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A GENERALIZED SOLUTION TO A CAHN-HILLIARD/ALLEN-CAHN SYSTEM

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Abstract. We study a system consisting of a Cahn-Hilliard and several Allen-Cahn type equations. This system was proposed by Fan, L.-Q. Chen, S. Chen and Voorhees for modelling Ostwald ripening in two-phase system. We prove the existence of a generalized solution whose concentration component is in L^∞.

1. Introduction

Ostwald ripening is a phenomenon observed in a wide variety of two-phase systems in which there is coarsening of one phase dispersed in the matrix of another. Because its practical importance, this process has been extensively studied in several degrees of generality. In particular for Ostwald ripening of anisotropic crystals, Fan et al. [6] presented a model taking in consideration both the evolution of the compositional field and of the crystallographic orientations. In the work of Fan et al. [6], there are also numerical experiments used to validate the model, but there is no rigorous mathematical analysis of the model. Our objective in this paper is to do such mathematical analysis.

By defining orientation and composition field variables, the kinetics of coupled grain growth can be described by their spatial and temporal evolution, which is related with the total free energy of the system. The microstructural evolution of Ostwald ripening can be described by the Cahn-Hilliard/Allen-Cahn system

$$\begin{gathered}\partial_t c = \nabla\cdot[D\nabla(\partial_c\mathcal{F}-\kappa_c\Delta c)],\quad (x,t)\in\Omega_T\\ \partial_t\theta_i = -L_i(\partial_{\theta_i}\mathcal{F}-\kappa_i\Delta\theta_i),\quad (x,t)\in\Omega_T\\ \partial_{\mathbf{n}}c=\partial_{\mathbf{n}}(\partial_c\mathcal{F}-\kappa_c\Delta c)=\partial_{\mathbf{n}}\theta_i=0,\quad (x,t)\in S_T\\ c(x,0)=c_0(x),\quad \theta_i(x,0)=\theta_{i0}(x),\quad x\in\Omega\end{gathered}\tag{1.1}$$

for $i=1,\dots,p$. Here, Ω is the physical region where the Ostwald process is occurring; $\Omega_T=\Omega\times(0,T)$; $S_T=\partial\Omega\times(0,T)$; $0<T<+\infty$; $\mathbf{n}$ denotes the unitary exterior normal vector and $\partial_{\mathbf{n}}$ is the exterior normal derivative at the boundary; $c(x,t)$ is the compositional field (fraction of the soluto with respect to

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the mixture) which takes one value within the matrix phase, another value within a second phase grain and $c(x,t)$ varies between these values at the interfacial region between the matrix phase and a second phase grain; $\theta_i(x,t)$, for $i=1,\dots,p$, are the crystallographic orientations fields; D, λ_c, L_i, λ_i are positive constants related to the material properties. The function $\mathcal{F}=\mathcal{F}(c,\theta_1,\dots,\theta_p)$ is the local free energy density which is given by

$$\begin{aligned}&\mathcal{F}(c,\theta_1,\dots,\theta_p)\\&=-\frac{A}{2}(c-c_m)^2+\frac{B}{4}(c-c_m)^4+\frac{D_\alpha}{4}(c-c_\alpha)^4\\&\quad+\frac{D_\beta}{4}(c-c_\beta)^4+\sum_{i=1}^{p}[-\frac{\gamma}{2}(c-c_\alpha)^2\theta_i^2+\frac{\delta}{4}\theta_i^4]+\sum_{i=1}^{p}\sum_{i\neq j=1}^{p}\frac{\varepsilon_{ij}}{2}\theta_i^2\theta_j^2.\end{aligned}\tag{1.2}$$

where c_α and c_β are the solubilities in the matrix phase and the second phase respectively, and $c_m=(c_\alpha+c_\beta)/2$. The positive coefficients A, B, D_α, D_β, γ, δ and ε_{ij} are phenomenological parameters.

In this paper we obtain a $(p+2)$-tuple which satisfies a variational inequality related to Problem (1.1) and also satisfies the physical requirement that the concentration takes values in the closed interval $[0,1]$.

Our approach to the problem is to analyze a three-parameter family of suitable systems which contain a logarithmic perturbation term and approximate the model presented by Fan et al. [6]. In this analysis, we show that the approximate solutions converge to a generalized solution of the original continuous model and this, in particular, will furnish a rigorous proof of the existence of generalized solutions (see the statement of Theorem 2.1). Our approach is similar to that used by Passo et al. [3] for an Cahn-Hilliard/Allen-Cahn system with degenerate mobility.

2. Existence of Solutions

Including the physical restriction on the concentration, Problem (1.1) is stated as follows:

$$\begin{gathered}\partial_t c=\nabla[D\nabla(\partial_c\mathcal{F}-\kappa_c\Delta c)],\quad (x,t)\in\Omega_T\\ \partial_t\theta_i=-L_i(\partial_{\theta_i}\mathcal{F}-\kappa_i\Delta\theta_i),\quad (x,t)\in\Omega_T\\ \partial_{\mathbf{n}}c=\partial_{\mathbf{n}}(\partial_c\mathcal{F}-\kappa_c\Delta c)=\partial_{\mathbf{n}}\theta_i=0,\quad (x,t)\in S_T\\ c(x,0)=c_0(x),\quad \theta_i(x,0)=\theta_{i0}(x),\quad x\in\Omega\\ 0\le c\le 1\quad (x,t)\in\Omega_T\end{gathered}\tag{2.1}$$

for $i=1,\dots,p$.

Throughout this paper, standard notation will be used for the several required functional spaces. We denote by $\overline{f}$ the mean value of f in Ω of a given $f\in L^1(\Omega)$. The duality pairing between $H^1(\Omega)$ and its dual is denoted by $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. We will prove the following:

Theorem 2.1. *Let $T>0$ and $\Omega\subset R^d$, $1\le d\le 3$ be a bounded domain with C^3-boundary. For all $c_0,\theta_{i0}, i=1,\dots,p$, satisfying $c_0,\theta_{i0}\in H^1(\Omega)$, for $i=1,\dots,p$, $0\le c_0\le 1$, there exists a unique $(p+2)$-tuple $(c,w-\overline{w},\theta_1,\dots,\theta_p)$ such that, for $i=1,\dots,p$,*

(a) $c,\ \theta_i\in L^\infty(0,T,H^1(\Omega))\cap L^2(0,T,H^2(\Omega))$,
(b) $w\in L^2(0,T,H^1(\Omega))$
(c) $\partial_t c\in L^2(0,T,[H^1(\Omega)]')$, $\partial_t\theta_i\in L^2(\Omega_T)$

(d) $0 \leq c \leq 1$ *a.e. in* Ω_T
(e) $c(x,0) = c_0(x)$, $\theta_i(x,0) = \theta_{i0}(x)$
(f) $\partial_c \mathcal{F}(c, \theta_1, \dots, \theta_p)$, $\partial_{\theta_i} \mathcal{F}(c, \theta_1, \dots, \theta_p) \in L^2(\Omega_T)$
(g) $\partial_{\mathbf{n}} c_{|_{S_T}} = \partial_{\mathbf{n}} \theta_{i|_{S_T}} = 0$ *in* $L^2(S_T)$
(h) $(c, w, \theta_1, \dots, \theta_p)$ *satisfies*

$$\int_0^T \langle \partial_t c, \phi \rangle dt = -\iint_{\Omega_T} \nabla w \nabla \phi, \quad \forall \phi \in L^2(0,T,H^1(\Omega)), \tag{2.2}$$

$$\int_0^T \xi(t) \left\{ \kappa_c D(\nabla c, \nabla \phi - \nabla c) - (w - D\partial_c \mathcal{F}(c, \theta_1, \dots, \theta_p), \phi - c) \right\} dt \geq 0, \tag{2.3}$$

for all $\xi \in C[0,T), \xi \geq 0$, $\forall \phi \in K = \{\eta \in H^1(\Omega),\ 0 \leq \eta \leq 1,\ \overline{\eta} = \overline{c_0}\}$, *and*

$$\iint_{\Omega_T} \partial_t \theta_i \psi_i = -\iint_{\Omega_T} L(\partial_{\theta_i} \mathcal{F}(c, \theta_1, \dots, \theta_p) - \kappa_i \Delta \theta_i) \psi_i, \tag{2.4}$$

for all $\psi_i \in L^2(\Omega_T), i = 1, \dots, p$, *where* $\mathcal{F}$ *is given by (1.2).*

Remark 2.2. The inequality obtained (2.3) is similar to one obtained by Elliott and Luckhaus [5] in the case of the deep quench limit problem for a system of nonlinear diffusion equations.

