Electronic Journal of Differential Equations, Vol. 2004(2004), No. 117, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ON THE INSTABILITY OF CERTAIN SIXTH-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

CEMIL TUNÇ

Abstract. In this paper, we give an instability criteria for the nonlinear sixth-order vector differential equation

$$\begin{aligned}X^{(6)}+AX^{(5)}+B(t)\Phi(X,\dot X,\ddot X,\dddot X,X^{(4)},X^{(5)})X^{(4)}+C(t)\Psi(\ddot X)\dddot X\\ +D(t)\Omega(X,\dot X,\ddot X,\dddot X,X^{(4)},X^{(5)})\ddot X+E(t)G(\dot X)+H(X)=0\,.\end{aligned}$$

The result extends and includes earlier results [6, 16, 20].

1. Introduction

It is well known from the relevant literature that there have been intensive studies on the qualitative behavior of solutions of certain nonlinear ordinary differential equations in recent years. Meanwhile, many articles have been devoted to the investigation of the instability properties of solutions for various third-, fourth-, fifth-, sixth-, seventh and eighth order certain nonlinear differential equations; see for instance [1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 19, 20, 21] and the references cited therein. However, to the best our knowledge for the case $n=1$, there exist only two works on the instability properties of solutions of certain nonlinear differential equations of the sixth order. Namely, in the case $n=1$, Ezeilo [6] and Tiryaki [16] studied the instability of the zero solution $x=0$ of the following nonlinear differential equations:

$$\begin{aligned}x^{(6)}+a_1x^{(5)}+a_2x^{(4)}+e(x,\dot x,\ddot x,\dddot x,x^{(4)},x^{(5)})\dddot x\\ +f(\dot x)\ddot x+g(x,\dot x,\ddot x,\dddot x,x^{(4)},x^{(5)})\dot x+h(x)=0\end{aligned}$$

and

$$\begin{aligned}x^{(6)}+a_1x^{(5)}+f_1(x,\dot x,\ddot x,\dddot x,x^{(4)},x^{(5)})x^{(4)}\\ +f_2(\ddot x)\dddot x+f_3(x,\dot x,\ddot x,\dddot x,x^{(4)},x^{(5)})\ddot x+f_4(\dot x)+f_5(x)=0\end{aligned}$$

respectively.

2000 *Mathematics Subject Classification.* 34D05, 34D20.
Key words and phrases. Nonlinear differential equations of sixth order; instability; Lyapunov function.

Submitted July 13, 2004. Published October 7, 2004.

Recently, the author in [20] investigated the same subject for the sixth order nonlinear vector differential equations of the form:

$$\begin{aligned} X^{(6)} + AX^{(5)} + \Phi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})X^{(4)} + \Psi(\ddot{X})\dddot{X} \\ + F(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})\ddot{X} + G(\dot{X}) + H(X) = 0. \end{aligned} \tag{1.1}$$

In this paper we are concerned with the instability of the trivial solution $X = 0$ of the nonlinear vector differential equations of the form:

$$\begin{aligned} X^{(6)} + AX^{(5)} + B(t)\Phi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})X^{(4)} + C(t)\Psi(\ddot{X})\dddot{X} \\ + D(t)\Omega(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}, X^{(5)})\ddot{X} + E(t)G(\dot{X}) + H(X) = 0 \end{aligned} \tag{1.2}$$

in which $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$ and $X \in \mathbb{R}^n$; A is a constant $n\times n$-symmetric matrix; B, Φ, C, Ψ, D, Ω and E are continuous $n\times n$ -symmetric matrices depending, in each case, on the arguments shown; $G : \mathbb{R}^n \to \mathbb{R}^n$, $H : \mathbb{R}^n \to \mathbb{R}^n$ and $G(0) = H(0)=0$. It is supposed that the functions G and H are continuous. Let $J_H(X)$, $J_G(\dot{X})$ and $J_\Psi(\ddot{X})$ denote the Jacobian matrices corresponding to the $H(X)$, $G(\dot{X})$ and $\Psi(\ddot{X})$, respectively, that is,

$$J_H(X) = \Big(\frac{\partial h_i}{\partial x_j}\Big), \quad J_G(\dot{X}) = \Big(\frac{\partial g_i}{\partial \dot{x}_j}\Big), \quad J_\Psi(\ddot{X}) = \Big(\frac{\partial \psi_i}{\partial \ddot{x}_j}\Big), \quad (i, j = 1, 2, \dots, n),$$

