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NONLINEAR SUBELLIPTIC SCHRÖDINGER EQUATIONS WITH EXTERNAL MAGNETIC FIELD

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Abstract. To account for an external magnetic field in a Hamiltonian of a quantum system on a manifold (modelled here by a subelliptic Dirichlet form), one replaces the the momentum operator $\frac{1}{i}d$ in the subelliptic symbol by $\frac{1}{i}d - \alpha$, where $\alpha \in TM^*$ is called a magnetic potential for the magnetic field $\beta = d\alpha$.

We prove existence of ground state solutions (Sobolev minimizers) for nonlinear Schrödinger equation associated with such Hamiltonian on a generally, non-compact Riemannian manifold, generalizing the existence result of Esteban-Lions [5] for the nonlinear Schrödinger equation with a constant magnetic field on $\mathbb{R}^N$ and the existence result of [6] for a similar problem on manifolds without a magnetic field. The counterpart of a constant magnetic field is the magnetic field, invariant with respect to a subgroup of isometries. As an example to the general statement we calculate the invariant magnetic fields in the Hamiltonians associated with the Kohn Laplacian and for the Laplace-Beltrami operator on the Heisenberg group.

1. Introduction

In this paper we study nonlinear Schrödinger equations with external magnetic field on (generally) non-compact Riemannian manifolds. A summary exposition on the magnetic Schrödinger operator can be found in [1]. The scope of the paper includes subelliptic Hamiltonians.

Let M be a differentiable n-dimensional Riemannian manifold and let α be a 1-form on M. We consider the quadratic form

$$E_0 = \int_M a\Big(\frac{1}{i}du - u\alpha, \frac{1}{i}du - u\alpha\Big)d\mu \tag{1.1}$$

where μ is the Riemannian measure of M and $a \in TM^{2,0}$ (called the *symbol* of the quadratic form), is a smooth Hermitian bilinear form with real-valued coefficients defined on fibers TM_x^*.

The form E is understood in physics as a generalized Hamiltonian for a quantum particle on M in presence of the external magnetic field $\beta = d\alpha$. In general, a

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magnetic field is a closed 2-form that does not have to be exact. Quantization of systems with a non-potential magnetic field is more complicated (see [10] and references therein) and is not considered here. The potential α is defined by β up to an arbitrary closed form and the energy is invariant under the gauge transformation $(\alpha, u) \mapsto (\alpha + d\varphi, e^{i\varphi}u)$.

The (stationary) nonlinear Schrödinger equation for complex-valued functions on M in the weak form is:

$$\int_M \big(a(\frac{1}{i}du - u\alpha, \frac{1}{i}dv - v\alpha) + \lambda uv - |u|^{q-2}uv\big)d\mu = 0, \tag{1.2}$$

$v \in C_0^\infty(M)$. In what follows we will use the notation $a[\alpha] := a(\alpha,\alpha)$, $E_0[u] := E_0(u,u)$ etc. for quadratic forms.

Let $H^1(M)$ be the Hilbert space defined as the closure of $C_0^\infty(M;\mathbb{C})$ with respect to the Hilbert norm $\big(\int_M(|du|^2+|u|^2)d\mu\big)^{1/2}$. For an open set $\Omega \subset M$ the subspace $H^1(\Omega)$ will be the closure of $C_0^\infty(\Omega)$ in $H^1(M)$.

We assume that the symbol a and the number 2^* are related via the Sobolev inequality for the real-valued functions $u \in H^1(M)$:

$$\int_M (a[du] + |u|^2)d\mu \ge c\|u\|^2_{L^q(M,d\mu)}, \quad q \in [2, 2^*], \tag{1.3}$$

and that, in restriction to $H^1(\Omega)$ with any bounded $\Omega \subset M$, and with $q \in (2, 2^*)$, this imbedding is compact.

This is true, for example, when $a[\xi] \ge c|\xi|^2$ with some $c > 0$ (the uniformly elliptic case) and when M satisfies the assumption (1.8) below. In this case $2^* = \frac{2n}{n-2}$ for $n > 2$, and $2^* = \infty$ for $n = 2$. The relation (1.3) holds as well when M is a Lie group and the symbol of E_0 is $a = \sum_j X_j \otimes X_j$, where $X_j \in TM$, $j = 1, \dots, m$, are left-invariant vector fields. If the subsequent commutators of X_j span the whole tangent space of M (Hörmander condition), then there exists a $N \ge n$, called homogeneous dimension, such that (1.3) holds with $2^* = \frac{2N}{N-2}$ ([8, 9, 19] and references therein).

