = h_0^* F_{h_0(\Omega_0)} {h_0^*}^{-1} u_0(x_0),$$
$$h^* \mathcal{B}_{h(\Omega)} {h^*}^{-1} u(x) = h_0^* \mathcal{B}_{h_0(\Omega_0)} {h_0^*}^{-1} u_0(x_0),$$

using the normal

$$N_{h_0(\Omega_0)}(h_0(x_0)) = N_{h(\Omega)}(h(x)).$$

This "change of origin" is used frequently in the sequel, as it permits us to compute derivatives in h at $h = i_\Omega$, where the formulas are simpler.

2.4. **The Transversality Theorem.** A basic tool for our results will be the Transversality Theorem in the form below, due to D. Henry [8]. We first recall some definitions.

A map $T \in \mathcal{L}(X, Y)$ where X and Y are Banach spaces is a *semi-Fredholm map* if the range of T is closed and at least one (or both, for Fredholm) of $\dim \mathcal{N}(T)$, $\operatorname{codim} \mathcal{R}(T)$ is finite; the index of T is then

$$\operatorname{ind}(T) = \dim \mathcal{N}(T) - \operatorname{codim} \mathcal{R}(T).$$

We say that a subset F of a topological space X is *rare* if its closure has empty interior and *meager* if it is contained in a countable union of rare subsets of X. We say that F is *residual* if its complement in X is meager. We also say that X is a *Baire space* if any residual subset of X is dense.

Let f be a C^k map between Banach spaces. We say that x is a *regular point* of f if the derivative $f'(x)$ is surjective and its kernel is finite-dimensional. Otherwise, x is called a *critical* point of f. A point is *critical* if it is the image of some critical point of f.

Let now X be a Baire space and $I = [0, 1]$. For any closed or σ-closed $F \subset X$ and any nonnegative integer m we say that the codimension of F is greater or equal to m (*codim* $F \geq m$) if the subset $\{\phi \in C(I^m, X) : \phi(I^m) \cap F$ is non-empty $\}$ is meager in $C(I^m, X)$. We say $\operatorname{codim} F = k$ if k is the largest integer satisfying $\operatorname{codim} F \geq m$.

Theorem 2.8. *Suppose given positive numbers k and m; Banach manifolds X, Y, Z of class C^k; an open set $A \subset X \times Y$; a C^k map $f : A \mapsto Z$ and a point $\xi \in Z$. Assume for each (x,y) $\in f^{-1}(\xi)$ that:*

(1) $\frac{\partial f}{\partial x}(x, y) : T_x X \mapsto T_\xi Z$ *is semi-Fredholm with index* $< k$.
(2) *Either*
   (α) $Df(x, y) = \big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\big) : T_x X \times T_y Y \mapsto T_\xi Z$ *is surjective; or*
   (β) $\dim \big\{\mathcal{R}\big(Df(x, y)\big)/\mathcal{R}\big(\frac{\partial f}{\partial x}(x, y)\big)\big\} \geq m + \dim \mathcal{N}\big(\frac{\partial f}{\partial x}(x, y)\big)$.
(3) $(x, y) \mapsto y : f^{-1}(\xi) \mapsto Y$ *is σ-proper,* $f^{-1}(\xi) = \bigcup_{j=1}^\infty \mathcal{M}_j$ *is a countable union of sets $\mathcal{M}_j$ such that $(x, y) \mapsto y : \mathcal{M}_j \mapsto Y$, is a proper map for each j. [Given $(x_\nu, y_\nu) \in \mathcal{M}_j$ such that y_ν converges in Y, there exists a subsequence (or subnet) with limit in $\mathcal{M}_j$].*

We note that (3) holds if $f^{-1}(\xi)$ is Lindelöf [every open cover has a countable subcover] or, more specifically, if $f^{-1}(\xi)$ is a separable metric space, or if X, Y are separable metric spaces.

Let $A_y = \{x | (x, y) \in A\}$ and

$$Y_{\text{crit}} = \{y : \xi \textit{ is a critical value of } f(\cdot, y) : A_y \mapsto Z\}.$$

Then $Y_{\rm crit}$ is a meager set in Y and, if (x,y) $\mapsto y : f^{-1}(\xi) \mapsto Y$ is proper, $Y_{\rm crit}$ is also closed. If $\operatorname{ind}\frac{\partial f}{\partial x} \le -m < 0$ *on $f^{-1}(\xi)$, then (2(α)) implies (2(β)) and*

$$Y_{\rm crit} = \{y : \xi \in f(A_y, y)\}$$

has codimension less than or equal to m in Y. [Note $Y_{\rm crit}$ is meager if and only if $\operatorname{codim} Y_{\rm crit} \ge 1$*].*

Remark 2.9. The usual hypothesis is that ξ is a regular value of f, so (2(α)) holds. If (2(β)) holds at some point then $\operatorname{ind}(\frac{\partial f}{\partial x}) \le -m$ at this point, since $\operatorname{codim}\mathcal{R}\big(\frac{\partial f}{\partial x}\big) \ge \dim\{\frac{\mathcal{R}(Df)}{\mathcal{R}\left(\frac{\partial f}{\partial x}\right)}\}$. If $\operatorname{ind}\big(\frac{\partial f}{\partial x}\big) \le -m$ and (2(α)) holds, then (2(β)) also holds. Thus (2(β)) is more general for the case of negative index.

3. Continuous dependence of a finite system of eigenvalues

Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded, C^4-regular region. It is well known that the problem (1.1) possesses an enumerable sequence of negative eigenvalues $0 > \lambda_0 > \lambda_1 > \cdots \to -\infty$. In this section, we will show the continuous dependence of the eigenvalues of

$$\begin{gathered} (\Delta^2 + \lambda)v = 0 \quad \text{in } h(\Omega) \\ v = \frac{\partial v}{\partial N} = 0 \quad \text{on } \partial h(\Omega) \end{gathered} \tag{3.1}$$

with respect to variation of $h \in \operatorname{Diff}^4(\Omega)$. More precisely, we show that a part of spectrum of (3.1) consisting of finite system of eigenvalues changes continuously with h.

To accomplish that, we use the theory described in section 2.2 rewriting(3.1) as

$$\begin{gathered} h^*(\Delta^2 + \lambda){h^*}^{-1}u = 0 \quad \text{in } \Omega \\ u = \frac{\partial u}{\partial N} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.2}$$

where $u = h^*v$. Observe that the problem (3.1) is equivalent to (3.2). In fact, v is a solution of (3.1) if and only if u is a solution of

$$\begin{gathered} h^*(\Delta^2 + \lambda){h^*}^{-1}u = 0 \quad \text{in } \Omega \\ u = h^*\frac{\partial}{\partial N_{h(\Omega)}}{h^*}^{-1}u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.3}$$

where $N_{h(\Omega)}$ is the normal of the region $h(\Omega)$ defined by (2.2). Now,

$$\begin{aligned} \Big(h^*\frac{\partial}{\partial N_{h(\Omega)}}{h^*}^{-1}u\Big)(x) &= \sum_{i=1}^n \Big(h^*\frac{\partial}{\partial y_i}{h^*}^{-1}u\Big)(x)(N_{h(\Omega)})_i(h(x)) \\ &= \sum_{i,j=1}^n b_{ij}(x)\frac{\partial u}{\partial x_j}(x)(N_{h(\Omega)})_i(h(x)) \\ &= N_{h(\Omega)}(h(x))b(x)\nabla u(x) \end{aligned}$$

where $b_{ij}(x) = (h_x^{-1})_{ji}(x)$ [the i,j-th entry in the transposed inverse of the Jacobian Matrix of h] and $b(x) = (b_{ij})(x)$ with $x \in \Omega$. Since $u = 0$ on $\partial\Omega$ we have

$$\nabla u = \frac{\partial u}{\partial N}N \quad \text{on } \partial\Omega.$$

Observe that, for all $x \in \Omega$, $b(x)$ is a non singular matrix and $b(x)N(x)$ is in the direction of $N_{h(\Omega)}(h(x))$. Thus

$$h^* \frac{\partial}{\partial N_{h(\Omega)}} {h^*}^{-1} u = 0 \quad \text{on } \partial\Omega \iff \frac{\partial u}{\partial N} = 0 \qquad \text{on } \partial\Omega,$$

that is, v is solution of (3.1) if and only if $u = h^* v$ is solution of (3.2).