Remark 2.3. We observe that (2.3) comes from the fact that classically w is expected to be equal to $D(\partial_c \mathcal{F} - \kappa_c \Delta c)$ up to a function of time.

Remark 2.4. The solution presented in Theorem 2.1 is a generalized solution of (2.1). In fact, as will be shown at the end of this article, (2.3) holds as an equality in the region where $0 < c(x,t) < 1$ for almost all times.

We start by proving the uniqueness referred to in Theorem 2.1.

Lemma 2.5. *Consider a solution of (2.2)–(2.4) as in Theorem 2.1. Under the hypotheses (a)–(e) and (h) of Theorem 2.1, the components* c, θ_1, …, θ_p *are uniquely determined; the component* w *is uniquely determined up to a function of time.*

Proof. We argue as Elliott and Luckhaus [5]. We introduce the Green's operator G: given $f \in [H^1(\Omega)]'_{null} = \{f \in [H^1(\Omega)]',\ \langle f, 1 \rangle = 0\}$, we define $Gf \in H^1(\Omega)$ as the unique solution of

$$\int_\Omega \nabla Gf \nabla \psi = \langle f, \psi \rangle, \quad \forall \psi \in H^1(\Omega) \quad \text{and} \quad \int_\Omega Gf = 0.$$

Let $z^c = c_1 - c_2$, $z^w = w_1 - w_2$ and $z^{\theta_i} = \theta_{i1} - \theta_{i2}$, $i = 1, \dots, p$ be the differences of two pair of solutions to (2.2)–(2.4) as in Theorem 2.1. Since equation (2.2) implies that the mean value of the composition field in Ω is conserved, we have that $(z^c, 1) = 0$ and we find from (2.2) that

$$-Gz_t^c = \overline{z^w}.$$

The definition of the Green operator and the fact that $(z^c, 1) = 0$ give

$$-(\nabla G z_t^c, \nabla G z^c) = -(G z_t^c, z^c) = (\overline{z^w}, z^c) = (z^w, z^c).$$

Since c_1, $c_2 \in K = \{\eta \in H^1(\Omega),\ 0 \leq \eta \leq 1,\ \overline{\eta} = \overline{c_0}\}$, we find from (2.3) that

$$-\kappa_c D |\nabla z^c|^2 + (z^w, z^c) - D(\partial_c \mathcal{F}(c_1, \theta_{11}, \dots, \theta_{p1}) - \partial_c \mathcal{F}(c_2, \theta_{12}, \dots, \theta_{p2}), z^c) \geq 0.$$

Thus, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla G z^c| + \kappa_c D|\nabla z^c|^2 \\
&\leq -D(\partial_c \mathcal{F}(c_1,\theta_{11},\dots,\theta_{p1}) - \partial_c \mathcal{F}(c_2,\theta_{12},\dots,\theta_{p2}) - \kappa_c \Delta z^c, z^c).
\end{aligned}$$

We find from (2.4) that

$$\begin{aligned}
&\frac{D}{2L_i}\frac{d}{dt}|z^{\theta_i}|^2 + D\lambda_i|\nabla z^{\theta_i}|^2 \\
&+ D\big(\partial_{\theta_i}\mathcal{F}(c_1,\theta_{11},\dots,\theta_{p1}) - \partial_{\theta_i}\mathcal{F}(c_2,\theta_{12},\dots,\theta_{p2}), z^{\theta_i}\big) = 0.
\end{aligned}$$

By adding the above equations, using the convexity of $[\mathcal{F}+H](c,\theta_1,\dots,\theta_p)$,

$$[\mathcal{F}+H](c,\theta_1,\dots,\theta_p) = \frac{D_\alpha}{4}(c-c_\alpha)^4 + \frac{D_\beta}{4}(c-c_\beta)^4 + \frac{\delta}{4}\sum_{i=1}^p \theta_i^4$$

where

$$H(c,\theta_1,\dots,\theta_p) = \frac{A}{2}(c-c_m)^2 + \frac{\gamma}{2}\sum_{i=1}^p c^2\theta_i^2 - \sum_{i=1}^p\sum_{i\neq j=1}^p \frac{\varepsilon_{ij}}{2}\theta_i^2\theta_j^2,$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla G z^c|^2 + \kappa_c D|\nabla z^c|^2 + \sum_{i=1}^p [\frac{D}{2L_i}\frac{d}{dt}|z^{\theta_i}|^2 + D\lambda_i|\nabla z^{\theta_i}|^2] \\
&\leq \big(\nabla(H(c_1,\theta_{11},\dots,\theta_{p1}) - H(c_2,\theta_{12},\dots,\theta_{p2}))\cdot(z^c, z^{\theta_1},\dots,z^{\theta_p}),1\big)
\end{aligned} \tag{2.5}$$

To estimate the right-hand side of the above inequality, we use the regularity of c_i and θ_{ik}. Then

$$\begin{aligned}
&(\nabla(\theta_{i1}^2\theta_{j1}^2 - \theta_{i2}^2\theta_{j2}^2)\cdot(z^{\theta_i}, z^{\theta_j}),1) \\
&= 2((\theta_{i1}\theta_{j1}^2 - \theta_{i2}\theta_{j2}^2, \theta_{i1}^2\theta_{j1} - \theta_{i2}^2\theta_{j2})\cdot(z^{\theta_i}, z^{\theta_j}),1) \\
&= 2((z^{\theta_i}\theta_{j1}^2 + \theta_{i2}(\theta_{j1}^2 - \theta_{j2}^2), \theta_{i1}^2\theta_{j1} - \theta_{i2}^2\theta_{j2})\cdot(z^{\theta_i}, z^{\theta_j}),1) \\
&= 2((z^{\theta_i}\theta_{j1}^2 + \theta_{i2}(\theta_{j1}+\theta_{j2})z^{\theta_j}, z^{\theta_j}\theta_{i1}^2 + \theta_{j2}(\theta_{i1}+\theta_{i2})z^{\theta_i})\cdot(z^{\theta_i}, z^{\theta_j}),1) \\
&\leq C[|z^{\theta_i}|^2 + |z^{\theta_j}|^2] + \frac{D\lambda_i}{8(p-1)}|\nabla z^{\theta_i}|^2 + \frac{D\lambda_j}{8(p-1)}|\nabla z^{\theta_j}|^2
\end{aligned}$$

and

$$\gamma(\nabla(c_1^2\theta_{i1}^2 - c_2^2\theta_{i2}^2)\cdot(z^c, z^{\theta_i}),1) \leq C[|z^c|^2 + |z^{\theta_i}|^2] + \frac{\kappa_c D}{2p}|\nabla z^c|^2 + \frac{D\lambda_i}{4}|\nabla z^{\theta_i}|^2.$$

The above inequalities and (2.5) imply

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla G z^c|^2 + \frac{\kappa_c D}{2}|\nabla z^c|^2 + \sum_{i=1}^p \big[\frac{D}{2L_i}\frac{d}{dt}|z^{\theta_i}|^2 + \frac{D\lambda_i}{2}|\nabla z^{\theta_i}|^2\big] \\
&\leq C\big[\|z^c\|_{L^2(\Omega)}^2 + \sum_{i=1}^p \|z^{\theta_i}\|_{L^2(\Omega)}^2\big].
\end{aligned}$$

From the definition of the Green operator, $|z^c|^2 = (\nabla G z^c, \nabla z^c)$. Using the Hölder inequality, we rewrite the above inequality as

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\nabla G z^c|^2 + \frac{\kappa_c D}{4}|\nabla z^c|^2 + \sum_{i=1}^{p}\big[\frac{D}{2L_i}\frac{d}{dt}|z^{\theta_i}|^2 + \frac{D\lambda_i}{2}|\nabla z^{\theta_i}|^2\big]\\
&\le C\big[\|\nabla G z^c\|^2_{L^2(\Omega)} + \sum_{i=1}^{p}\|z^{\theta_i}\|^2_{L^2(\Omega)}\big].
\end{aligned}
$$

A standard Gronwall argument then yields

$$\nabla G z^c = 0 \quad \text{and} \quad z^{\theta_i} = 0, \quad i = 1, \dots, p,$$

since

$$G z^c(0) = 0 \quad \text{and} \quad z^{\theta_i}(0) = 0, \quad i = 1, \dots, p.$$

The uniqueness is proved since $|z^c|^2 = (\nabla \mathcal{G} z^c, \nabla z^c) = 0$. □

As a corollary of this, we have the following:

Lemma 2.6. *Under the conditions of Theorem 2.1, when either $c_0 \equiv 0$ or $c_0 \equiv 1$ almost everywhere on Ω, there is a solution of* (2.2)–(2.4).