Let now $H^1_\alpha(M)$ (resp. $H^1_\alpha(\Omega)$) be the closure of $C_0^\infty(M;\mathbb{C})$ (resp. $C_0^\infty(\Omega;\mathbb{C})$) in the metric of

$$E[u] := E_0[u] + \|u\|^2_{L^2(M,d\mu)}. \tag{1.4}$$

The following inequality is an elementary generalization of the diamagnetic inequality, well known for the Euclidean case (see e.g. [13]).

Lemma 1.1. *Let $\alpha \in TM^*$ and let a be as above. The following inequality is true for every $u \in C_0^\infty(M;\mathbb{C})$ at every point where $u \neq 0$:*

$$a[du - iu\alpha] \ge a[d|u|]. \tag{1.5}$$

Proof. Let v, w be the real and the imaginary parts of u. The assertion follows from the following chain of identities that use the bilinearity of a and the chain rule:

$$\begin{aligned} a[du - iu\alpha] - a[d|u|] &= a[du] + |u|^2a[\alpha] - 2va(\alpha, dw) + 2wa(\alpha, dv) \\ &\quad - |u|^{-2}\left\{v^2a[dv] + w^2a[dw] + 2vwa(dv, dw)\right\} \\ &= |u|^{-2}\left\{a[vdw - wdv] + 2|u|^2a(\alpha, wdv - vdw) + |u|^4a[\alpha]\right\} \\ &= |u|^{-2}a[wdv - vdw + |u|^2\alpha] \ge 0. \end{aligned}$$

□

Proposition 1.2. *The following inequality holds:*

$$E(u) \geq \||u|\|^2_{H^1(M)}, \quad u \in H^1_\alpha(M). \tag{1.6}$$

Moreover, the space $H^1_\alpha(M)$ is continuously imbedded into $L^q(M,\mu)$, $q \in (2, 2^*$, and for any bounded open $\Omega \subset M$ the imbedding of $H^1_\alpha(\Omega)$ into $L^q(\Omega,\mu)$ is compact.

Proof. Using approximation operators $T_\epsilon : C^1_0(M) \to C^1_0(M)$, $T_\epsilon u := (u^2+\epsilon^2)^{1/2} - \epsilon$, one can immediately deduce from Lemma 1.5 (see for details the proof of Lemma 7.6 in [7]) that $u \in H^1_\alpha(M) \Rightarrow |u| \in H^1(M)$ with $E_0(u) \geq \||u|\|^2_{H^1(M)}$. Thus, by (1.3) applied to $|u|$, the space $H^1_\alpha(M)$ is continuously embedded into $L^q(M,\mu)$ and for any open bounded Ω, the subspace $H^1_\alpha(\Omega)$ is compactly embedded into $L^q(\Omega,\mu)$. □

Critical points of the map $u \mapsto (E(u), \int |u|^q d\mu)$, $H \to \mathbb{R}^2$ provide solutions of the equation (1.2) (up to a scalar multiple). We look here for solutions of the ground state type, that is, the minimizers in the problem

$$c_q := \inf_{\int_M |u|^q d\mu = 1} E[u], q \in (2, 2^*). \tag{1.7}$$

By analogy with the semilinear elliptic problem for the Laplacian on $\mathbb{R}^n$ without a magnetic field, the minimum in the problem (1.7) is not expected to exist without substantial additional assumption. Existence of a minimizer is known for (1.7) in the Euclidean case with a constant magnetic field ([5]). If the field is not constant, or a potential term is added to the equation, existence of minimum has been derived from various penalty conditions at infinity, typically involving a potential term $\int V(x)|u|^2$ in the energy (see [12]). One may also observe absence of minimizer if the penalty condition is appropriately reversed ([5]). In this paper we consider invariant (which, in case of a discrete group, means space-periodic) magnetic fields on manifolds that are co-compact with respect to their isometry groups, a class that includes homogeneous Riemannian spaces and in particular, Lie groups.