The next Lemma is essential in the proof of the continuous dependence of a finite system of eigenvalues of (3.1) with respect to variation of $h \in \mathrm{Diff}^4(\Omega)$.

Lemma 3.1. *Given $h_0 \in \mathrm{Diff}^4(\Omega)$, there exists a neighbourhood V_0 of h_0 in $\mathrm{Diff}^4(\Omega)$ such that, for all $h \in V_0$ and $u \in H^4 \cap H_0^2(\Omega)$*

$$\|(h^* \Delta^2 {h^*}^{-1} - h_0^* \Delta^2 {h_0^*}^{-1})u\|_{L^2(\Omega)} \leq \epsilon(h) \|h_0^* \Delta^2 {h_0^*}^{-1} u\|_{L^2(\Omega)}$$

with $\epsilon(h) \to 0$ as $h \to h_0$ in $C^4(\Omega, \mathbb{R}^n)$.

Proof. It is clearly sufficient to consider the case $h_0 = i_\Omega$. We have

$$h^* \frac{\partial}{\partial y_i} {h^*}^{-1} u(x) = \frac{\partial}{\partial y_i}(u \circ h^{-1})(h(x)) = \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x) (h_x^{-1})_{ji}(x) = \sum_{j=1}^n b_{ij}(x) \frac{\partial u}{\partial x_j}(x)$$

where $b_{ij}(x) = (h_x^{-1})_{ji}(x)$, that is, $b_{ij}(x)$ is the i,j-th entry in the transposed inverse of the Jacobian matrix of $h_x = (\frac{\partial h_i}{\partial x_j})_{i,j=1}^n$. Therefore,

$$\begin{aligned}
h^* \frac{\partial^2}{\partial y_i^2} {h^*}^{-1} u(x) &= \sum_{k=1}^n b_{ik}(x) \frac{\partial}{\partial x_k} \Big(\sum_{j=1}^n b_{ij}(x) \frac{\partial u}{\partial x_j}(x) \Big) \\
&= \sum_{k=1}^n b_{ik}(x) \sum_{j=i}^n \Big[\frac{\partial}{\partial x_k}(b_{ij}(x)) \frac{\partial u}{\partial x_j}(x) + b_{ij}(x) \frac{\partial^2 u}{\partial x_k \partial x_j}(x) \Big] \\
&= \sum_{j,k=1}^n b_{ik}(x) b_{ij}(x) \frac{\partial^2 u}{\partial x_k \partial x_j}(x) + \sum_{j,k=1}^n b_{ik}(x) \frac{\partial}{\partial x_k}(b_{ij}(x)) \frac{\partial u}{\partial x_j}(x),
\end{aligned}$$

$$\begin{aligned}
& h^* \frac{\partial^3}{\partial y_s \partial {y_i}^2} {h^*}^{-1} u(x) \\
&= \sum_{l=1}^n b_{sl}(x) \frac{\partial}{\partial x_l} \Big(\frac{\partial^2}{\partial y_i^2} (u \circ h^{-1})(h(x)) \Big) \\
&= \sum_{l,j,k=1}^n b_{sl}(x) b_{ik}(x) b_{ij}(x) \frac{\partial^3 u}{\partial x_l \partial x_j \partial x_k}(x) \\
&\quad + \sum_{l,j,k=1}^n b_{sl}(x) b_{ik}(x) \frac{\partial}{\partial x_k}(b_{ij}(x)) \frac{\partial^2 u}{\partial x_l \partial x_j}(x) \\
&\quad + \sum_{l,j,k=1}^n \Big[b_{ij}(x) \frac{\partial}{\partial x_k}(b_{ik}(x)) + b_{ik}(x) \frac{\partial}{\partial x_k}(b_{ij}(x)) \Big] b_{sl}(x) \frac{\partial^2 u}{\partial x_k \partial x_j}(x) \\
&\quad + \sum_{l.j.k=1}^n \Big[\frac{\partial}{\partial x_l}(b_{ik}(x)) \frac{\partial}{\partial x_k}(b_{ij}(x)) + b_{ik}(x) \frac{\partial^2}{\partial x_l \partial x_k}(b_{ij}(x)) \Big] b_{sl}(x) \frac{\partial u}{\partial x_j}(x),
\end{aligned}$$

$$
\begin{aligned}
h^*\frac{\partial^4}{\partial {y_s}^2\partial {y_i}^2}h^{*-1}u(x) &= \frac{\partial}{\partial y_s}\frac{\partial^3}{\partial y_s\partial {y_i}^2}(u\circ h^{-1})(h(x))\\
&= \sum_{r=1}^n b_{rs}(x)\frac{\partial}{\partial x_r}\Big(\frac{\partial^3}{\partial y_s\partial {y_i}^2}(u\circ h^{-1})(h(x))\Big)\\
&= \frac{\partial^4 u}{\partial {x_s}^2\partial {x_i}^2}(x) + L^h_{si}(u)(x)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
L^h_{si}(u)(x) = {}&\Big(b_{ss}(x)^2 b_{ii}(x)^2 - 1\Big)\frac{\partial^4 u}{\partial {x_s}^2\partial {x_i}^2}(x)\\
&+\sum_{r,l,j,k=1}^n (1-\delta_{s,r,l}\delta_{i,j,k})b_{sl}(x)b_{sr}(x)b_{ik}(x)b_{ij}(x)\frac{\partial^4 u}{\partial x_r\partial x_l\partial x_j\partial x_k}(x)\\
&+\sum_{r,l,j,k=1}^n \frac{\partial}{\partial x_r}\Big[b_{sl}(x)b_{ik}(x)b_{ij}(x)\Big]b_{sr}(x)\frac{\partial^3 u}{\partial x_l\partial x_j\partial x_k}(x)\\
&+\sum_{r,l,j,k=1}^n \frac{\partial}{\partial x_l}\Big[b_{ik}(x)b_{ij}(x)\Big]b_{sr}(x)b_{sl}(x)\frac{\partial^3 u}{\partial x_r\partial x_k\partial x_j}(x)\\
&+\sum_{r,l,j,k=1}^n b_{sr}(x)b_{sl}(x)b_{ik}(x)\frac{\partial}{\partial x_k}(b_{ij}(x))\frac{\partial^3 u}{\partial x_r\partial x_l\partial x_j}(x)\\
&+\sum_{r,l,j,k=1}^n b_{sr}(x)\frac{\partial}{\partial x_r}\Big[\frac{\partial}{\partial x_l}\Big(b_{ij}(x)b_{ik}(x)\Big)b_{sl}(x)\Big]\frac{\partial^2 u}{\partial x_k\partial x_j}(x)\\
&+\sum_{r,l,j,k=1}^n b_{sr}(x)b_{sl}(x)\frac{\partial}{\partial x_l}\Big(b_{ik}(x)\frac{\partial}{\partial x_k}(b_{ij}(x))\Big)\frac{\partial^2 u}{\partial x_r\partial x_j}(x)\\
&+\sum_{r,l,j,k=1}^n b_{sr}(x)\frac{\partial}{\partial x_r}\Big[b_{sl}(x)b_{ik}(x)\frac{\partial}{\partial x_r}(b_{ij}(x))\Big]\frac{\partial^2 u}{\partial x_l\partial x_j}(x)\\
&+\sum_{r,l,j,k=1}^n b_{sr}(x)\frac{\partial}{\partial x_r}\Big[\frac{\partial}{\partial x_l}\Big(b_{ik}(x)\frac{\partial}{\partial x_k}(b_{ij}(x))\Big)b_{sl}(x)\Big]\frac{\partial u}{\partial x_j}(x).
\end{aligned}
$$