Proof. In such cases, since $0 \le c_0(x) \le 1$, we have in fact that either $c_0(x) = 0$ or $c_0(x) = 1$. Now, take c identically zero or one, respectively. Then, equation (2.2) is trivially satisfied and will imply that w is a constant. Otherwise, (2.3) is also trivially satisfied and to obtain a solution of the Problem (2.2)–(2.4), we just have to solve the nonlinear parabolic system (2.4). But this system can be solved rather easily by standard methods, like Galerkin method, for instance, since the nonlinearities have the right sign and thus furnish suitable estimates. □

By the above lemma, we have uniqueness of c in the cases where either $c_0 \equiv 0$ or $c_0 \equiv 1$ almost everywhere on Ω. Thus, to prove Theorem 2.1, it just remain to deal with the cases where the mean value of the initial condition c_0 is strictly between to zero and one. Thus, in the following we assume that

$$
\begin{aligned}
c_0,\ \theta_{i0} \in H^1(\Omega), \quad i = 1, \dots, p,\\
0 \le c_0 \le 1, \quad \overline{c_0} \in (0,1),
\end{aligned}
\tag{2.6}
$$

To obtain the result in Theorem 2.1, we approximate system (2.1) by a three-parameter family of suitable systems which contain a logarithmic perturbation term and then pass to the limit. In Section 3, we use the results of Passo et al. [3] to construct such perturbed systems and together with some ideas presented by Copetti and Elliott [2] and by Elliott and Luckhaus [5], we take the limit in these systems in the last three sections.

For sake of simplicity of exposition, without loosing generality, we develop the proof for the case of dimension one and for only one orientation field variable, that is, when Ω is a bounded open interval and p is equal to one, and thus we have just one orientation field that we denote θ. In this case, the local free energy density is reduced to

$$
\begin{aligned}
\mathcal{F}(c,\theta) = &-\frac{A}{2}(c-c_m)^2 + \frac{B}{4}(c-c_m)^4 + \frac{D_\alpha}{4}(c-c_\alpha)^4 + \frac{D_\beta}{4}(c-c_\beta)^4\\
&+ \frac{\delta}{4}\theta^4 - \frac{\gamma}{2}(c-c_\alpha)^2\theta^2.
\end{aligned}
\tag{2.7}
$$

We remark that, even though the cross terms of (1.2) involving the orientation field variables are absent in above expression, their presences when p is greater than one will not bring any difficulty for extending the result, as we will point out at the end of the paper.

3. Perturbed Systems

In this section we construct a three-parameter family of perturbed systems. The auxiliary parameter M controls a truncation of the local free energy $\mathcal{F}$ which will permit the application of an existence result of Passo et al. [3]. The parameters σ and ε are associated to the logarithmic term, their introduction will enable us to guarantee that the composition field variable c takes values in the closure of the set $I = (0,1)$.

For each positive constants σ, M and $\varepsilon \in (0,1)$, we define the perturbed local free energy density as follows:

$$\mathcal{F}_{\sigma\varepsilon M}(c,\theta) = f(c) + g_M(\theta) + h_M(c,\theta) + \varepsilon[F_\sigma(c) + F_\sigma(1-c)]. \tag{3.1}$$

where the first three terms give a truncation of the original $\mathcal{F}(c,\theta)$ given in (2.7), and the last term is a logarithmic perturbation. To obtain a truncation of the local free energy density, we introduce bounded functions whose summation coincides with $\mathcal{F}$ for $(c,\theta) \in [0,1] \times [-M,M]$. Let f, g_M and h_M be such that

$$f(c) = -\frac{A}{2}(c-c_m)^2 + \frac{B}{4}(c-c_m)^4 + \frac{D_\alpha}{4}(c-c_\alpha)^4 + \frac{D_\beta}{4}(c-c_\beta)^4, \quad 0 \le c \le 1,$$
$$g_M(\theta) = \frac{\delta}{4}h_2^2(M;\theta) \quad \text{and} \quad h_M(c,\theta) = h_1(c)h_2(M;\theta)$$

with

$$h_1(c) = -\frac{\gamma}{2}(c-c_\alpha)^2, \quad 0 \le c \le 1,$$
$$h_2(M;\theta) = \theta^2, \quad -M \le \theta \le M.$$

Outside the intervals $[0,1]$ and $[-M,M]$, we extend the above functions to satisfy

$$\|f\|_{C^2(\mathbf{R})} \le U_0, \quad \|g_M\|_{C^2(\mathbf{R})} \le V_0(M), \tag{3.2}$$
$$\|h_1\|_{C^2(\mathbf{R})} \le W_0, \quad \|h_M\|_{C^2(\mathbf{R}^2)} \le Z_0(M), \tag{3.3}$$
$$|h_2(M;\theta)| \le K\theta^2, \quad |h_2'(M;\theta)| \le K|\theta|, \quad \forall M > 0, |\, \forall\theta \in \mathbf{R}, \tag{3.4}$$

where $U_0, W_0, K > 0$ are constants and, for each M, $V_0(M)$ and $Z_0(M)$ are also constants.

We took the logarithmic term $\varepsilon[F_\sigma(c) - F_\sigma(1-c)]$ as in Passo et al. [3]. Let us denote

$$F(s) = s\ln s.$$

For $\sigma \in (0, 1/e)$, we choose $F'_\sigma(s)$ such that

$$F'_\sigma(s) = \begin{cases} \frac{\sigma}{2\sigma - s} + \ln\sigma, & \text{if } s < \sigma, \\ \ln s + 1, & \text{if } \sigma \le s \le 1-\sigma, \\ f_\sigma(s), & \text{if } 1-\sigma < s < 2, \\ 1, & \text{if } s \ge 2, \end{cases}$$

where $f_\sigma \in C^1([1-\sigma, 2])$ is chosen having the following properties:

$$\begin{gathered} f_\sigma \leq F', \quad f'_\sigma \geq 0, \\ f_\sigma(1-\sigma) = F'(1-\sigma), \quad f_\sigma(2) = 1, \\ f'_\sigma(1-\sigma) = F''(1-\sigma), \quad f'_\sigma(2) = 0. \end{gathered}$$

Defining

$$F_\sigma(s) = -\frac{1}{e} + \int_{\frac{1}{e}}^{s} F'_\sigma(\xi) d\xi,$$

we have $F_\sigma \in C^2(\mathbf{R})$ and $F''_\sigma \geq 0$.

Clearly, $\mathcal{F}_{\sigma\varepsilon M}$ has a lower bound which is independent of σ and ε. We claim that $\mathcal{F}_{\sigma\varepsilon M}$ can also be bounded from below independently of M. To prove this fact, we just have to estimate $g_M(\theta) + h_M(c, \theta)$. We have

$$\begin{aligned} g_M(\theta) + h_M(c,\theta) &= \frac{\delta}{4} h_2^2(M;\theta) + h_1(c) h_2(M;\theta) \\ &= \frac{\delta}{4} h_2(M;\theta)\big[h_2(M;\theta) + \frac{4}{\delta} h_1(c)\big] \geq -\frac{h_1^2(c)}{\delta} \geq -\frac{W_0^2}{\delta} \end{aligned}$$

Therefore,

$$\begin{gathered} -U_0 - \frac{W_0^2}{\delta} - \frac{2}{e} \leq \mathcal{F}_{\sigma\varepsilon M}(c,\theta) \quad \text{in } \mathbf{R}^2, \\ \mathcal{F}_{\sigma\varepsilon M}(c,\theta) < U_0 + g_M(\theta) - h_M(c,\theta) \quad \text{in cl } I. \end{gathered} \tag{3.5}$$

Then, the perturbed systems we will consider are

$$\begin{gathered} \partial_t c = D(\partial_c \mathcal{F}_{\sigma\varepsilon M}(c,\theta) - \kappa_c c_{xx})_{xx}, \quad (x,t) \in \Omega_T \\ \partial_t \theta = -L[\partial_\theta \mathcal{F}_{\sigma\varepsilon M}(c,\theta) - \kappa \theta_{xx}], \quad (x,t) \in \Omega_T \\ \partial_{\mathbf{n}} c = \partial_{\mathbf{n}}(\partial_c \mathcal{F}_{\sigma\varepsilon M}(c,\theta) - \kappa_c c_{xx}) = \partial_{\mathbf{n}} \theta = 0 \quad (x,t) \in S_T \\ c(x,0) = c_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \Omega \end{gathered} \tag{3.6}$$

To solve the above problem, we shall use the next proposition which is an existence result stated by Passo et al. [3] for the system

$$\begin{gathered} \partial_t u = [q_1(u,v)(f_1(u,v) - \kappa_1 u_{xx})_x]_x, \quad (x,t) \in \Omega_T \\ \partial_t v = -q_2(u,v)[f_2(u,v) - \kappa_2 v_{xx}], \quad (x,t) \in \Omega_T \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} u_{xx} = \partial_{\mathbf{n}} v = 0 \quad (x,t) \in S_T \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega \end{gathered} \tag{3.7}$$

where q_i and f_i satisfy the following hypotheses:

- (H1) $q_i \in C(\mathbf{R}^2, \mathbf{R}^+)$, with $q_{\min} \leq q_i \leq q_{\max}$ for some $0 < q_{\min} \leq q_{\max}$;
- (H2) $f_1 \in C^1(\mathbf{R}^2, \mathbf{R})$ and $f_2 \in C(\mathbf{R}^2, \mathbf{R})$, with $\|f_1\|_{C^1} + \|f_2\|_{C^0} \leq F_0$ for some $F_0 > 0$.