Let I be a subgroup of the isometry group of M, closed in the CO-topology. We assume that there is a compact set $K \subset M$ such that

$$\bigcup_{\eta \in I} \eta K = M. \tag{1.8}$$

We assume that the symbol a is invariant with respect to the transformations $\eta \in I$. This is true, in particular, if it is the symbol of the Laplace-Beltrami operator or of an invariant subelliptic operator as defined above.

Consider now the condition of invariance of the magnetic field β. The invariance relation $\forall\eta,\ \eta\beta = \beta$, where $\eta : TM^{0,2}_{\eta\cdot} \to TM^{0,2}_{\cdot}$ is the natural action of the isometry $\eta \in I$ on 2-forms, written in terms of the magnetic potential α is equivalent to $d(\eta\alpha - \alpha) = 0$ where $\eta : TM^*_{\eta x} \to TM^*_x$ is the natural action of $\eta \in I$ on the cotangent bundle of M. For a technical reason (existence of global magnetic shifts) we put a somewhat stronger condition on α, namely that

$$\forall \eta \in I, \eta\alpha - \alpha \text{ is exact}. \tag{1.9}$$

This will allow to construct global magnetic shifts relative to $\eta \in I$ in the next section.

The main result of this paper is

Theorem 1.3. *Let a and μ be invariant under the action of the group I. Assume* (1.3)*,* (1.8)*,* (1.9)*. Then the problem* (1.7) *has a point of minimum which, up to the constant multiple is a non-trivial solution of* (1.2)*.*

Remark 1.4. The statement of the theorem remains true if one replaces in the energy the term $\int |u|^2$ in $E[u]$ with $\int V(x)|u|^2 d\mu$, $V \in L^1_{loc}(M,\mu)$, $\inf_M V > 0$ provided that $V \circ \eta = V$, $\eta \in I$. This generalization does not require any essential changes in the proof.

The proof of the existence of the minimum in (1.7) is based on the concentration compactness principle (see [14, 15] for a fundamental exposition for the subcritical case). One can use here the approach of [3, 18], and we give an essentially equivalent proof, using a general "multi-bump" expansion for bounded sequences (in the spirit of [16]) from [17].

In what follows we assume conditions of Theorem 1.3.

2. Concentration compactness with magnetic shifts

By (1.9), for every $\eta \in I$ there exists a $\psi_\eta \in C^\infty(M)$ such that

$$\eta\alpha - \alpha = d\psi_\eta. \tag{2.1}$$

This implies that $d\psi_{\mathrm{id}} = 0$, so that ψ_{id} is constant on connected components of M. Since the relation (2.1) is satisfied by $\psi_\eta - \psi_{\mathrm{id}}$, we normalize ψ_η by setting

$$\psi_{\mathrm{id}}(x) = 0, \quad x \in M. \tag{2.2}$$

Let

$$g_\eta u = e^{i\psi_\eta} u \circ \eta, \quad u \in C_0^\infty(M). \tag{2.3}$$

The action g_η on $u \in C_0^\infty(M)$ (as well as its continuous extension below) is called a magnetic shift. We set

$$D := \{g_\eta\}_{\eta \in I}. \tag{2.4}$$

Lemma 2.1. *Every operator $g \in D$ extends by continuity to a unitary operator on $H^1_\alpha(M)$. The (renamed) set D of extended operators is a multiplicative operator group on $H^1_\alpha(M)$.*

Proof. It suffices to prove that

$$g_{\eta^{-1}} = g_\eta^{-1}, \tag{2.5}$$

$$g_{\eta^{-1}} = g_\eta^* \tag{2.6}$$

for every $\eta \in I$. To prove (2.5), note that from (2.1) and (2.2) it follows immediately that

$$\psi_\eta = -(\psi_{\eta^{-1}} \circ \eta). \tag{2.7}$$

Then solving the equation $g_\eta u = v$, one has $v = e^{-i\psi_\eta \circ \eta^{-1}} u \circ \eta^{-1} = e^{i\psi_{\eta^{-1}}} u \circ \eta^{-1}$.