Thus

$$
h^*\Delta^2 h^{*-1}u = \Delta^2 u + L^h u \tag{3.4}
$$

with $L^h u = \sum_{s,i=1}^n L^h_{is}u$. Since $b_{ij}\to\delta_{ij}$ in $C^4(\Omega,\mathbb{R}^n)$ when $h\to i_\Omega$ in $C^4(\Omega,\mathbb{R}^n)$, the coefficients of L^h go to 0 uniformly in x as $h\to i_\Omega$ in $C^4(\Omega,\mathbb{R}^n)$. It follows that,

$$
\|L^h u\|_{L^2(\Omega)} \le \epsilon(h)\|\Delta^2 u\|_{L^2(\Omega)} \tag{3.5}
$$

where $\epsilon(h)$ goes to 0 as $h\to i_\Omega$ in $C^4(\Omega,\mathbb{R}^n)$. $\square$

Theorem 3.2. *Let λ be an eigenvalue of (3.1) in Ω with multiplicity m and I an interval such that λ is the unique element of the spectrum in I. Then, for any $\eta>0$ and any interval $J\subset I$ with $\lambda\in J$, there exists a neighbourhood $\mathcal{V}$ of i_Ω in $\mathrm{Diff}^k(\Omega)$ ($k\ge 4$) such that if $h\in\mathcal{V}$ there exist exactly m eigenvalues (counted with multiplicity) $\lambda_1(h),\dots,\lambda_m(h)$ of (3.1) in J depending continuously of h with $\lambda_i(i_\Omega)=\lambda$ for all $1\le i\le m$. Moreover, the projection $P(h)$ of $L^2(\Omega)$ onto the sum* *of the associated eigenspaces of* $\lambda_1(h), \dots, \lambda_m(h)$ *satisfies* $\|P(h) - P(i_\Omega)\| < \eta$ *in* $\mathcal{V}$.

Proof. To prove this Theorem we use the theory of perturbation for unbounded operators developed in chapter IV of [9]. By Lemma 3.1, proved above, $A_h = h^* \Delta^2 h^{*-1} - \Delta^2$ is Δ^2-bounded (that is, $D(A_h) = D(\Delta^2) = H^4 \cap H_0^2(\Omega)$ in $L^2(\Omega)$ and $\|A_h u\|_{L^2(\Omega)} \le \epsilon(h) \|\Delta^2 u\|_{L^2(\Omega)}$ for all $u \in H^4 \cap H_0^2(\Omega)$ with $\epsilon(h) \to 0$ as $h \to i_\Omega$). Moreover, there exists a neighbourhood V of i_Ω in $\mathrm{Diff}^k(\Omega)$ such that $\epsilon(h) < 1$ for all $h \in V$. Therefore, by Theorem IV 2.14 of [9], we have that $A_h + \Delta^2 = h^* \Delta^2 h^{*-1}$ is a closed operator in $L^2(\Omega)$ with

$$\widehat{\delta}(h^* \Delta^2 h^{*-1}, \Delta^2) \le (1 - \epsilon(h))^{-1} \epsilon(h) \quad \forall h \in V \tag{3.6}$$

where $\widehat{\delta}$ is the *gap* between closed operators defined in [9]. If J is an open interval satisfying the hypotheses above, we can find a closed curve γ in $\mathbb{C}$ with *int* $\gamma \cap \mathbb{R} = J$. Since $\bar{\delta}(h^* \Delta^2 h^{*-1}, \Delta^2) \to 0$ as $h \to i_\Omega$ in C^k, it follows, from Theorem IV 3.16 of [9] that, if $\|i_\Omega - h\|_{C^k(\Omega)}$ is small enough, $h^* \Delta^2 h^{*-1}$ posses exactly m eigenvalues $\lambda_1(h), \dots, \lambda_m(h)$ counted with multiplicity in the interior of γ. Being real, they must lie in J as required. Furthermore, by the same result, it follows that $P(h) \to P(i_\Omega)$ in norm as $h \to i_\Omega$ in C^k as asserted. $\square$

Corollary 3.3. *The set*

$$\begin{aligned}\mathcal{D}_m = \big\{ h \in \mathrm{Diff}^{\,4}(\Omega) : -M \textit{ is not eigenvalue of (1.1) in } h(\Omega) \\ \textit{and all the eigenvalues } \lambda \in (-M, 0) \textit{ in } h(\Omega) \textit{ are simple} \big\}\end{aligned}$$

is open in $\mathrm{Diff}^4(\Omega)$ *for all* $M \in \mathbb{N}$.

Proof. Let $h_0 \in \mathcal{D}_M$ and $\lambda_1, \dots, \lambda_k$ be the (simple) eigenvalues of Δ^2 in $h_0(\Omega)$ greater $-M$. Let also γ be the circle of radius M with center in the origin.

From the previous Theorem, for each $1 \le i \le k$ there exists a neighborhood $V_i \subset \mathrm{Diff}^4(\Omega)$ of h_0 and continuous functions $\Lambda_i : V_i \to (-M, 0)$ such that $\Lambda_i(h)$ is a simple eigenvalue of $h^* \Delta^2 h^{*-1}$ for any $h \in V_i$ with $\Lambda_i(h_0) = \lambda_i$ and the sets $\Lambda_i(V_i)$ are pairwise disjoint. Define $V = \bigcap_{i=0}^k V_i$ and observe that $\forall h \in V$, $h^* \Delta^2 h^{*-1}$ has k eigenvalues greater $-M$, which are all simple. Therefore, $\mathcal{D}_M$ is open. $\square$

4. Perturbation of simple eigenvalues

Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded, C^4-regular region and λ_0 a simple eigenvalue of the equation

$$\begin{aligned}(\Delta^2 + \lambda) u &= 0 \quad \text{in } \Omega \\ u = \frac{\partial u}{\partial N} &= 0 \quad \text{on } \partial\Omega\end{aligned} \tag{4.1}$$

with corresponding eigenvalue u_0, with $\int_\Omega u_0^2 = 1$.

Consider the map $F : H^4 \cap H_0^2(\Omega) \times \mathbb{R} \times \mathrm{Diff}^4(\Omega) \to L^2(\Omega) \times \mathbb{R}$ defined by

$$F(u, \lambda, h) = (h^* (\Delta^2 + \lambda) h^{*-1} u, \int_\Omega u^2 \det h').$$

Then F is analytic by section 2.2 and $F(u, \lambda, h) = (0, 1)$ if and only if $v = h^{*-1} u \in H^4 \cap H_0^2(h(\Omega))$ is a solution of (3.1) with $\int_{h(\Omega)} v^2 = 1$.

Observe that $F(u_0,\lambda_0,i_\Omega)=(0,1)$ and the operator

$$\frac{\partial F}{\partial(u,\lambda)}(u_0,\lambda_0,i_\Omega): H^4\cap H_0^2(\Omega)\times\mathbb{R}\to L^2(\Omega)\times\mathbb{R}$$
$$(\dot u,\dot\lambda)\to\Big((\Delta^2+\lambda_0)\dot u+\dot\lambda u_0, 2\int_\Omega u_0\dot u\Big)$$

is an isomorphism. In fact, since λ_0 is a simple eigenvalue of Fredholm operator with index zero

$$\Delta^2: H^4\cap H_0^2(\Omega)\to L^2(\Omega),$$

we have

$$\mathcal{R}(\Delta^2+\lambda_0)\oplus[u_0]=L^2(\Omega).$$

So, given $(f,\alpha)\in L^2(\Omega)\times\mathbb{R}$ there exists a unique

$$(\dot u,\dot\lambda)=(\frac{\alpha}{2}u_0+w,\int_\Omega u_0 f)\in H^4\cap H_0^2(\Omega)\times\mathbb{R},$$

where $w\in H^4\cap H_0^2(\Omega)$ is a solution of

$$(\Delta^2+\lambda_0)w=f-\dot\lambda u_0$$

with $w\perp u_0$, such that

$$\frac{\partial F}{\partial(u,\lambda)}(u_0,\lambda_0,i_\Omega)(\dot u,\dot\lambda)=((\Delta^2+\lambda_0)\dot u+\dot\lambda u_0, 2\int_\Omega u_0\dot u)=(f,\alpha).$$

It follows that $\frac{\partial F}{\partial(u,\lambda)}(u_0,\lambda_0,i_\Omega)$ is a continuous bijection, therefore an isomorphism, by the Closed Graph Theorem. Thus, by the Implict Function Theorem there exists a neighbourhood V of i_Ω in $\mathrm{Diff}^4(\Omega)$ and analytic functions $u(h)$ and $\lambda(h)$ in V such that $F(u(h),\lambda(h),h)=(0,1)$ for all $h\in V$. In fact, we can say more, $\frac{\partial F}{\partial(u,\lambda)}(u(h),\lambda(h),h)$ is an isomorphism for all $h\in V$, that is, $\lambda(h)$ is simple eigenvalue in V. Therefore, we have the following result.