Proposition 3.1. *Assuming (H1), (H2) and that $u_0,\ v_0 \in H^1(\Omega)$, there exists a pair of functions (u,v) such that*

1. $u \in L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^3(\Omega)) \cap C([0,T];H^\lambda(\Omega))$, $\lambda < 1$
2. $v \in L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega)) \cap C([0,T];H^\lambda(\Omega))$, $\lambda < 1$
3. $\partial_t u \in L^2(0,T,[H^1(\Omega)]')$, $\quad \partial_t v \in L^2(\Omega_T)$
4. $u(0) = u_0$ *and* $v(0) = v_0$ *in* $L^2(\Omega)$
5. $\partial_{\mathbf{n}} u_{|S_T} = \partial_{\mathbf{n}} v_{|S_T} = 0$ *in* $L^2(S_T)$

(6) (u,v) *solves (3.7) in the following sense*

$$\int_0^t \langle \partial_t u, \phi\rangle = -\iint_{\Omega_t} q_1(u,v)(f_1(u,v)-\kappa_1 u_{xx})_x\phi_x, \quad \forall\phi\in L^2(0,T,H^1(\Omega))$$

$$\iint_{\Omega_t} \partial_t v\psi = -\iint_{\Omega_t} q_2(u,v)(f_2(u,v)-\kappa_2 v_{xx})\psi, \quad \forall\phi\in L^2(\Omega_T).$$

where $\Omega_t=\Omega\times(0,t)$ *and* $\iint_{\Omega_t}$ *is the integral over* Ω_t.

Remark 3.2. The regularity of the test functions with respect to t allow us to obtain the integrals over $(0,t)$, instead of $(0,T)$ as originally presented by Passo et al. [3].

Applying the above proposition, for each $\varepsilon,\sigma,M>0$ there exists a solution $(c_{\sigma\varepsilon M},\theta_{\sigma\varepsilon M})$ of Problem (3.6) in the following sense

$$\int_0^t \langle \partial_t c_{\sigma\varepsilon M}, \phi\rangle = -\iint_{\Omega_t} D(\partial_c\mathcal{F}_{\sigma\varepsilon M}(c_{\sigma\varepsilon M},\theta_{\sigma\varepsilon M})-\kappa_c[c_{\sigma\varepsilon M}]_{xx})_x\phi_x, \tag{3.8}$$

for $\phi\in L^2(0,T,H^1(\Omega))$ and

$$\iint_{\Omega_t} \partial_t\theta_{\sigma\varepsilon M}\psi = -\iint_{\Omega_t} L(\partial_\theta\mathcal{F}_{\sigma\varepsilon M}(c_{\sigma\varepsilon M},\theta_{\sigma\varepsilon M})-\kappa[\theta_{\sigma\varepsilon M}]_{xx})\psi, \tag{3.9}$$

for $\psi\in L^2(\Omega_T)$. Let us observe that equation for c in equation (3.8) implies that the mean value of $c_{\sigma\varepsilon M}$ in Ω is

$$\overline{c_{\sigma\varepsilon M}(t)} = \overline{c_0}\in(0,1) \tag{3.10}$$

4. Limit as $M\to\infty$

In this section we obtain some a priori estimates that will allow us to take the limit in the parameter M. Actually, some of these estimates are also independent of the parameters σ and ε and will be useful in next sections.

Lemma 4.1. *There exists a constant* C_1 *independent of* M *(sufficiently large),* σ *(sufficiently small) and* ε *such that*

(1) $\|c_{\sigma\varepsilon M}\|_{L^\infty(0,T,H^1(\Omega))}\le C_1$
(2) $\|\theta_{\sigma\varepsilon M}\|_{L^\infty(0,T,H^1(\Omega))}\le C_1$
(3) $\|(\partial_c\mathcal{F}_{\sigma\varepsilon M}-\kappa_c(c_{\sigma\varepsilon M})_{xx})_x\|_{L^2(\Omega_T)}\le C_1$
(4) $\|\partial_\theta\mathcal{F}_{\sigma\varepsilon M}-\kappa(\theta_{\sigma\varepsilon M})_{xx}\|_{L^2(\Omega_T)}\le C_1$
(5) $\|\partial_t c_{\sigma\varepsilon M}\|_{L^\infty(0,T,[H^1(\Omega)]')}\le C_1$
(6) $\|\partial_t\theta_{\sigma\varepsilon M}\|_{L^2(\Omega_T)}\le C_1$
(7) $\|\mathcal{F}_{\sigma\varepsilon M}(c_{\sigma\varepsilon M},\theta_{\sigma\varepsilon M})\|_{L^\infty(0,T,L^1(\Omega))}\le C_1$

Proof. To obtain items 3, 4 and 7, we argue as Passo et al. [3] and Elliott and Garcke [4]. First, we observe that by the regularity of $c_{\sigma\varepsilon M}$ and $\theta_{\sigma\varepsilon M}$, we could take

$$\partial_c\mathcal{F}_{\sigma\varepsilon M}-\kappa_c(c_{\sigma\varepsilon M})_{xx} \quad\text{and}\quad \partial_\theta\mathcal{F}_{\sigma\varepsilon M}-\kappa(\theta_{\sigma\varepsilon M})_{xx}$$

as test functions in the equations (3.8) and (3.9), respectively. By adding the resulting identities, we obtain

$$\begin{aligned} &\int_0^t \langle \partial_t c_{\sigma\varepsilon M}, \partial_c \mathcal{F}_{\sigma\varepsilon M} - \kappa_c (c_{\sigma\varepsilon M})_{xx} \rangle + \iint_{\Omega_t} \partial_t \theta_{\sigma\varepsilon M} \partial_\theta \mathcal{F}_{\sigma\varepsilon M} - \kappa(\theta_{\sigma\varepsilon M})_{xx} \\ &= - \iint_{\Omega_t} D[(\partial_c \mathcal{F}_{\sigma\varepsilon M} - \kappa_c (c_{\sigma\varepsilon M})_{xx})_x]^2 - \iint_{\Omega_t} L[\partial_\theta \mathcal{F}_{\sigma\varepsilon M} - \kappa(\theta_{\sigma\varepsilon M})_{xx}]^2. \end{aligned} \tag{4.1}$$

Now, given a small $h > 0$, we consider the functions

$$c_{\sigma\varepsilon M h}(x,t) = \frac{1}{h} \int_{t-h}^t c_{\sigma\varepsilon M}(\tau, x) d\tau$$

where we set $c_{\sigma\varepsilon M}(x,t) = c_0(x)$ for $t \leq 0$. Since $\partial_t c_{\sigma\varepsilon M h}(x,t) \in L^2(\Omega_T)$, we have

$$\begin{aligned} &\int_0^T \langle (c_{\sigma\varepsilon M h})_t, [\partial_c \mathcal{F}_{\sigma\varepsilon M h} - \kappa_c (c_{\sigma\varepsilon M h})_{xx}] \rangle dt \\ &+ \iint_{\Omega_T} (\theta_{\sigma\varepsilon M})_t [\partial_\theta \mathcal{F}_{\sigma\varepsilon M h} - \kappa(\theta_{\sigma\varepsilon M})_{xx}] \\ &= \int_\Omega [\frac{\kappa_c}{2} |[c_{\sigma\varepsilon M h}(t)]_x|^2 + \frac{\kappa}{2} |[\theta_{\sigma\varepsilon M}]_x(t)|^2 + \mathcal{F}_{\sigma\varepsilon M h}(t)] \\ &\quad - \int_\Omega [\frac{\kappa_c}{2} |[c_0]_x|^2 + \frac{\kappa}{2} |[\theta_0]_x|^2 + \mathcal{F}_{\sigma\varepsilon M}(c_0, \theta_0)]. \end{aligned}$$