In order to prove (2.6), consider the following calculations, taking into account invariance properties of a and μ, (2.7) and (1.9):

$$\begin{aligned}
E_0(u, g_\eta v) &= \int_M e^{-i\psi_\eta} a\,(du + iu\alpha, d(v\circ\eta) - id\psi_\eta v\circ\eta + i(v\circ\eta)\alpha)\,d\mu \\
&= \int_M e^{-i\psi_\eta\circ\eta^{-1}} a\,\big((du)\circ\eta^{-1} + i(u\circ\eta^{-1})\eta^{-1}\alpha, dv + iv\alpha\big)\,d\mu \\
&= \int_M e^{i\psi_{\eta^{-1}}} a\,\big(d(u\circ\eta^{-1}) + i(u\circ\eta^{-1})(\alpha + d\psi_{\eta^{-1}}), dv + iv\alpha\big)\,d\mu \\
&= E_0(g_{\eta^{-1}}u, v),\;\; u, v \in C_0^\infty(M).
\end{aligned}$$

□

Lemma 2.2. *The group D on $H^1_\alpha(M)$ is a set of dislocations according to* [17], *i.e. a set of unitary operators on a separable Hilbert space satisfying the condition:*

(*) *Any sequence $g_k \in D$ that does not converge to zero weakly has a strongly convergent subsequence.*

We recall that a sequence of operators g_k in a Banach space E is called strongly convergent if for every $x \in E$, $g_k x$ converges.

Proof. Assume that $g_{\eta_k} \not\rightharpoonup 0$. Then there exist $u, v \in C_0^\infty(M)$ and a renamed subsequence of η_k, such that $(g_{\eta_k}u, v) \not\to 0$, so that $\eta_k^{-1}(\operatorname{supp} u) \cap \operatorname{supp} v \neq \emptyset$. Let $x_k \in \operatorname{supp} u$ be such that $\eta_k x_k \in \operatorname{supp} v$. Since $\operatorname{supp} u$ is compact, a renamed subsequence of x_k converges to some $x \in \operatorname{supp} u$. Since $\operatorname{supp} v$ is compact and η_k are isometries, a renamed subsequence of $\eta_k x$ converges, and therefore η_k converges to some $\eta \in I$ in the compact-open topology (cf. [11]) and therefore uniformly on compact sets. Then $g_{\eta_k} v$ converges for any $v \in C_0^\infty(M)$ by convergence of integrals under uniform convergence.

Since operators in D are unitary, it suffices to verify the strong operator convergence on $C_0^\infty(M)$, which in turn follows from convergence of integrals under uniform convergence. □

Definition 2.3. Let $u, u_k \in H^1_\alpha(M)$. We will say that u_k converges to u *D-weakly*, which we will denote as $u_k \overset{D}{\rightharpoonup} u$, if for all $\varphi \in H^1_\alpha(M)$,

$$\lim_{k\to\infty} \sup_{g\in D} (g(u_k - u), \varphi) = 0. \tag{2.8}$$

Lemma 2.4. *Let $u_k \in H^1_\alpha(M)$ be a bounded sequence. Then*

$$u_k \overset{D}{\rightharpoonup} 0 \Rightarrow u_k \to 0 \quad \text{in } L^q(M, \mu), q \in (2, 2^*). \tag{2.9}$$

Proof. If $g_{\eta_k} u_k \rightharpoonup 0$, then due to the inequality (1.6), $|u_k| \circ \eta_k \rightharpoonup 0$ in $H^1(M)$. Then $|u_k| \to 0$ in $L^q(M, \mu)$ by [2, Lemma 3.7] (when a is uniformly elliptic, one can also refer to [6, Lemma 2.6]). □

Theorem 2.5 ([17]). *Let $u_k \in H$ be a bounded sequence. Then there exist $w^{(n)} \in H$, $g_k^{(n)} \in D$, $k, n \in \mathbb{N}$, such that for a renumbered subsequence*

$$g_k^{(1)} = id,\ {g_k^{(n)}}^{-1} g_k^{(m)} \rightharpoonup 0 \quad \text{for } n \neq m, \tag{2.10}$$

$$w^{(n)} = \mathrm{w} - \lim {g_k^{(n)}}^{-1} u_k \tag{2.11}$$

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|^2 \le \limsup \|u_k\|^2 \tag{2.12}$$

$$u_k - \sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)} \overset{D}{\rightharpoonup} 0. \tag{2.13}$$