Proposition 4.1. *Let λ_0 be a simple eigenvalue of (4.1). Then, there exists a neighbourhood V of i_Ω in $\mathrm{Diff}^4(\Omega)$ and analytic functions $u(h)$ and $\lambda(h)$ from V into $H^4\cap H_0^2(\Omega)$ and $\mathbb{R}$ respectively, satisfying (3.2) for all $h\in V$. Moreover, $\lambda(h)$ is a simple eigenvalue for all $h\in V$ with $\lambda(i_\Omega)=\lambda_0$.*

5. The generic simplicity of the eigenvalues

Let $\mathcal{P}$ be a property depending of a parameter $x\in X$, where X is a Baire topological space. We say that $\mathcal{P}$ is generic (in x) if it holds for all x in a residual set of X.

In our application, X will be the class of regions C^4-diffeomorphic to a fixed region Ω_0 of class C^4, that is,

$$X=\{h(\Omega_0): h\in\mathrm{Diff}^4(\Omega)\}.$$

We introduce a topology in this set by defining a (sub-basis of) the neighborhoods of a given Ω by

$$\{h(\Omega);\|h-i_\Omega\|_{C^4(\Omega,\mathbb{R}^n)}<\varepsilon,\text{ with }\varepsilon>0\text{ suficiently small}\}.$$

When $\|h-i_\Omega\|_{C^4(\Omega,\mathbb{R}^n)}$ is small, h is a C^4 imbedding of Ω in $\mathbb{R}^n$, a C^4 diffeomorphism to its image $h(\Omega)$. Michelleti [11] shows this topology is metrizable, and the set of regions C^4-diffeomorphic to Ω may be considered a separable metric space of Baire.

In fact, we will prove in our application that the property $\mathcal{P}$ holds for all $h \in \operatorname{Diff}^4(\Omega)$ except a meager set $\mathcal{F} \subset \operatorname{Diff}^4(\Omega)$. However, the set $\mathcal{F}$ of imbeddings excluded will be always defined by properties of then image (therefore, it is invariant by composition with C^4-diffeomorphisms of Ω_0 into Ω_0). This imply that (see [8]) the set of the regions defined by $\mathcal{F}$ are also meager in our space X.

In this section, we show that, generically in the set of open, connected, bounded, C^4-regular regions $\Omega \subset \mathbb{R}^n$, $n \geq 2$, all eigenvalues of (1.1) are simple. In order to apply transversality arguments, we first show that our generic property is equivalent of zero being a regular value for an appropriate mapping. More precisely, we have

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, connected, bounded, C^4-regular region. Then, all eigenvalues of (1.1) are simple if and only if zero is a regular value of the mapping $\phi : H^4 \cap H_0^2(\Omega) \times \mathbb{R} \to L^2(\Omega)$ defined by*

$$\phi(u, \lambda) = (\Delta^2 + \lambda)u.$$

Proof. In fact, 0 is a regular value of ϕ if and only if for all $(u, \lambda) \in H^4 \cap H_0^2(\Omega) \times \mathbb{R}$ with $\phi(u, \lambda) = 0$

$$D\phi(u, \lambda)(\dot{u}, \dot{\lambda}) = (\Delta^2 + \lambda)\dot{u} + \dot{\lambda}u$$

is onto. Now, since the operator $(\Delta^2 + \lambda) : H^4 \cap H_0^2(\Omega) \to L^2(\Omega)$ is selfadjoint and Fredholm with index zero we have

$$L^2(\Omega) = \mathcal{R}(\Delta^2 + \lambda) \oplus \mathcal{N}(\Delta^2 + \lambda).$$

Thus, $D\phi(u, \lambda)$ is onto if and only if

$$\mathcal{R}(\Delta^2 + \lambda) \oplus [u] = L^2(\Omega),$$

that is, if and only if λ is simple eigenvalue of (1.1). $\square$

Consider the map $F : H^4 \cap H_0^2(\Omega) \times \mathbb{R} \times \operatorname{Diff}^4(\Omega) \to L^2(\Omega)$ defined by

$$F(u, \lambda, h) = h^*(\Delta^2 + \lambda){h^*}^{-1}u.$$

Observe that zero is regular value of the map

$$(u, \lambda) \to F(u, \lambda, h) \tag{5.1}$$

for $h \in \operatorname{Diff}^4(\Omega)$ if and only if $0 \in L^2(h(\Omega))$ is regular value of

$$\phi_h : H^4 \cap H_0^1(h(\Omega)) \times \mathbb{R} \to L^2(h(\Omega))$$

(In fact, h^* is an isomorphism.) So, since that the problem (3.1) is equivalent to (3.2), we have by Proposition 5.1 that all eigenvalues of (1.1) in $h(\Omega)$ are simple if and only if $0 \in L^2(\Omega)$ is regular value of the map (5.1). Therefore, to prove the generic simplicity of eigenvalues of the Dirichlet Problem for Bilaplacian it is sufficient to show zero is regular value of (5.1) for most $h \in \operatorname{Diff}^4(\Omega)$, or to show zero is regular value of F verifying the hypotheses of Theorem Transversality. We try to do that and fail. For certain $h \in \operatorname{Diff}^4(\Omega)$, zero may be a critical value of F. But the critical point has special properties, namely, if (u, λ, h) is a critical point of F there must exist another eigenfunction v of (4.1) in $h(\Omega)$ such that $\Delta u \Delta v \equiv 0$ on $\partial h(\Omega)$. Then, we have to show that the special properties can only occur in a "exceptional" set of regions diffeomorphic to Ω. To do this, we consider the mapping

$$\begin{aligned} Q : &H^4 \cap H_0^1(\Omega)^2 \times \mathbb{R} \times \operatorname{Diff}^4(\Omega) \\ &\to L^2(\Omega) \times H^{\frac{5}{2}}(\partial\Omega) \times L^2(\Omega) \times H^{\frac{5}{2}}(\partial\Omega) \times L^1(\partial\Omega) \end{aligned}$$

defined by

$$Q(u,v,\lambda,h)=\Big(h^*(\Delta^2+\lambda)h^{*-1}u, h^*\frac{\partial}{\partial N}h^{*-1}u, h^*(\Delta^2+\lambda)h^{*-1}v,\\ h^*\frac{\partial}{\partial N}h^{*-1}v, \Big\{h^*\Delta h^{*-1}uh^*\Delta h^{*-1}v\Big\}\Big|_{\partial\Omega}\Big)$$

and then we use the condiction $2(\beta)$ of Transversality Theorem.
Observe that $(u,v,\lambda,h)\in Q^{-1}(0,0,0,0,0)$ if and only if u,v are eigenfunctions of (4.1) in $h(\Omega)$ satisfying $\Delta u\Delta v\equiv 0$ in $\partial h(\Omega)$. So, we show there exists an open dense set in $\mathrm{Diff}^4(\Omega)$ such that the restriction of F on this set has zero as regular value, proving the result.