Taking the limit as h tends to zero in the above expression and using the result in (4.1), we obtain

$$\begin{aligned} &\iint_{\Omega_t} D[(\partial_c \mathcal{F}_{\sigma\varepsilon M} - \kappa_c (c_{\sigma\varepsilon M})_{xx})_x]^2 + \iint_{\Omega_t} L[\partial_\theta \mathcal{F}_{\sigma\varepsilon M} - \kappa(\theta_{\sigma\varepsilon M})_{xx}]^2 \\ &\quad + \frac{\kappa_c}{2} \|[c_{\sigma\varepsilon M}]_x(t)\|^2_{L^2(\Omega)} + \frac{\kappa}{2} \|[\theta_{\sigma\varepsilon M}]_x(t)\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{F}_{\sigma\varepsilon M}(t) \\ &\quad = \frac{\kappa_c}{2} \|[c_0]_x\|^2_{L^2(\Omega)} + \frac{\kappa}{2} \|[\theta_0]_x\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{F}_{\sigma\varepsilon M}(c_0, \theta_0) \end{aligned}$$

for almost every $t \in (0, T]$. Using (2.6) and (3.5), we could choose M_0 and σ_0, depending only on the initial conditions, to obtain for all $M > M_0$ and all $\sigma < \sigma_0$

$$\begin{aligned} &\iint_{\Omega_T} D[(\partial_c \mathcal{F}_{\sigma\varepsilon M} - \kappa_c (c_{\sigma\varepsilon M})_{xx})_x]^2 + \iint_{\Omega_T} L[\partial_\theta \mathcal{F}_{\sigma\varepsilon M} - \kappa(\theta_{\sigma\varepsilon M})_{xx}]^2 \\ &\quad + \frac{\kappa_c}{2} \|[c_{\sigma\varepsilon M}]_x(t)\|^2_{L^2(\Omega)} + \frac{\kappa}{2} \|[\theta_{\sigma\varepsilon M}]_x(t)\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{F}_{\sigma\varepsilon M}(t) \leq C_1 \end{aligned} \tag{4.2}$$

which implies items 3, 4 and 7 since we have (3.5). Using the Poincaré's inequality, (3.5) and (3.10), Item 1 is also verified.

To prove Item 6, we choose $\psi = \partial_t \theta_{\sigma\varepsilon M}$ as a test function in (3.9), which yields

$$\begin{aligned} \iint_{\Omega_T} [\partial_t \theta_{\sigma\varepsilon M}]^2 &= - \iint_{\Omega_T} (\partial_\theta \mathcal{F}_{\sigma\varepsilon M} - \kappa(\theta_{\sigma\varepsilon M})_{xx}) \partial_t \theta_{\sigma\varepsilon M} \\ &\leq \Big(\iint_{\Omega_T} (\partial_\theta \mathcal{F}_{\sigma\varepsilon M} - \kappa(\theta_{\sigma\varepsilon M})_{xx})^2 \Big)^{1/2} \Big(\iint_{\Omega_T} [\partial_t \theta_{\sigma\varepsilon M}]^2 \Big)^{1/2}. \end{aligned}$$

Since

$$\int_\Omega \theta^2_{\sigma\varepsilon M} \leq 2 \int_\Omega |\theta_0|^2 + 2t \iint_{\Omega_T} (\partial_t \theta_{\sigma\varepsilon M})^2 d\tau \leq C_2,$$

Item 6 and (4.2), it follows that Item 2 is verified. Finally, Item 5 follows since

$$\Big|\int_0^T \langle \partial_t c_{\sigma\varepsilon M}, \phi\rangle\Big| \leq \Big(\iint_{\Omega_T} [(\partial_c \mathcal{F}_{\sigma\varepsilon M} - \kappa_c (c_{\sigma\varepsilon M})_{xx})_x]^2\Big)^{1/2} \Big(\iint_{\Omega_T} (\phi_x)^2\Big)^{1/2}$$

for all $\phi \in L^2(0,T,H^1(\Omega))$. □

Remark 4.2. From (4.2), using (3.5), we obtain

$$\begin{aligned} &\iint_{\Omega_T} D[(\partial_c \mathcal{F}_{\sigma\varepsilon M} - \kappa_c (c_{\sigma\varepsilon M})_{xx})_x]^2 + \iint_{\Omega_T} L[\partial_\theta \mathcal{F}_{\sigma\varepsilon M} - \kappa(\theta_{\sigma\varepsilon M})_{xx}]^2 \\ &+ \frac{\kappa_c}{2}\|[c_{\sigma\varepsilon M}(t)]_x\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|[\theta_{\sigma\varepsilon M}]_x(t)\|^2_{L^2(\Omega)} \leq C_1. \end{aligned} \tag{4.3}$$

Lemma 4.3. *For M sufficiently large and σ sufficiently small, there exist a constant C_3 independent of σ, M and ε and a constant $C_3'(\sigma)$ independent of M and ε such that*

(1) $\|\partial_c \mathcal{F}_{\sigma\varepsilon M}\|_{L^2(0,T,H^1(\Omega))} \leq C_3'$,
(2) $\|\partial_\theta \mathcal{F}_{\sigma\varepsilon M}\|_{L^2(\Omega_T)} \leq C_3$,
(3) $\|[c_{\sigma\varepsilon M}]_{xx}\|_{L^2(\Omega_T)} \leq C_3$,
(4) $\|[\theta_{\sigma\varepsilon M}]_{xx}\|_{L^2(\Omega_T)} \leq C_3$,

Proof. First, we prove items 2 and 4. From Item 4 of Lemma 4.1, we have

$$\iint_{\Omega_T} (\partial_\theta \mathcal{F}_{\sigma\varepsilon M})^2 - 2\kappa \iint_{\Omega_T} \partial_\theta \mathcal{F}_{\sigma\varepsilon M}[\theta_{\sigma\varepsilon M}]_{xx} + \kappa^2 \iint_{\Omega_T} [\theta_{\sigma\varepsilon M}]^2_{xx} \leq C_3. \tag{4.4}$$

Since

$$\begin{aligned} \partial_\theta \mathcal{F}_{\sigma\varepsilon M}[\theta_{\sigma\varepsilon M}]_{xx} &= [g_M'(\theta_{\sigma\varepsilon M}) + \partial_\theta h_M(c_{\sigma\varepsilon M}, \theta_{\sigma\varepsilon M})][\theta_{\sigma\varepsilon M}]_{xx} \\ &= [g_M'(\theta_{\sigma\varepsilon M}) + h_1(c_{\sigma\varepsilon M}) h_2'(M;\theta_{\sigma\varepsilon M})][\theta_{\sigma\varepsilon M}]_{xx}, \end{aligned}$$

using (3.3) and (3.4), we obtain

$$2\kappa \partial_\theta \mathcal{F}_{\sigma\varepsilon M}[\theta_{\sigma\varepsilon M}]_{xx} \leq \frac{\kappa^2}{2}[\theta_{\sigma\varepsilon M}]^2_{xx} + C_3[\theta^6_{\sigma\varepsilon M} + W_0^2\theta^2_{\sigma\varepsilon M}].$$

Thus, from Item 2 of Lemma 4.1, it follows from (4.4) that

$$\iint_{\Omega_T} (\partial_\theta \mathcal{F}_{\sigma\varepsilon M})^2 + \frac{\kappa^2}{2}\iint_{\Omega_T} [\theta_{\sigma\varepsilon M}]^2_{xx} \leq C_3. \tag{4.5}$$

Now, we prove Item 3. Defining, $H_{\sigma\varepsilon M} = \partial_c \mathcal{F}_{\sigma\varepsilon M} - \kappa_c[c_{\sigma\varepsilon M}]_{xx}$, since $[c_{\sigma\varepsilon M}]_{x|S_T} = 0$, we have

$$\iint_{\Omega_T} H_{\sigma\varepsilon M} = \iint_{\Omega_T} \partial_c \mathcal{F}_{\sigma\varepsilon M},$$

and from Item 3 of Lemma 4.1,

$$\iint_{\Omega_T} [H_{\sigma\varepsilon M}]^2_x \leq C_1.$$

Using the definition of $\mathcal{F}_{\sigma\varepsilon M}$, given in (3.1), and an integration by parts, we obtain

$$\begin{aligned}
&\iint_{\Omega_T} H^2_{\sigma\varepsilon M}\\
&= \iint_{\Omega_T} (\partial_c \mathcal{F}_{\sigma\varepsilon M})^2 + 2\kappa_c\varepsilon \iint_{\Omega_T} [F''_\sigma(c_{\sigma\varepsilon M}) + F''_\sigma(1-c_{\sigma\varepsilon M})][c_{\sigma\varepsilon M}]^2_x\\
&\quad - 2\kappa_c L \iint_{\Omega_T} (f'(c_{\sigma\varepsilon M}) + h_1'(c_{\sigma\varepsilon M})h_2(M;\theta_{\sigma\varepsilon M}))[c_{\sigma\varepsilon M}]_{xx} + \kappa_c^2 \iint_{\Omega_T} [c_{\sigma\varepsilon M}]^2_{xx}.
\end{aligned}$$

On the other hand, we can write

$$\begin{aligned}
\iint_{\Omega_T} H^2_{\sigma\varepsilon M} &= \iint_{\Omega_T} [H_{\sigma\varepsilon M} - \overline{H_{\sigma\varepsilon M}}]^2 + \iint_{\Omega_T} \overline{H_{\sigma\varepsilon M}}^2\\
&\le C_P \iint_{\Omega_T} [H_{\sigma\varepsilon M}]^2_x + \iint_{\Omega_T} (\partial_c \mathcal{F}_{\sigma\varepsilon M})^2
\end{aligned}$$

where C_P denotes the constant appearing in Poincaré's inequality. From these two last results, Item 3 follows recalling that $[F''_\sigma(c_{\sigma\varepsilon M}) + F''_\sigma(1-c_{\sigma\varepsilon M})] \ge 0$ and using (3.2), (3.3), (3.4) and Item 2 of Lemma 4.1.