Lemma 2.6. *Let D be the group of magnetic shifts in $H^1_\alpha(M)$, let u_k be a bounded sequence in $H^1(M)$ and let $w^{(n)}$ be as in Theorem 2.5. Then the corresponded renamed subsequence u_k satisfies*

$$\int_M |u_k|^q d\mu = \sum_{n\in\mathbb{N}} \int_M |w^{(n)}|^q d\mu, \quad q \in (2, 2^*). \tag{2.14}$$

Proof. Apply Theorem 2.5 for the bounded (by (1.6) sequence $|u_k|$ in $H^1(M)$ equipped with the dislocation group $D_0 := \{v \to v \circ \eta, \eta \in I$. Since the weak convergence in both spaces H^1 and H^1_α implies convergence in measure, the weak limits (2.11) in the (H^1, D_0)-case, written in terms of those in the (H^1_α, D)-case, are $|w^{(n)}|$. Note now that $g_{\eta_k} \rightharpoonup 0$ (in (H^1, D_0)) implies that for any compact set $K \subset M$, $d(\eta_k K, 0) \to \infty$. Indeed, if $\eta_k x_k$ were bounded for some $x_k \in K$, then, since η_k are isometries, η_k converges in the CO topology (cf. [11]). Then the assertion of the lemma follows elementarily from restriction of $|w^{(n)}|$ to disjoint balls of arbitrarily large radius. ☐

3. Magnetic Schrödinger equation on the Heisenberg group

In this section we give an example of a manifold with a subelliptic energy form and a potential magnetic field to which Theorem 1.3 applies.

Let $\mathbb{H}^3$ be the space $\mathbb{R}^3$, whose elements we denote as $\eta = (x, y, t)$, equipped with the group operation

$$\eta \circ \eta' = (x + x', y + y', t + t' + 2(xy' - yx')). \tag{3.1}$$

This group multiplication endows $\mathbb{H}^3$ with the structure of a Lie group with $e = 0$. Two invariant vector fields $X = \partial_x + 2y\partial_t$ and $Y = \partial_y - 2x\partial_t$ satisfy the bracket condition, namely, together with $T = [X, Y]$ they form the basis in the tangent space, which yields the homogeneous dimension $N = 4$ and $2^* = 4$. The Riemannian structure is fixed by setting the scalar product at $T\mathbb{H}^3$ so that the given basis X, Y, T is orthonormal. The Riemannian measure and the left and the right Haar measure on $\mathbb{H}^3$ coincide with the Lebesgue measure.

The Sobolev inequality (1.3) holds with the subelliptic symbol $a = X\otimes X + Y\otimes Y$ for $2 < q < 4$ and with the elliptic symbol $X \otimes X + Y \otimes Y + T \otimes T$ for $2 < q < 6$, and for any open bounded Ω there is compactness in the Sobolev imbedding for functions with support in Ω [8, 4].

Every homogeneous magnetic field on the Heisenberg group has a form $\beta = Adt \wedge dx + Bdy \wedge dt + (C - 2By + 2Ax)dx \wedge dy$ with arbitrary constants $A, B, C \in \mathbb{R}$ (one can verify (1.9) by direct substitution, and the field is uniquely defined by its value at the origin due to transitivity). A magnetic potential α satisfying $\beta = d\alpha$, can be written in the following form, uniequely up to the differential of an arbitrary function:

$$\alpha = \frac{1}{2}A(tdx - xdt + x^2dy - 2xydx) + \frac{1}{2}B(ydt - tdy - 2xydy + y^2dx) + \frac{1}{2}C(xdy - ydx).$$

The function ψ_η that satisfies $\eta\alpha - \alpha = d\psi_\eta$, and the normalization condition $\psi_e(x, y, t) = 0$ is as follows:

$$\begin{aligned} &\psi_{(x',y',t')}(x, y, t) \\ &= \frac{1}{2}A(t'x - x't - {x'}^2y - y'x^2) + \frac{1}{2}B(y't - t'y - {y'}^2x - x'y^2) + \frac{1}{2}C(x'y - y'x). \end{aligned}$$