Remark 5.2. Without loss of generality, we can work with C^5-regular instead of C^4-regular regions. In fact, by Corollary 3.3, given $M\in\mathbb{N}$ the set $\mathcal{D}_M$ is open in $\mathrm{Diff}^4(\Omega)$. If we prove that this set is also dense in $\mathrm{Diff}^4(\Omega)$, the result follows taking intersection in $M\in\mathbb{N}$. Now, to prove density we can work with more regular regions since C^4-regions can be aproximated to C^k-regions with $k\geq 5$. This is necessary because we need to use Theorems of regularity to Elliptical Equations in ours proofs.

The next Lemma shows that the eigenfunctions u of (4.1) can not have $\Delta u\equiv 0$ genericaly in the set of the C^5-regular region with $n\geq 2$ in a nonempty open set of the boundary fixed. This one is necessary in the proof of the Lemma 5.4.

Lemma 5.3. *Let $\Omega\subset\mathbb{R}^n$ be an open, connected, bounded, C^5-regular region with $n\geq 2$ and $J\subset\partial\Omega$ a nonempty open set. Consider the analytic map*

$$G: B_M\times(-M,0)\times \mathrm{Diff}^{\,5}(\Omega)\to L^2(\Omega)\times H^{3/2}(J)$$

defined by

$$G(u,\lambda,h)=\Big(h^*(\Delta^2+\lambda)h^{*-1}u, h^*\Delta h^{*-1}u\Big|_J\Big)$$

where $B_M=\{u\in H^4\cap H^2_0(\Omega)-\{0\}\mid \|u\|\leq M\}$. Then the set

$$C^J_M=\{h\in \mathrm{Diff}^{\,5}(\Omega)\mid (0,0)\in G(B_M\times(-M,0),h)\}$$

is meager and closed in $\mathrm{Diff}^5(\Omega)$.

Proof. We apply the Transversality Theorem. By section 2.2, we have that the mapping G is analytic in h. It is clearly also analytic in the other variables. We verify hypothesis (3) of the Trasversality Theorem showing the mapping $(u,\lambda,h)\to h: G^{-1}(0,0)\to \mathrm{Diff}^5(\Omega)$ is proper. Let $\{(u_n,\lambda_n,h_n)\}_{n\in\mathbb{N}}\subset G^{-1}(0,0)$ with $h_n\to h_0=i_\Omega$ [the general case is analogous]. Since $\{u_n\}_{n\in\mathbb{N}}\subset B_M$ and $\{\lambda_n\}_{n\in\mathbb{N}}\subset(-M,0)$, we can suppose, by compactness, that there exist $u\in H^2_0(\Omega)$ and $\lambda\in(-M,0)$ such that $u_n\to u$ in $H^2_0(\Omega)$ and $\lambda_n\to\lambda$ in $(-M,0)$. During the proof of Lemma 3.1 we proved that $h^*\Delta^2h^{*-1}u=\Delta^2u+L^hu$ for all $h\in \mathrm{Diff}^5(\Omega)$ where L^hu is small when h is closed to i_Ω. So, we have, for all $v\in H^2_0(\Omega)$

$$\begin{aligned}0&=\lim_{n\to\infty}\int_\Omega v\{h_n^*(\Delta^2+\lambda_n)h_n^{*-1}u_n\}\\ &=\lim_{n\to\infty}\int_\Omega v\{(\Delta^2+\lambda_n)u_n+L^{h_n}u_n\}\\ &=\lim_{n\to\infty}\Big[\int_\Omega \Delta v\Delta u_n+\int_\Omega v\{\lambda_nu_n+L^{h_n}u_n\}\Big]\end{aligned}$$

$$= \int_\Omega \{\Delta v \Delta u + \lambda v u\},$$

since $u_n \to u$ in $H_0^2(\Omega)$, $\lambda_n \to \lambda$ in $(-M, 0)$, $h_n \to i_\Omega$ in $\mathrm{Diff}^5(\Omega)$ and

$$\Big| \int_\Omega v L^{h_n} u_n \Big| \le \|v\|_{L^2(\Omega)} \|L^{h_n} u_n\|_{L^2(\Omega)} \le M\epsilon(h_n) \|v\|_{L^2(\Omega)},$$

by equation (3.5), where $\lim_{n\to\infty} \epsilon(h_n) = 0$. Thus, $u \in B_M$ is a weak, therefore strong solution of (1.1). Now, we will show that $\{u_n\}_{n\in\mathbb{N}}$ converges in $H^4 \cap H_0^2(\Omega)$ to $u \in H^4 \cap H_0^2(\Omega)$. In fact, for all n, $m \in \mathbb{N}$

$$\begin{aligned} \|\Delta^2(u_n - u_m)\|_{L^2(\Omega)} &= \|L^{h_n}(u_n - u_m) + (L^{h_n} - L^{h_m})u_m \\ &\quad + \lambda_n(u_n - u_m) + (\lambda_n - \lambda_m)u_m\|_{L^2(\Omega)} \end{aligned}$$

following that

$$\|\Delta^2(u_n - u_m)\|_{L^2(\Omega)} \to 0 \text{ as } n, m \to +\infty. \tag{5.2}$$

Since that there exists $c_0 > 0$ such that

$$\|\Delta^2(u_n - u_m)\|_{L^2(\Omega)} \ge c_0 \|u_n - u_m\|_{H^4\cap H_0^2(\Omega)},$$

we have, by (5.2), that $\{u_n\}_{n\in\mathbb{N}}$ converges in $H^4 \cap H_0^2(\Omega)$ to $u \in H^4 \cap H_0^2(\Omega)$. This proves that the mapping $(u, \lambda, h) \to h : G^{-1}(0,0) \to \mathrm{Diff}^5(\Omega)$ is proper.

Let $(u, \lambda, h) \in G^{-1}(0,0)$. By section 2.3, we can suppose that $h = i_\Omega$. The partial derivative $\partial G/\partial(u,\lambda)(u, \lambda, i_\Omega)$ defined from $H^4 \cap H_0^2(\Omega) \times \mathbb{R}$ into $L^2(\Omega) \times H^{3/2}(J)$ is given by

$$\begin{aligned} \frac{\partial G}{\partial(u,\lambda)}(u, \lambda, i_\Omega)(\dot u, \dot\lambda) &= \Big(\frac{\partial G_1}{\partial u, \lambda)}(u, \lambda, i_\Omega)(\dot u, \dot\lambda), \frac{\partial G_2}{\partial(u,\lambda)}(u, \lambda, i_\Omega)(\dot u, \dot\lambda)\Big) \\ &= \Big((\Delta^2 + \lambda)\dot u + \dot\lambda u, \Delta \dot u\Big|_J\Big). \end{aligned}$$

Now, $DG(u, \lambda, i_\Omega)$ defined from $H^4 \cap H_0^2(\Omega) \times \mathbb{R} \times C^5(\Omega, \mathbb{R}^n)$ into $L^2(\Omega) \times H^{3/2}(J)$ can be computed by Theorem 2.4 like the Remark 2.5 and it is given by

$$DG(u, \lambda, i_\Omega)(\dot u, \dot\lambda, \dot h) = \Big(DG_1(u, \lambda, i_\Omega)(\dot u, \dot\lambda, \dot h), DG_2(u, \lambda, i_\Omega)(\dot u, \dot\lambda, \dot h)\Big)$$

where

$$\begin{aligned} DG_1(u, \lambda, i_\Omega)(\dot u, \dot\lambda, \dot h) &= (\Delta^2 + \lambda)(\dot u - \dot h \cdot \nabla u) + \dot h \cdot \nabla\Big[(\Delta^2 + \lambda)u\Big] + \dot\lambda u \\ &= (\Delta^2 + \lambda)(\dot u - \dot h \cdot \nabla u) + \dot\lambda u\,, \end{aligned}$$

$$\begin{aligned} DG_2(u, \lambda, i_\Omega)(\dot u, \dot\lambda, \dot h) &= \{\Delta(\dot u - \dot h \cdot \nabla u) + \dot h \cdot \nabla(\Delta u)\}\Big|_J \\ &= \Big\{\Delta(\dot u - \dot h \cdot \nabla u) + \frac{\partial \Delta u}{\partial N} \dot h \cdot N\Big\}\Big|_J \end{aligned}$$

since that $(\Delta^2 + \lambda)u = 0$ in Ω and $\Delta u|_{\partial\Omega} \equiv 0$. By [2], we have that $\Omega \subset \mathbb{R}^n$, C^5-regular implies that $u \in H^5(\Omega)$, so $\dot h \cdot \nabla u \in H^4(\Omega)$.