where $(x_1, x_2, \dots, x_n)$, $(\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n)$, $(\ddot{x}_1, \ddot{x}_2, \dots, \ddot{x}_n)$, $(h_1, h_2, \dots, h_n)$, $(g_1, g_2, \dots, g_n)$ and $(\psi_1, \psi_2, \dots, \psi_n)$ are the components of X, $\dot{X}$, $\ddot{X}$, H, G and Ψ, respectively. Other than these, it is also assumed that the Jacobian matrices $J_H(X)$, $J_G(\dot{X})$, $J_\Psi(\ddot{X})$ and the derivatives $\frac{d}{dt}C(t) = \dot{C}(t)$ and $\frac{d}{dt}E(t) = \dot{E}(t)$ exist and are continuous. Moreover, it is assumed that all matrices given in the pairs C, Ψ; C,J_Ψ; $\dot{C}$,Ψ; E, J_G; $\dot{E}$, J_G; B, Φ and D, Ω commute with each others.

The symbol $\langle X, Y\rangle$ corresponding to any pair$X$, Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^n x_i y_i$, and $\lambda_i(A)(i = 1, 2, \dots, n)$ are the eigenvalues of the $n\times n$-matrix A.

In what follows it will be convenient to use the equivalent differential system:

$$\begin{aligned} &\dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = W, \quad \dot{W} = U, \quad \dot{U} = V, \\ &\dot{V} = -AV - B(t)\Phi(X, Y, Z, W, U, V)U - C(t)\Psi(Z)W \\ &\qquad -D(t)\Omega(X, Y, Z, W, U, V)Z - E(t)G(Y) - H(X) \end{aligned} \tag{1.3}$$

which is obtained from (1.2) by setting $\dot{X} = Y$, $\ddot{X} = Z$, $\dddot{X} = W$, $X^{(4)} = U$ and $X^{(5)} = V$.

2. Main Result

The main result is the following theorem. This extends and includes the results of Ezeilo [6], Tiryaki [16] and Tunç [20].

Theorem 2.1. *Suppose that there are constants a_1, a_2 and a_4 with $a_4 > \frac{1}{4}a_2^2$ such that*

(i) *A, B, D, $J_H(X)$ are symmetric and $\lambda_i(A) \geq a_1 > 0$, $\lambda_i(B(t)) \geq 1$, $\lambda_i(D(t)) \geq 1$ for all $t \in \mathbb{R}^+$, $H(X) \neq 0$ for all $X \neq 0$, $X \in \mathbb{R}^n$, and $\lambda_i(J_H(X)) < 0$ for all $X \in \mathbb{R}^n$, $(i = 1, 2, \dots, n)$.*

(ii) *$\Phi(X,Y,Z,W,U,V)$, $\Omega(X,Y,Z,W,U,V)$ are symmetric and $\lambda_i(\Phi(X,Y,Z,W,U,V)) \le a_2$, $\lambda_i(\Omega(X,Y,Z,W,U,V)) \ge a_4$ for all X, Y, Z, W, U, $V \in \mathbb{R}^n$, $(i = 1,2,\dots,n)$.*

(iii) *$\dot{E}(t)$, $J_G(Y)$ and $\dot{C}(t)$, $\Psi(Z)$ are symmetric and have opposite-sign eigenvalues for all $t \in \mathbb{R}^+$ and Y, $Z \in \mathbb{R}^n$.*

Then the zero solution $X = 0$ of the system (1.3) is unstable.

Proof. Our main tool in the proof of the theorem is the Lyapunov function $\Gamma = \Gamma(t,X,Y,Z,W,U,V)$ given by

$$\begin{aligned}\Gamma = &-\langle V,Z\rangle - \langle Z,AU\rangle + \langle W,U\rangle + \frac{1}{2}\langle W,AW\rangle - \int_0^1 \langle E(t)G(\sigma Y),Y\rangle d\sigma \\ &- \int_0^1 \langle \sigma C(t)\Psi(\sigma Z)Z,Z\rangle d\sigma - \langle H(X),Y\rangle.\end{aligned} \tag{2.1}$$