Finally, recalling that for each σ, $F'_\sigma(s)$ is bounded in $\mathbf{R}$, using again the definition of f and h_M and Item 2 of Lemma 4.1, we obtain

$$\begin{aligned}
\|\partial_c \mathcal{F}_{\sigma\varepsilon M}\|^2_{L^2(\Omega_T)} &\le C \iint_{\Omega_T} \{[f'(c_{\sigma\varepsilon M})]^2 + [h_1'(c_{\sigma\varepsilon M})]^2 [h_2(M;\theta_{\sigma\varepsilon M})]^2\\
&\quad + \varepsilon^2 [F'_\sigma(c_{\sigma\varepsilon M}) - F'_\sigma(1-c_{\sigma\varepsilon M})]\}\\
&\le C\{[U_0^2|\Omega_T| + W_0^2 \|\theta_{\sigma\varepsilon M}\|^4_{L^4}] + C(\sigma)\} \le C_3'(\sigma).
\end{aligned}$$

A similar argument shows that $\|[\partial_c \mathcal{F}_{\sigma\varepsilon M}]_x\|^2_{L^2(\Omega_T)}$ is also bounded by a constant which depends only on σ. Thus, we have proved the Item 1. $\square$

We can now state the following result.

Proposition 4.4. *For σ (sufficiently small), there exists a pair $(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon})$ such that:*

1. $c_{\sigma\varepsilon} \in L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^3(\Omega))$
2. $\theta_{\sigma\varepsilon} \in L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega))$
3. $\partial_t c_{\sigma\varepsilon} \in L^2(0,T,[H^1(\Omega)]')$, $\quad \partial_t \theta_{\sigma\varepsilon} \in L^2(\Omega_T)$
4. $\partial_c \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon})$, $\partial_\theta \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon}) \in L^2(\Omega_T)$
5. $c_{\sigma\varepsilon}(0) = c_0$ *and* $\theta_{\sigma\varepsilon}(0) = \theta_0$ *in* $L^2(\Omega)$
6. $[c_{\sigma\varepsilon}]_{x|_{S_T}} = [\theta_{\sigma\varepsilon}]_{x|_{S_T}} = 0$ *in* $L^2(S_T)$
7. $(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon})$ *solves the perturbed system (3.6) in the following sense:*

$$\int_0^T \langle \partial_t c_{\sigma\varepsilon}, \phi\rangle = -\iint_{\Omega_T} D[\partial_c \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon}) - \kappa_c (c_{\sigma\varepsilon})_{xx}]_x \phi_x \tag{4.6}$$

for all $\phi \in L^2(0,T,H^1(\Omega))$, *and*

$$\iint_{\Omega_T} \partial_t \theta_{\sigma\varepsilon} \psi = -\iint_{\Omega_T} L(\partial_\theta \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon}) - \kappa(\theta_{\sigma\varepsilon})_{xx})\psi \tag{4.7}$$

for all $\psi \in L^2(\Omega_T)$, *and* $\mathcal{F}_{\sigma\varepsilon}$ *is given by*

$$\mathcal{F}_{\sigma\varepsilon}(c,\theta) = f(c) + \frac{\delta}{4}\theta^4 + h_1(c)\theta^2 + \varepsilon[F_\sigma(c) + F_\sigma(1-c)].$$

Proof. First, let us observe that from Item 3 of Lemma 4.1 and Item 1 of Lemma 4.3, the norm of $[c_{\sigma\varepsilon M}]_{xxx}$ in $L^2(\Omega_T)$ is bounded by a constant which does not depend on M. This fact, the estimates of Lemmas 4.1 and 4.3 together with a compactness argument imply that there exists a subsequence (still denoted by $\{(c_{\sigma\varepsilon M}, \theta_{\sigma\varepsilon M})\}$) that satisfies (as M goes to infinity)

$$\begin{gathered}
c_{\sigma\varepsilon M},\ \theta_{\sigma\varepsilon M} \quad \text{converge weakly-* to} \quad c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon} \quad \text{in} \quad L^\infty(0,T,H^1(\Omega)),\\
c_{\sigma\varepsilon M}, \quad \text{converges weakly to} \quad c_{\sigma\varepsilon} \quad \text{in} \quad L^2(0,T,H^3(\Omega)),\\
\theta_{\sigma\varepsilon M}, \quad \text{converges weakly to} \quad \theta_{\sigma\varepsilon} \quad \text{in} \quad L^2(0,T,H^2(\Omega)),\\
\partial_t c_{\sigma\varepsilon M}, \quad \text{converges weakly to} \quad \partial_t c_{\sigma\varepsilon} \quad \text{in} \quad L^2(0,T,[H^1(\Omega)]'),\\
\partial_t \theta_{\sigma\varepsilon M}, \quad \text{converges weakly to} \quad \partial_t \theta_{\sigma\varepsilon} \quad \text{in} \quad L^2(\Omega_T)\\
c_{\sigma\varepsilon M},\ \theta_{\sigma\varepsilon M} \quad \text{converge to} \quad c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon} \quad \text{in} \quad L^2(\Omega_T).
\end{gathered}$$

By recalling Lemmas 4.1 and 4.3, items 1–3 now follow. Now, items 1 and 2 of Lemma 4.3 imply that

$$\begin{gathered}
\partial_c \mathcal{F}_{\sigma\varepsilon M}(c_{\sigma\varepsilon M}, \theta_{\sigma\varepsilon M}) \quad \text{converges weakly to} \quad \mathcal{G} \quad \text{in} \quad L^2(\Omega_T),\\
\partial_\theta \mathcal{F}_{\sigma\varepsilon M}(c_{\sigma\varepsilon M}, \theta_{\sigma\varepsilon M}) \quad \text{converges weakly to} \quad \mathcal{H} \quad \text{in} \quad L^2(\Omega_T).
\end{gathered}$$

Since the strong convergence of the sequence $(c_{\sigma\varepsilon M})$ implies that (at least for a subsequence) $\partial_c \mathcal{F}_{\sigma\varepsilon M}(c_{\sigma\varepsilon M}, \theta_{\sigma\varepsilon M})$ converges pointwise in Ω_T, it follows from Lions [7, Lemma 1.3], that $\mathcal{G} = \partial_c \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon})$. Similarly, we have $\mathcal{H} = \partial_\theta \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon})$. Thus Item 4 is proved.

Item 5 is straightforward. Now, by compactness we have that

$$\begin{gathered}
c_{\sigma\varepsilon M} \quad \text{converges to} \quad c_{\sigma\varepsilon} \quad \text{in} \quad L^2(0,T,H^{2-\lambda}(\Omega)), \quad \lambda > 0,\\
\theta_{\sigma\varepsilon M} \quad \text{converges to} \quad \theta_{\sigma\varepsilon} \quad \text{in} \quad L^2(0,T,H^{2-\lambda}(\Omega)), \quad \lambda > 0,
\end{gathered}$$

which imply Item 6. To prove Item 7, by using the previous convergences, we pass to the limit as M goes to infinity in the equations (3.8) and (3.9). □

5. Limit as $\sigma \to 0^+$

In this section we obtain some a priori estimates that allow taking the limit in the parameter σ.

First, let us note that (4.6) implies that the mean value of $c_{\sigma\varepsilon}$ in Ω is given by

$$\overline{c_{\sigma\varepsilon}(t)} = \overline{c_0} \in (0,1), \tag{5.1}$$

We start with the following Lemma.