Once we evaluate $\alpha(X) = \frac{1}{2}A(t - 4xy) + \frac{3}{4}By^2 - \frac{1}{2}Cy$ and $\alpha(Y) = \frac{3}{4}Ax^2 + \frac{1}{2}B(-t - 4xy) + \frac{1}{2}Cx$, we can write the invariant subelliptic energy functional E_0 on the Heisenberg group as

$$\begin{aligned} E_0[u] = \int_M \Big(&\Big|\frac{1}{i}\frac{\partial u}{\partial x} + \frac{2}{i}y\frac{\partial u}{\partial t} - (\frac{1}{2}A(t - 4xy) + \frac{3}{4}By^2 - \frac{1}{2}Cy)u\Big|^2 \\ &+ \Big|\frac{1}{i}\frac{\partial u}{\partial y} - \frac{2}{i}x\frac{\partial u}{\partial t} - (\frac{3}{4}Ax^2 + \frac{1}{2}B(-t - 4xy) + \frac{1}{2}Cx)u\Big|^2\Big) dx\, dy\, dt, \end{aligned}$$

so that Theorem 1.3 gives existence of the minimizer in the inequality

$$E_0[u] + \int |u|^2 dz \ge c\|u\|^2_{L^q(M,\mu)} \tag{3.2}$$

for $2 < q < 4$.

For the same reason one has existence of the minimizer with $2 < q < 4$ that corresponds to

$$\begin{aligned} E_0[u] = \int_M \Big(&P(x, y, t)\Big|\frac{1}{i}\frac{\partial u}{\partial x} + \frac{2}{i}y\frac{\partial u}{\partial t} - (\frac{1}{2}A(t - 4xy) + \frac{3}{4}By^2 \\ &- \frac{1}{2}Cy)u\Big|^2 + Q(x, y, t)\Big|\frac{1}{i}\frac{\partial u}{\partial y} - \frac{2}{i}x\frac{\partial u}{\partial t} \\ &- (\frac{3}{4}Ax^2 + \frac{1}{2}B(-t - 4xy) + \frac{1}{2}Cx)u\Big|^2\Big) dx\, dy\, dt, \end{aligned}$$

where P, Q are bounded positive measurable functions, bounded away from zero, periodic with respect to the group shifts with $x', y', z' \in \mathbb{Z}$.

The existence result applied to the uniformly elliptic case involves the functional

$$\begin{aligned} E_0[u] = \int_M \Big(&P(x, y, t)\Big|\frac{1}{i}\frac{\partial u}{\partial x} + \frac{2}{i}y\frac{\partial u}{\partial t} - (\frac{1}{2}A(t - 4xy) + \frac{3}{4}By^2 - \frac{1}{2}Cy)u\Big|^2 \\ &+ Q(x, y, t)\Big|\frac{1}{i}\frac{\partial u}{\partial y} - \frac{2}{i}x\frac{\partial u}{\partial t} - (\frac{3}{4}Ax^2 + \frac{1}{2}B(-t - 4xy) + \frac{1}{2}Cx)u\Big|^2 \\ &+ R(x, y, t)\Big|\frac{1}{i}\frac{\partial u}{\partial t} - \frac{1}{2}(By - Ax)u\Big|^2\Big) dx\, dy\, dt, \end{aligned}$$

with $2 < q < 6$ (we used here the evaluation $\alpha(\partial_t) = \frac{1}{2}(By - Ax)$), assuming that P, Q, R satisfy the same conditions as P, Q in the previous example.

4. Proof of Theorem 1.3

Proof. Let u_k be a minimizing sequence for the relation (1.7) We apply Theorem 2.5:

$$\sum \|w^{(n)}\|^2_{H^1_\alpha(M)} \le c_q. \tag{4.1}$$

At the same time we have (2.14). From (2.14) and (4.1) follows that

$$\sum \|w^{(n)}\|^2_{H^1_\alpha(M)} \le c_q \sum t_n^{2/q}, \tag{4.2}$$

where $t_n = \|w^{(n)}\|^q_{L^p(X,\mu)}$. Note now that (2.14) can be written as $\sum t_n = 1$, so that, since $q > 2$, $\sum t_n^{2/q} = 1$ only if all but one of t_n, say for $n = n_0$, equals zero. We conclude that $w^{(n_0)}$ is the minimizer for (1.7). □

Remark 4.1. We note that from the proof of Theorem 1.3 follows that that for any minimizing sequence u_k for (1.7) there is a sequence η_k, such that $g_{\eta_k} u_k$ converges to the minimizer in $H^1_\alpha(M)$. Indeed, with $\eta_k = (\eta_k^{n_0})^{-1}$ as above we have a weak convergence and convergence of the norms, and thus the norm convergence.

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