Now, we can easily see that the hypothesis (1) of the Transversality Theorem is satisfied. In fact $\frac{\partial G_1}{\partial(u,\lambda)}(u, \lambda, i_\Omega)$ is a Fredholm map, so we have that $\frac{\partial G_2}{\partial(u,\lambda)}(u, \lambda, i_\Omega)$ is a semi-Fredholm map with index $< +\infty$.

We now prove that (2β) also holds, that is, we show that

$$\dim\Big\{\frac{R(DG(u, \lambda, i_\Omega))}{R(\frac{\partial G}{\partial(u,\lambda)}(u, \lambda, i_\Omega))}\Big\} = \infty.$$

Suppose this is not true. Then there exist $\theta_1,\dots,\theta_m \in L^2(\Omega)\times H^{3/2}(J)$ such that for all $\dot h \in C^5(\Omega,\mathbb{R}^n)$ there exist $\dot u, \dot\lambda$ and $c_1,\dots,c_m$ such that

$$DG(u,\lambda,i_\Omega)(\dot u,\dot\lambda,\dot h) = \sum_{j=1}^m c_j\theta_j, \quad \theta_j = (\theta_j^1,\theta_j^2),$$

that is,

$$\Big((\Delta^2+\lambda)(\dot u - \dot h\cdot\nabla u) + \dot\lambda u, \Delta(\dot u - \dot h\cdot\nabla u) + \frac{\partial\Delta u}{\partial N}\dot h\cdot N\Big) = \sum_{j=1}^m c_j\theta_j. \tag{5.3}$$

Consider the operator $\mathcal{S}_{\Delta^2+\lambda} : L^2(\Omega)\to H^4\cap H_0^2(\Omega)$ defined by

$$v = \mathcal{S}_{\Delta^2+\lambda} f \quad \text{where } (\Delta^2+\lambda)v - f \in \mathcal{N}(\Delta^2+\lambda), v\perp\mathcal{N}(\Delta^2+\lambda).$$

Observe that operator $\mathcal{S}_{\Delta^2+\lambda}$ is well defined. In fact,

$$(\Delta^2+\lambda) : H^4\cap H_0^2(\Omega)\to L^2(\Omega)$$

is a Fredholm map with index zero. Then, we have that

$$\mathcal{R}(\Delta^2+\lambda)\oplus\mathcal{N}(\Delta^2+\lambda) = L^2(\Omega).$$

So, given $f = f_1+f_2 \in L^2(\Omega)$, $f_1\in\mathcal{R}(\Delta^2+\lambda)$ e $f_2\in\mathcal{N}(\Delta^2+\lambda)$, there exists unique $v\in H^4\cap H_0^2(\Omega)$ such that $(\Delta^2+\lambda)v = f_1$ with $v\perp\mathcal{N}(\Delta^2+\lambda)$.

Choosing $\dot h\in C^5(\Omega,\mathbb{R}^n)$ with $\dot h\equiv 0$ on $\partial\Omega - \{J\}$, we can solve the first component of equation (5.3) modulo the finite dimensional subspace $\mathcal{N}(\mathcal{S}_{\Delta^2+\lambda}) = \mathcal{N}(\Delta^2+\lambda)$, since $\Delta u|_J = 0$. In fact,

$$\dot u - \dot h\cdot\nabla u = \sum_{j=1}^l \xi_j u_j + \sum_{j=1}^m c_j\mathcal{S}_{\Delta^2+\lambda}\theta_j^1\,, \tag{5.4}$$

where $\{u_1,\dots,u_l\}$ is an orthonormal basis of $\mathcal{N}(\Delta^2+\lambda)$. Substituting (5.4) in the second component of (5.3) we obtain that

$$\frac{\partial\Delta u}{\partial N}\dot h\cdot N\big|_J$$

belongs to a finite dimensional subspace of $H^{3/2}(J)$ for each $\dot h\in C^5(\Omega,\mathbb{R}^n)$ with $\dot h\equiv 0$ on $\partial\Omega - J$. But this can only occur, in $\dim\Omega\geq 2$, if

$$\frac{\partial\Delta u}{\partial N}\big|_J \equiv 0. \tag{5.5}$$

So, the solution u satisfies the hypothesis of the Theorem 2.3 which implies $u\equiv 0$. Since $u\in H^4\cap H_0^2(\Omega)-\{0\}$ we have a contradiction, proving the Lemma. □

Lemma 5.4. *Let $\Omega\subset\mathbb{R}^n$ be an open, connected, bounded, C^5-regular region with $n\geq 2$. Consider the analytic mapping*

$$\begin{aligned} Q :& B_M\times B_M\times(-M,0)\times D_M \\ &\to L^2(\Omega)\times H^{\frac52}(\partial\Omega)\times L^2(\Omega)\times H^{\frac52}(\partial\Omega)\times L^1(\partial\Omega)\end{aligned}$$

defined by

$$\begin{aligned} Q(u,v,\lambda,h) = \Big(& h^*(\Delta^2+\lambda)h^{*-1}u, h^*\frac{\partial}{\partial N}h^{*-1}u, h^*(\Delta^2+\lambda)h^{*-1}v, h^*\frac{\partial}{\partial N}h^{*-1}v, \\ &\Big\{h^*\Delta h^{*-1}u h^*\Delta h^{*-1}v\Big\}\Big|_{\partial\Omega}\Big)\end{aligned}$$

where $B_M = \{u \in H^4 \cap H_0^1(\Omega) - \{0\} \mid \|u\| \le M\}$ *and* $D_M = \mathrm{Diff}^5(\Omega) - C_M^{\partial\Omega}$. *[*$C_M^{\partial\Omega}$ *is the meager and closed set that exists by Lemma 5.3]. Then, the set*

$$E_M = \{h \in D_M \mid (0,0,0,0,0) \in Q(B_M \times B_M \times (-M,0), h)\}$$

is meager and closed in $\mathrm{Diff}^5(\Omega)$.

Proof. We apply the Transversality Theorem. The hypotesis (3) can hold like in Lemma 5.3. Let $(u, v, \lambda, h) \in Q^{-1}(0,0,0,0,0)$. By "change of origin", we can suppose without loss of generality that $h = i_\Omega$.