It should be noted that this function and its total time derivative satisfy some fundamental properties: It is clear from (2.1) that $\Gamma(0,0,0,0,0,0,0) = 0$. Obviously, it also follows from the assumptions of the theorem and (2.1) that

$$\Gamma(0,0,0,0,\varepsilon,\varepsilon,0) = \langle \varepsilon,\varepsilon\rangle + \frac{1}{2}\langle \varepsilon, A\varepsilon\rangle \ge \|\varepsilon\|^2 + \frac{1}{2}a_1\|\varepsilon\|^2 > 0,$$

for all $\varepsilon \neq 0$ in $\mathbb{R}^n$. Next let $(X,Y,Z,W,U,V) = (X(t),Y(t),Z(t),W(t),U(t),V(t))$ be an arbitrary solution of (1.3). Differentiating (2.1) we obtain

$$\begin{aligned}\dot{\Gamma} &= \frac{d}{dt}\Gamma(t,X,Y,Z,W,U,V) \\ &= \langle U,U\rangle + \langle B(t)\Phi(X,Y,Z,W,U,V)U,Z\rangle + \langle Z,D(t)\Omega(X,Y,Z,W,U,V)Z\rangle \\ &\quad - \langle J_H(X)Y,Y\rangle + \langle E(t)G(Y),Z\rangle + \langle C(t)\Psi(Z)W,Z\rangle \\ &\quad - \frac{d}{dt}\int_0^1 \langle E(t)G(\sigma Y),Y\rangle d\sigma - \frac{d}{dt}\int_0^1 \langle \sigma C(t)\Psi(\sigma Z)Z,Z\rangle d\sigma.\end{aligned} \tag{2.2}$$

Recall that

$$\begin{aligned}&\frac{d}{dt}\int_0^1 \langle E(t)G(\sigma Y),Y\rangle d\sigma \\ &= \int_0^1 \langle \dot{E}(t)G(\sigma Y),Y\rangle d\sigma + \int_0^1 \sigma\langle E(t)J_G(\sigma Y)Z,Y\rangle d\sigma + \int_0^1 \langle E(t)G(\sigma Y),Z\rangle d\sigma \\ &= \int_0^1 \langle \dot{E}(t)G(\sigma Y),Y\rangle d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle E(t)G(\sigma Y),Z\rangle d\sigma + \int_0^1 \langle E(t)G(\sigma Y),Z\rangle d\sigma \\ &= \int_0^1 \langle \dot{E}(t)G(\sigma Y),Y\rangle d\sigma + \sigma\langle E(t)G(\sigma Y),Z\rangle\,\big|_0^1 \\ &= \int_0^1 \langle \dot{E}(t)G(\sigma Y),Y\rangle d\sigma + \langle E(t)G(Y),Z\rangle.\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \langle \sigma C(t)\Psi(\sigma Z)Z, Z\rangle d\sigma \\
&= \int_0^1 \langle \sigma \dot{C}(t)\Psi(\sigma Z)Z, Z\rangle d\sigma + \int_0^1 \langle \sigma C(t)\Psi(\sigma Z)Z, W\rangle d\sigma \\
&\quad + \int_0^1 \sigma^2 \langle C(t)J_\Psi(\sigma Z)ZW, Z\rangle d\sigma + \int_0^1 \langle \sigma C(t)\Psi(\sigma Z)W, Z\rangle d\sigma \\
&= \int_0^1 \langle \sigma \dot{C}(t)\Psi(\sigma Z)Z, Z\rangle d\sigma + \int_0^1 \sigma \frac{\partial}{\partial\sigma}\langle \sigma C(t)\Psi(\sigma Z)W, Z\rangle d\sigma \\
&\quad + \int_0^1 \langle \sigma C(t)\Psi(\sigma Z)W, Z\rangle d\sigma \\
&= \int_0^1 \langle \sigma \dot{C}(t)\Psi(\sigma Z)Z, Z\rangle d\sigma + \sigma^2\langle C(t)\Psi(\sigma Z)W, Z\rangle \Big|_0^1 \\
&= \int_0^1 \langle \sigma \dot{C}(t)\Psi(\sigma Z)Z, Z\rangle d\sigma + \langle C(t)\Psi(Z)W, Z\rangle.
\end{aligned} \tag{2.4}$$