Lemma 5.1. *There exists a constant C_1 independent of ε and σ (sufficiently small) such that*

(1) $\|c_{\sigma\varepsilon}\|_{L^\infty(0,T,H^1(\Omega))} \le C_1$
(2) $\|\theta_{\sigma\varepsilon}\|_{L^\infty(0,T,H^1(\Omega))} \le C_1$
(3) $\|[\partial_c \mathcal{F}_{\sigma\varepsilon} - \kappa_c (c_{\sigma\varepsilon})_{xx}]_x\|_{L^2(\Omega_T)} \le C_1$
(4) $\|\partial_\theta \mathcal{F}_{\sigma\varepsilon} - \kappa(\theta_{\sigma\varepsilon})_{xx}\|_{L^2(\Omega_T)} \le C_1$
(5) $\|\partial_t c_{\sigma\varepsilon}\|_{L^\infty(0,T,[H^1(\Omega)]')} \le C_1$
(6) $\|\partial_t \theta_{\sigma\varepsilon}\|_{L^2(\Omega_T)} \le C_1$
(7) $\|\mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon})\|_{L^\infty(0,T,L^1(\Omega))} \le C_1$

Proof. Let us observe that in the proof of Proposition 4.4, we have identified the weak limits, when M goes to infinity, of the sequences $\partial_c\mathcal{F}_{\sigma\varepsilon M}$ and $\partial_\theta\mathcal{F}_{\sigma\varepsilon M}$ as $\partial_c\mathcal{F}_{\sigma\varepsilon}$ and $\partial_\theta\mathcal{F}_{\sigma\varepsilon}$, respectively. Thus, by taking the inferior limit as M goes to infinity, of estimate (4.3), we obtain

$$\begin{aligned}&\frac{\kappa_c}{2}\|(c_{\sigma\varepsilon})_x\|^2_{L^\infty(0,T,L^2(\Omega))}+\frac{\kappa}{2}\|(\theta_{\sigma\varepsilon})_x\|^2_{L^\infty(0,T,L^2(\Omega))}\\&+D\|[\partial_c\mathcal{F}_{\sigma\varepsilon}-\kappa_c(c_{\sigma\varepsilon})_{xx}]_x\|^2_{L^2(\Omega_T)}+L\|\partial_\theta\mathcal{F}_{\sigma\varepsilon}-\kappa(\theta_{\sigma\varepsilon})_{xx}\|^2_{L^2(\Omega_T)}\le C_1.\end{aligned}\tag{5.2}$$

The items 3 and 4 follow from (5.2). Using (5.2), Poincaré's inequality and (5.1), we obtain Item 1. To prove items 2, 5 and 6, we just take the inferior limit of items 2, 5 and 6 of Lemma 4.1. Finally, using (3.2), (3.3), (3.4) and the estimates of Lemma 4.1, we can estimate $\|f(c)+\frac{\delta}{4}\theta^4+h_1(c)\theta^2\|_{L^\infty(0,T,L^1(\Omega))}$. Also with the estimates of Lemma 4.1, we get the strong convergence of a subsequence of $(c_{\sigma\varepsilon})$. Using this convergence and the Fatou's Lemma, we get a bound for $\|\varepsilon[F_\sigma(c)+F_\sigma(1-c)]\|_{L^\infty(0,T,L^1(\Omega))}$ which together with the previous estimate yield Item 7. $\square$

As Passo et al. [3], by arguing in a standard way (see Bernis and Friedman [1] for a proof, p. 183), we obtain

Corolary 5.2. *There exists a constant C_2 independent of ε and σ (sufficiently small) such that*

$$\|c_{\sigma\varepsilon}\|_{C^{0,\frac12,\frac18}(\operatorname{cl}\Omega_T)}\le C_2\quad\textit{and}\quad\|\theta_{\sigma\varepsilon}\|_{C^{0,\frac12,\frac18}(\operatorname{cl}\Omega_T)}\le C_2\tag{5.3}$$

By Corollary 5.2, we can extract a subsequence (still denoted by $(c_{\sigma\varepsilon},\theta_{\sigma\varepsilon})$) such that

$$\begin{aligned}&(c_{\sigma\varepsilon},\theta_{\sigma\varepsilon})\text{ converges uniformly to }(c_\varepsilon,\theta_\varepsilon)\text{ in }\operatorname{cl}\Omega_T\text{ as }\sigma\text{ approaches zero,}\\&c_\varepsilon\in C^{0,\frac12,\frac18}(\operatorname{cl}\Omega_T)\quad\text{and}\quad\theta_\varepsilon\in C^{0,\frac12,\frac18}(\operatorname{cl}\Omega_T).\end{aligned}\tag{5.4}$$

We now demonstrate that the limit c_ε lies within the interval

$$I=\{c\in\mathbb{R},0<c<1\}.$$

Lemma 5.3. $|\Omega_T\setminus\mathcal{B}(c_\varepsilon)|=0$ *with* $\mathcal{B}(c)=\{(x,t)\in\operatorname{cl}\Omega_T,\ c(x,t)\in I\}$.

Proof. Arguing as Passo et al. [3], let N denote the operator defined as minus the inverse of the Laplacian with zero Neumann boundary conditions. That is, given $f\in[H^1(\Omega)]'_{null}=\{f\in[H^1(\Omega)]',\ \langle f,1\rangle=0\}$, we define $Nf\in H^1(\Omega)$ as the unique solution of

$$\int_\Omega(Nf)'\psi'=\langle f,\psi\rangle,\quad\forall\psi\in H^1(\Omega)\quad\text{and}\quad\int_\Omega Nf=0.\tag{5.5}$$

By (5.1) and Item 1 of Lemma 5.1, $N(c_{\sigma\varepsilon}-\overline{c_{\sigma\varepsilon}})$ is well defined. Choosing $\phi=N(c_{\sigma\varepsilon}-\overline{c_{\sigma\varepsilon}})$ as a test function in the equation (4.6), we have

$$\begin{aligned}&\int_0^T\langle\partial_tc_{\sigma\varepsilon},N(c_{\sigma\varepsilon}-\overline{c_{\sigma\varepsilon}})\rangle dt\\&=-\iint_{\Omega_T}D[\partial_c\mathcal{F}_{\sigma\varepsilon}-\kappa_c(c_{\sigma\varepsilon})_{xx}]_x[N(c_{\sigma\varepsilon}-\overline{c_{\sigma\varepsilon}})]_x\\&=-\iint_{\Omega_T}D(c_{\sigma\varepsilon}-\overline{c_{\sigma\varepsilon}})\partial_c\mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon},\theta_{\sigma\varepsilon})-D\kappa_c\iint_{\Omega_T}[(c_{\sigma\varepsilon})_x]^2\end{aligned}$$

Now, estimates in Lemma 5.1 and the definition of N imply

$$\iint_{\Omega_T} (c_{\sigma\varepsilon} - \overline{c_{\sigma\varepsilon}})\partial_c \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon}) \leq C_4. \tag{5.6}$$

We observe that the following identity holds for any $m \in \mathbf{R}$,

$$\begin{aligned}
&(c-m)\partial_c \mathcal{F}_{\sigma\varepsilon}(c,\theta) \\
&= (c-m)[f'(c) + h_1'(c)\theta^2 + \varepsilon(F_\sigma'(c) - F_\sigma'(1-c))] \\
&= \varepsilon\{c[F_\sigma'(c) - 1] + (1-c)[F_\sigma'(1-c) - 1] + 2\} \\
&\quad + (c-m)[f'(c) + h_1'(c)\theta^2] - \varepsilon - \varepsilon m F_\sigma'(c) - \varepsilon(1-m)F_\sigma'(1-c)
\end{aligned} \tag{5.7}$$

We observe that the terms inside the braces are bounded from below since for any $\sigma \in (0, 1/e)$, we have

$$\begin{gathered}
-1/e \leq \sigma \ln \sigma \leq s[F_\sigma'(s) - 1] \leq 0, \quad s \leq \sigma, \\
-1/e \leq s \ln s = s[F_\sigma'(s) - 1] \leq 0, \quad \sigma \leq s \leq 1-\sigma, \\
-2 \leq s[F_\sigma'(s) - 1] \leq 0, \quad 1-\sigma \leq s \leq 2, \\
0 = s[F_\sigma'(s) - 1], \quad s \geq 2.
\end{gathered}$$