The partial derivative $\partial Q(u, v, \lambda, i_\Omega)/\partial(u, v, \lambda)$ defined from $H^4 \cap H_0^1(\Omega) \times H^4 \cap H_0^1(\Omega) \times \mathbb{R}$ into $L^2(\Omega) \times H^{\frac{5}{2}}(\partial\Omega) \times L^2(\Omega) \times H^{\frac{5}{2}}(\partial\Omega) \times L^1(\partial\Omega)$ is given by

$$\begin{aligned}
&\frac{\partial Q}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\cdot)\\
&= \Big(\frac{\partial Q_1}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\cdot), \frac{\partial Q_2}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\cdot),\\
&\quad \frac{\partial Q_3}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\cdot), \frac{\partial Q_4}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\cdot), \frac{\partial Q_5}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\cdot)\Big)\,;
\end{aligned}$$

$$\begin{aligned}
\frac{\partial Q_1}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda) &= (\Delta^2+\lambda)\dot u + \dot\lambda u\,,\\
\frac{\partial Q_2}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda) &= \frac{\partial}{\partial N}\dot u\,,\\
\frac{\partial Q_3}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda) &= (\Delta^2+\lambda)\dot v + \dot\lambda v\,,\\
\frac{\partial Q_4}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda) &= \frac{\partial}{\partial N}\dot v\,,\\
\frac{\partial Q_5}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda) &= \big\{\Delta u\Delta\dot v + \Delta v\Delta\dot u\big\}\Big|_{\partial\Omega}\,.
\end{aligned}$$

Now, $DQ(u, v, \lambda, i_\Omega)$ defined from $H^4 \cap H_0^1(\Omega) \times H^4 \cap H_0^1(\Omega) \times \mathbb{R} \times C^5(\Omega, \mathbb{R}^n)$ into $L^2(\Omega) \times H^{\frac{5}{2}}(\partial\Omega) \times L^2(\Omega) \times H^{\frac{5}{2}}(\partial\Omega) \times L^1(\partial\Omega)$ can be computed by Theorems 2.4 and 2.7 and is given by

$$\begin{aligned}
DQ(u,v,\lambda,i_\Omega)(\cdot) = \Big(&DQ_1(u,v,\lambda,i_\Omega)(\cdot), DQ_2(u,v,\lambda,i_\Omega)(\cdot),\\
&DQ_3(u,v,\lambda,i_\Omega)(\cdot), DQ_4(u,v,\lambda,i_\Omega)(\cdot), DQ_5(u,v,\lambda,i_\Omega)(\cdot)\Big)
\end{aligned}$$

$$\begin{aligned}
DQ_1(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda,\dot h) &= (\Delta^2+\lambda)(\dot u - \dot h\cdot\nabla u) + \dot\lambda u\\
DQ_2(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda,\dot h) &= \frac{\partial}{\partial N}(\dot u - \dot h\cdot\nabla u) + \dot h\cdot\nabla\big(\frac{\partial u}{\partial N}\big) - \nabla u\cdot\nabla(\dot h\cdot N)\\
&= \frac{\partial}{\partial N}(\dot u - \dot h\cdot\nabla u) + \dot h\cdot N\Delta u\\
DQ_3(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda,\dot h) &= (\Delta^2+\lambda)(\dot v - \dot h\cdot\nabla v) + \dot\lambda v\\
DQ_4(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda,\dot h) &= \frac{\partial}{\partial N}(\dot v - \dot h\cdot\nabla v) + \dot h\cdot N\Delta v
\end{aligned}$$

since $\nabla u = \frac{\partial u}{\partial N} = \nabla v = \frac{\partial v}{\partial N} = 0$ on $\partial\Omega$ and $\Delta|_{\partial\Omega} = \frac{\partial^2}{\partial N^2}$ on $\partial\Omega$;

$$\begin{aligned}&DQ_5(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda,\dot h)\\ &= \Big\{\Delta u\big[\Delta(\dot v - \dot h\cdot\nabla v) + \dot h\cdot\nabla(\Delta v)\big] + \Delta v\big[\Delta(\dot u - \dot h\cdot\nabla u) + \dot h\cdot\nabla(\Delta u)\big]\Big\}\Big|_{\partial\Omega}.\end{aligned}$$

Observe that the hypothesis (1) of Transversality Theorem is easy to verify. Now, we prove (2β), that is, we show that

$$\dim\Big\{\frac{R(DQ(u,v,\lambda,i_\Omega))}{R(\frac{\partial Q}{\partial(u,v,\lambda)}(u,v,\lambda,i_\Omega))}\Big\} = \infty.$$

Suppose this is not true. Then there exist $\theta_1,\dots,\theta_m \in L^2(\Omega)\times H^{\frac52}(\partial\Omega)\times L^2(\Omega)\times H^{\frac52}(\partial\Omega)\times L^1(\partial\Omega)$ such that, for all $\dot h \in C^5(\Omega,\mathbb{R}^n)$ there exist $\dot u,\dot v,\dot\lambda$ and $c_1,\dots,c_m$ such that

$$DQ(u,v,\lambda,i_\Omega)(\dot u,\dot v,\dot\lambda,\dot h) = \sum_{j=1}^m c_j\theta_j, \quad \theta_j = (\theta_j^1,\theta_j^2,\theta_j^3,\theta_j^4,\theta_j^5),$$

that is,

$$(\Delta^2+\lambda)(\dot u - \dot h\cdot\nabla u) + \dot\lambda u = \sum_{j=1}^m c_j\theta_j^1\,, \tag{5.6}$$

$$\frac{\partial}{\partial N}(\dot u - \dot h\cdot\nabla u) + \dot h\cdot N\Delta u = \sum_{j=1}^m c_j\theta_j^2\,, \tag{5.7}$$

$$(\Delta^2+\lambda)(\dot v - \dot h\cdot\nabla v) + \dot\lambda v = \sum_{j=1}^m c_j\theta_j^3\,, \tag{5.8}$$

$$\frac{\partial}{\partial N}(\dot v - \dot h\cdot\nabla v) + \dot h\cdot N\Delta v = \sum_{j=1}^m c_j\theta_j^4\,, \tag{5.9}$$

$$\begin{aligned}&\Big\{\Delta u\big[\Delta(\dot v - \dot h\cdot\nabla v) + \dot h\cdot\nabla(\Delta v)\big] + \Delta v\big[\Delta(\dot u - \dot h\cdot\nabla u) + \dot h\cdot\nabla(\Delta u)\big]\Big\}\Big|_{\partial\Omega}\\ &= \sum_{j=1}^m c_j\theta_j^5.\end{aligned} \tag{5.10}$$

Let $\{u_1,\dots,u_p\}\subset L^2(\Omega)$ be an orthonormal basis constituted by eigenfunctions of (1.1) with corresponding eigenvalues λ and consider the linear operators

$$\mathcal{A}_{\Delta^2+\lambda} : L^2(\Omega) \to H^4\cap H_0^1(\Omega)\,,$$
$$\mathcal{C}_{\Delta^2+\lambda} : H^{\frac52}(\partial\Omega) \to H^4(\Omega)\cap H_0^1(\Omega)$$

defined by

$$w = \mathcal{A}_{\Delta^2+\lambda}f + \mathcal{C}_{\Delta^2+\lambda}g \in H^4\cap H_0^1(\Omega)$$

where

$$(\Delta^2+\lambda)w - f \in [u_1,\dots,u_p], \quad w\perp[u_1,\dots,u_p], \quad \frac{\partial w}{\partial N} = g \quad \text{on } \partial\Omega.$$

We obtain, by equations (5.6), (5.7), (5.8) and (5.9) that

$$\dot u - \dot h\cdot\nabla u = \sum_{j=1}^p \xi_j u_j + \sum_{j=1}^m c_j\{\mathcal{A}_{\Delta^2+\lambda}\theta_j^1 + \mathcal{C}_{\Delta^2+\lambda}\theta_j^2\} - \mathcal{C}_{\Delta^2+\lambda}\big(\dot h\cdot N\Delta u\big)\,, \tag{5.11}$$

$$\dot{v} - \dot{h}\cdot\nabla v = \sum_{j=1}^{l}\eta_i u_i + \sum_{j=1}^{m} c_j\{\mathcal{A}_{\Delta^2+\lambda}\theta_j^3 + \mathcal{C}_{\Delta^2+\lambda}\theta_j^4\} - \mathcal{C}_{\Delta^2+\lambda}\big(\dot{h}\cdot N\Delta v\big). \quad (5.12)$$

Substituting these equations in (5.10) we have

$$\Delta u\Big[\dot{h}\cdot\nabla(\Delta v) - \Delta\big(\mathcal{C}_{\Delta^2+\lambda}\big(\dot{h}\cdot N\Delta v\big)\big)\Big] + \Delta v\Big[\dot{h}\cdot\nabla(\Delta u) - \Delta\big(\mathcal{C}_{\Delta^2+\lambda}\big(\dot{h}\cdot N\Delta u\big)\big)\Big]\Big|_{\partial\Omega} \quad (5.13)$$

belongs to a finite dimensional subspace for all $\dot{h}\in C^5(\Omega,\mathbb{R}^n)$. Since $\Delta u\Delta v\equiv 0$ on $\partial\Omega$ and $i_\Omega\in D_M$, $J=\{x\in\partial\Omega \mid \Delta u\neq 0\}$ is a nonempty open set in $\partial\Omega$ and $\Delta v\equiv 0$ on J.