On gathering the estimates (2.3) and (2.4) into (2.2), we obtain

$$\begin{aligned}
\dot{\Gamma} &= \langle U, U\rangle + \langle B(t)\Phi(X,Y,Z,W,U,V)Z, U\rangle \\
&\quad + \langle Z, D(t)\Omega(X,Y,Z,W,U,V)Z\rangle - \langle J_H(X)Y, Y\rangle \\
&\quad - \int_0^1 \langle \dot{E}(t)G(\sigma Y), Y\rangle d\sigma - \int_0^1 \langle \sigma \dot{C}(t)\Psi(\sigma Z)Z, Z\rangle d\sigma.
\end{aligned} \tag{2.5}$$

Since

$$G(0) = 0, \quad \frac{\partial}{\partial\sigma}G(\sigma Y) = J_G(\sigma Y)Y, \quad G(Y) = \int_0^1 J_G(\sigma Y)Y\,d\sigma,$$

the assumption (iii) of the theorem shows that

$$\int_0^1 \langle \dot{E}(t)G(\sigma Y), Y\rangle d\sigma = \int_0^1\int_0^1 \langle \sigma_1 \dot{E}(t)J_G(\sigma_1\sigma_2 Y)Y, Y\rangle d\sigma_2 d\sigma_1 \le 0. \tag{2.6}$$

Next, by the assumption of (iii) of the theorem it is also clear that

$$\int_0^1 \langle \sigma \dot{C}(t)\Psi(\sigma Z)Z, Z\rangle d\sigma \le 0. \tag{2.7}$$

Combining the estimates (2.6) and (2.7) into (2.5) we obtain

$$\begin{aligned}
\dot{\Gamma} &\ge \|U + \frac{1}{2}B(t)\Phi(X,Y,Z,W,U,V)Z\|^2 + \langle Z, D(t)\Omega(X,Y,Z,W,U,V)Z\rangle \\
&\quad - \langle J_H(X)Y, Y\rangle - \frac{1}{4}\langle B(t)\Phi(X,Y,Z,W,U,V)Z, B(t)\Phi(X,Y,Z,W,U,V)Z\rangle \\
&\ge \langle Z, D(t)\Omega(X,Y,Z,W,U,V)Z\rangle - \langle J_H(X)Y, Y\rangle \\
&\quad - \frac{1}{4}\langle B(t)\Phi(X,Y,Z,W,U,V)Z, B(t)\Phi(X,Y,Z,W,U,V)Z\rangle
\end{aligned}$$

Hence, it follows from the assumptions (i) and (ii) that

$$\dot{\Gamma} \ge \langle Z, a_4 Z\rangle - \frac{1}{4}\langle a_2 Z, a_2 Z\rangle = \big(a_4 - \frac{1}{4}a_2^2\big)\|Z\|^2 > 0.$$

Thus, the assumptions of the theorem show that $\dot{\Gamma} \geq 0$ for all $t \geq 0$, that is, $\dot{\Gamma}$ is positive semi-definite. Furthermore, $\dot{\Gamma} = 0 (t \geq 0)$ necessarily implies that $Y = 0$ for all $t \geq 0$, and therefore also that $X = \xi$ (a constant vector), $Z = \dot{Y} = 0$, $W = \ddot{Y} = 0$, $U = \dddot{Y} = 0$, $V = Y^{(4)} = 0$ for all $t \geq 0$. The substitution of the estimates

$$X = \xi, \quad Y = Z = W = U = V = 0$$

in (1.3) leads to the result $H(\xi) = 0$ which by assumption (i) of the theorem implies (only) that $\xi = 0$. Hence $\dot{\Gamma} = 0 \quad (t \geq 0)$ implies that

$$X = Y = Z = W = U = V = 0 \quad \text{for all } t \geq 0.$$

Therefore, the function Γ has the requisites for Krasovskii criterion [7] if the conditions of the theorem hold. Thus, the basic properties of $\Gamma(t, X, Y, Z, W, U, V)$, which are proved just above verify that the zero solution of the system (1.3) is unstable. (See Reissig et al [10, Theorem 1.15] and Krasovskii [7]). The system of equations (1.3) is equivalent to the differential equation (1.2). Consequently, this completes the proof of the theorem. $\square$

Acknowledgements. The author would like to express sincere thanks to the anonymous referee for his/her valuable comments and suggestions.