Choose $\dot{h}\in C^5(\Omega,\mathbb{R}^n)$ satisfying $\dot{h}\equiv 0$ on $\partial\Omega - J$. Thus, using (5.13), it follows that

$$\Big\{\Delta u\big(\dot{h}\cdot\nabla(\Delta v)\big) - \Delta v\Delta\big(\mathcal{C}_{\Delta^2+\lambda}\big(\dot{h}\cdot N\Delta u\big)\big)\Big\}\Big|_{\partial\Omega} \quad (5.14)$$

belongs to a finite dimensional subspace when $\dot{h}$ varies. In fact, for these choices of $\dot{h}$ we have that

$$\mathcal{C}_{\Delta^2+\lambda}\big(\dot{h}\cdot N\Delta v\big) = \mathcal{C}_{\Delta^2+\lambda}(0)$$

belongs to a finite dimensional subspace and $\Delta v\Big(\dot{h}\cdot\nabla(\Delta u)\Big)\equiv 0$ on $\partial\Omega$.

Now, observe that

$$\big\{\Delta u\big(\dot{h}\cdot\nabla(\Delta v)\big) - \Delta v\Delta\big(\mathcal{C}_{\Delta^2+\lambda}\big(\dot{h}\cdot N\Delta u\big)\big)\big\}\Big|_J = \Delta u\big(\dot{h}\cdot\nabla(\Delta v)\big)\Big|_J.$$

Thus, by equation (5.14) we obtain that the mapping

$$\Sigma : \dot{h}\to\Delta u\big(\dot{h}\cdot\nabla(\Delta v)\big)\big|_J,$$

defined for $\dot{h}\in C^5(\Omega,\mathbb{R}^n)$ with $\dot{h}\equiv 0$ on $\partial\Omega - J$, has finite rank. But this can only occur [in $\dim\Omega\geq 2$] if $\nabla(\Delta v)\equiv 0$ on $J\subset\partial\Omega$. Observe that, in this case

$$0 = \Delta v = \frac{\partial^2 v}{\partial N^2} \quad \text{on } J\,,$$
$$0 = \nabla(\Delta v) = \nabla\Big(\frac{\partial^2 v}{\partial N^2}\Big) \quad \text{on } J.$$

Thus the eigenfunction v of (1.1) satisfies

$$\frac{\partial^2 v}{\partial N^2} = \frac{\partial^3 v}{\partial N^3} = 0 \quad \text{on } J,$$

that is, v satisfies (2.1) and by the Theorem 2.3 $v\equiv 0$ in Ω. Then we obtain a contradiction, proving the result. □

Theorem 5.5. *Generically in* $\mathrm{Diff}^4(\Omega)$ *all eigenvalues of (1.1) are simple.*

Proof. By Remark (5.2), we can suppose that region Ω is C^5-regular. Consider the map $F : B_M\times(-M,0)\times U_M\to L^2(\Omega)$ defined by

$$F(u,\lambda,h) = h^*(\Delta^2+\lambda)h^{*-1}u$$

where $U_M = D_M - E_M$. We will show, using the Transversality Theorem, that the set

$$\{h\in U_M : 0 \text{ is not a regular value of } (u,\lambda)\to F(u,\lambda,h)\}$$

is meager and closed in U_M. From this, Proposition 5.1 and Baire's Theorem the result follows taking intersection with $M\in\mathbb{N}$.

We can easily see that the hypotheses (1) and (3) of the Theorem are satisfied. It remains only (2α) to be proved. We reason by contradiction. Suppose there exists a critical point with $F(u,\lambda,h)=0$. Without loss generality we can suppose $h=i_\Omega$. Then, there exists $\psi\in L^2(\Omega)$, such that

$$\langle DF(u,\lambda,i_\Omega)(\dot u,\dot\lambda,\dot h),\psi\rangle=0 \tag{5.15}$$

for all $(\dot u,\dot\lambda,\dot h)\in H^4\cap H^2_0(\Omega)\times\mathbb{R}\times C^5(\Omega,\mathbb{R}^n)$ where $DF(u,\lambda,i_\Omega):H^4\cap H^2_0(\Omega)\times\mathbb{R}\times C^5(\Omega,\mathbb{R}^n)\to L^2(\Omega)$ is given by

$$DF(u,\lambda,i_\Omega)(\dot u,\dot\lambda,\dot h)=(\Delta^2+\lambda)(\dot u-\dot h\cdot\nabla u)+\dot\lambda u.$$

If $\dot\lambda=\dot h=0$ in (5.15), we have

$$\int_\Omega\psi(\Delta^2+\lambda)\dot u=0\ \ \forall\dot u\in H^4\cap H^2_0(\Omega),$$

that is, $\psi\in R(\Delta^2+\lambda)^\perp=\mathcal{N}(\Delta^2+\lambda)$. Since $\partial\Omega$ is of class C^5, we have $\psi\in H^5(\Omega)$ and satisfies

$$\begin{gathered}(\Delta^2+\lambda)\psi=0\quad\text{in }\Omega\\ \psi=\frac{\partial\psi}{\partial N}=0\quad\text{on }\partial\Omega.\end{gathered}$$

If $\dot h=\dot u=0$ and $\dot\lambda\in\mathbb{R}$ we have $\int_\Omega u\psi=0$. If $\dot\lambda=\dot u=0$ and $\dot h\in C^5(\Omega,\mathbb{R}^n)$, we obtain

$$\begin{aligned}0&=-\int_\Omega\psi(\Delta^2+\lambda)(\dot h\cdot\nabla u)\\
&=\int_\Omega\{(\dot h\cdot\nabla u)(\Delta^2+\lambda)\psi-\psi(\Delta^2+\lambda)(\dot h\cdot\nabla u)\}\\
&=\int_{\partial\Omega}\{(\dot h\cdot\nabla u)\frac{\partial}{\partial N}(\Delta\psi)-\Delta\psi\frac{\partial}{\partial N}(\dot h\cdot\nabla u)-\psi\frac{\partial}{\partial N}(\Delta(\dot h\cdot\nabla u))+\Delta(\dot h\cdot\nabla u)\frac{\partial\psi}{\partial N}\}\\
&=\int_{\partial\Omega}\{(\dot h\cdot\nabla u)\frac{\partial}{\partial N}(\Delta\psi)-\Delta\psi\frac{\partial}{\partial N}(\dot h\cdot\nabla u)\}\\
&=\int_{\partial\Omega}\{\dot h\cdot N\frac{\partial\Delta\psi}{\partial N}\frac{\partial u}{\partial N}-\Delta\psi\frac{\partial}{\partial N}\Big(\dot h\cdot N\frac{\partial u}{\partial N}\Big)\}\\
&=-\int_{\partial\Omega}\dot h\cdot N\Delta\psi\Delta u\quad\forall\dot h\in C^5(\Omega,\mathbb{R}^n).\end{aligned}$$

So,

$$\int_{\partial\Omega}\dot h\cdot N\Delta\psi\Delta u=0\quad\forall\dot h\in C^5(\Omega,\mathbb{R}^n)$$

which implies $\Delta\psi\Delta u\equiv0$ on $\partial\Omega$. Since $i_\Omega\in U_M$, we obtain a contradiction. $\square$

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The author wants to correct some typing mistakes found in Section 5. The *open* interval $(-M, 0)$ must be replaced by the *closed* interval $[-M, 0]$ in the following definitions: G in Lemma 5.3, Q in Lemma 5.4, and F in Theorem 5.5. This condition is used in their proofs.

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