References

[1] Berketoolu, H.; *On the instability of trivial solutions of a class of eighth-order differential equations*. Indian J. Pure. Appl. Math. 22 (1991), no.3, 199-202.

[2] Ezeilo, J. O. C.; *An instability theorem for a certain fourth order differential equation*. Bull. London Math. Soc. 10 (1978), no. 2, 184-185.

[3] Ezeilo, J. O. C.; *Instability theorems for certain fifth-order differential equations*. Math. Proc. Cambridge Philos. Soc. 84 (1978), no. 2, 343-350.

[4] Ezeilo, J. O. C.; *A further instability theorem for a certain fifth-order differential equation*. Math. Proc. Cambridge Philos. Soc. 86 (1979), no. 3, 491-493.

[5] Ezeilo, J. O. C.; *Extension of certain instability theorems for some fourth and fifth order differential equations*. Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur. (8) 66 (1979), no. 4, 239-242.

[6] Ezeilo, J. O. C.; *An instability theorem for a certain sixth order differential equation*. J. Austral. Math. Soc. Ser. A 32 (1982), no. 1, 129-133.

[7] Krasovskii, N. N.; *On conditions of inversion of A. M. Lyapunov's theorems on instability for stationary systems of differential equations*. (Russian) Dokl. Akad. Nauk. SSSR (N.S.) 101, (1955). 17-20.

[8] Li, W. J.; Yu, Y. H.; *Instability theorems for some fourth-order and fifth-order differential equations*. (Chinese) J. Xinjiang Univ. Natur. Sci. 7 (1990), no. 2, 7-10.

[9] Li, W. J.; Duan, K. C.; *Instability theorems for some nonlinear differential systems of fifth order*. J. Xinjiang Univ. Natur. Sci. 17 (2000), no. 3, 1-5.

[10] Reissig, R.; Sansone, G.; Conti, R.; *Non-linear differential equations of higher order*. Translated from the German. Noordhoff International Publishing, Leyden, (1974).

[11] Sadek, A. I.; *An instability theorem for a certain seventh-order differential equation*. Ann. Differential Equations 19 (2003), no. 1, 1-5.

[12] Skrapek, W. A.; *Instability results for fourth-order differential equations*. Proc. Roy. Soc. Edinburgh Sect. A 85 (1980), no. 3-4, 247-250.

[13] Skrapek, W. A.; *Some instability theorems for third order ordinary differential equations*. Math. Nachr. 96 (1980), 113-117.

[14] Tejumola, H. O.; *Instability and periodic solutions of certain nonlinear differential equations of orders six and seven*. Ordinary differential equations (Abuja, 2000), 56-65, Proc. Natl. Math. Cent. Abuja Niger., 1.1, Natl. Math. Cent., Abuja, 2000.

[15] Tiryaki, A.; *Extension of an instability theorem for a certain fourth order differential equation*. Bull. Inst. Math. Acad. Sinica. 16 (1988), no.2, 163-165.

[16] Tiryaki, A.; *An instability theorem for a certain sixth order differential equation*. Indian J. Pure. Appl. Math. 21 (1990), no.4, 330-333.
[17] Tiryaki, A.; *Extension of an instability theorem for a certain fifth order differential equation*. National Mathematics Symposium (Trabzon, 1987). J. Karadeniz Tech. Univ. Fac. Arts Sci. Ser. Math. Phys. 11 (1988), 225-227 (1989).
[18] Tunç, C.; *An instability theorem for a certain vector differential equation of the fourth order*. Journal of Inequalities in Pure and Applied Mathematics 5 (2004), no. 1, 1-5.
[19] Tunç, C.; Tunç, E.; *An instability theorem for a certain eighth order differential equation*. Differential Equations (Differ. Uravn.) (2004) (in press).
[20] Tunç, C.; *An instability result for certain system of sixth order differential equations*. Applied Mathematics and Computation 157 (2004), no.2, 477-481.
[21] Tunç, C.; *On the instability of solutions of certain nonlinear vector differential equations of fifth order*, Panamerican Mathematical Journal, (2004) (in press).

Department of Mathematics, Faculty of Arts and Sciences, Yüzüncü Yil University, 65080, VAN – Turkey
E-mail address: cemtunc@yahoo.com