We now recall that the mean value of $c_{\sigma\varepsilon}$ in Ω is conserved and is equal to $\overline{c_0}$ which belongs to the interval $(0,1)$. Thus, since f', h_1' are uniformly bounded, using the estimates in Lemma 5.1, by setting $m = \overline{c_{\sigma\varepsilon}} = \overline{c_0}$ in (5.7) and noting $F_\sigma' \leq 1$, it follows from (5.6) that

$$-\varepsilon \iint_{\Omega_T} [F_\sigma'(c_{\sigma\varepsilon}) + F_\sigma'(1 - c_{\sigma\varepsilon})] \leq C_4 \tag{5.8}$$

To complete the proof, suppose by contradiction that the set $\Omega_T \setminus \mathcal{B}(c_\varepsilon)$ has a positive measure. We start supposing that

$$A = \{(x,t) \in \Omega_T,\ c_\varepsilon \leq 0\}$$

has positive measure. Since $F_\sigma' \leq 1$, the estimate (5.8) gives

$$-\varepsilon \iint_A F_\sigma'(c_{\sigma\varepsilon}) \leq C_4.$$

Note, however, that the uniform convergence of $c_{\sigma\varepsilon}$ implies that

$$\forall \lambda > 0,\ \exists \sigma_\lambda, \quad c_{\sigma\varepsilon} \leq \lambda, \quad \forall (x,t) \in A,\ \ \sigma < \sigma_\lambda$$

therefore, due to the convexity of F_σ, we have $F_\sigma'(c_{\sigma\varepsilon}) \leq F_\sigma'(\lambda)$. Hence

$$-\varepsilon |A|(\ln \lambda + 1) = -\varepsilon \lim_{\sigma \to 0^+} \iint_A F_\sigma'(\lambda) \leq C_4$$

which leads to a contradiction for $\lambda \in (0,1)$ sufficiently small. The same argument shows that $B = \{(x,t) \in \Omega_T,\ c_\varepsilon \geq 1\}$ has zero measure. □

In the next lemma we derive additional estimates which allow us to pass to the limit as σ tends to zero. Its proof follows directly from the estimates of Lemma 5.1.

Lemma 5.4. *There exists a constant C_3 which is independent of ε and σ (sufficiently small) such that*

(1) $\|\partial_\theta \mathcal{F}_{\sigma\varepsilon}\|_{L^2(\Omega_T)} \leq C_3$,
(2) $\|[c_{\sigma\varepsilon}]_{xx}\|_{L^2(\Omega_T)} \leq C_3$,
(3) $\|[\theta_{\sigma\varepsilon}]_{xx}\|_{L^2(\Omega_T)} \leq C_3$,

To pass to the limit as σ goes to zero, we need an estimate of $\partial_c \mathcal{F}_{\sigma\varepsilon}$ that is independent of σ. We cannot repeat the argument that we used in Lemma 4.3 because there we obtained with a constant that depends on σ. The desired estimate will be obtained by using the next lemma, presented by Copetti and Elliott [2, p. 48], and by Elliott and Luckhaus [5, p. 23].

Lemma 5.5. *Let $v \in L^1(\Omega)$ such that there exist positive constants δ_1 and δ_1' satisfying*

$$8\delta_1 < \frac{1}{|\Omega|}\int_\Omega v dx < 1 - 8\delta_1, \tag{5.9}$$

$$\frac{1}{|\Omega|}\int_\Omega ([v-1]_+ + [-v]_+) dx < \delta_1'. \tag{5.10}$$

If $16\delta_1' < \delta_1^2$ then

$$|\Omega_{\delta_1}^+| = |\{x \in \Omega, \quad v(x) > 1 - 2\delta_1\}| < (1-\delta_1)|\Omega|$$

and

$$|\Omega_{\delta_1}^-| = |\{x \in \Omega, \quad v(x) < 2\delta_1\}| < (1-\delta_1)|\Omega|.$$

Our task is now to verify the hypothesis of this lemma for the functions $c_{\sigma\varepsilon}$. To obtain (5.10), we note that items 2 and 7 of Lemma 5.1, (3.2) and (3.3) imply that, for almost every $t \in [0,T]$,

$$\begin{aligned}
&\varepsilon \int_\Omega [F_\sigma(c_{\sigma\varepsilon}) + F_\sigma(1 - c_{\sigma\varepsilon})] dx \\
&\le C_1 + \left\| f(c_{\sigma\varepsilon}) + \delta\theta_{\sigma\varepsilon}^4/4 + h_1(c_{\sigma\varepsilon})\theta_{\sigma\varepsilon}^2 \right\|_{L^\infty(0,T,L^1(\Omega))} \le C.
\end{aligned}$$

From the definition of $F_\sigma(s)$ for $s < \sigma$ (see page 6), we obtain

$$\begin{aligned}
&\int_{\{c_{\sigma\varepsilon}<0\}} F_\sigma(c_{\sigma\varepsilon}) dx \\
&\ge |\ln\sigma| \int_\Omega [-c_{\sigma\varepsilon}(\cdot,t)]_+ dx - \sigma[|\ln\sigma| + 2\sigma]|\Omega| - \sigma\|c_{\sigma\varepsilon}(t)\|_{L^2(\Omega)}|\Omega|^{1/2}.
\end{aligned}$$

Hence, since $F_\sigma(s) \ge -1/e$, we have

$$\begin{aligned}
&\int_\Omega F_\sigma(c_{\sigma\varepsilon}) dx \\
&\ge |\ln\sigma| \int_\Omega [-c_{\sigma\varepsilon}(\cdot,t)]_+ dx - \sigma[|\ln\sigma| + 2\sigma + e^{-1}]|\Omega| - \sigma\|c_{\sigma\varepsilon}(t)\|_{L^2(\Omega)}|\Omega|^{1/2}.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
\int_\Omega F_\sigma(1-c_{\sigma\varepsilon}) dx \ge |\ln\sigma| \int_\Omega [c_{\sigma\varepsilon}(\cdot,t) - 1]_+ dx - \sigma[|\ln\sigma| + 2\sigma - 1 \\
+ e^{-1}]|\Omega| - \sigma\|c_{\sigma\varepsilon}(t)\|_{L^2(\Omega)}|\Omega|^{1/2}.
\end{aligned}$$

Thus, using the above estimates and Item 1 of Lemma 5.1, we obtain

$$\int_\Omega [c_{\sigma\varepsilon}(\cdot,t) - 1]_+ dx + \int_\Omega [-c_{\sigma\varepsilon}(\cdot,t)]_+ dx \le \frac{C}{\varepsilon|\ln\sigma|}$$

The equation (5.1) says that the mean value of $c_{\sigma\varepsilon}$ is equal to $\overline{c_0}$ which belongs to $(0,1)$. Thus there exists $\delta_1 > 0$, such that $8\delta_1 < \overline{c_0} < 1 - 8\delta_1$. Using Lemma 5.5, for σ sufficiently small, we have for almost every $t \in [0,T]$

$$\begin{gathered} |\Omega^+_{\sigma\delta_1}| = \{x \in \Omega, \quad c_{\sigma\varepsilon}(x,t) > 1 - 2\delta_1\} < (1-\delta_1)|\Omega|, \\ |\Omega^-_{\sigma\delta_1}| = \{x \in \Omega, \quad c_{\sigma\varepsilon}(x,t) < 2\delta_1\} < (1-\delta_1)|\Omega|. \end{gathered} \tag{5.11}$$

We are now in position to estimate $\partial_c \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon})$.

Lemma 5.6. *There exists a constant C_4 which is independent of ε and σ (sufficiently small) such that*

$$\|\partial_c \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon})\|_{L^2(\Omega_T)} \le C_4.$$

Proof. First, let us recall that

$$\partial_c \mathcal{F}_{\sigma\varepsilon} = f'(c_{\sigma\varepsilon}) + h_1'(c_{\sigma\varepsilon})\theta^2_{\sigma\varepsilon} + \varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})].$$

In view of Item 2 of Lemma 5.1, (3.2) and (3.3), to obtain the desired estimate it is enough to obtain a bound for the norm of $\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]$ in $L^2(\Omega_T)$. Arguing as Copetti and Elliott [2], we obtain this bound by using the next equality

$$\begin{aligned} &\Big\|\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})] - \overline{\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]}\Big\|^2_{L^2(\Omega_T)} \\ &= \|\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]\|^2_{L^2(\Omega_T)} - \iint_{\Omega_T} \Big(\overline{\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]}\Big)^2 \end{aligned} \tag{5.12}$$

and estimating the term at the left hand side and the last term at the right hand side of the above equation.

Let us note that using Poincaré's inequality and Item 3 of Lemma 5.1, we obtain

$$\|\partial_c \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon}) - \kappa_c (c_{\sigma\varepsilon})_{xx} - \overline{\partial_c \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon}) - \kappa_c (c_{\sigma\varepsilon})_{xx}}\|_{L^2(\Omega_T)} \le C_1.$$

Recalling that $c_{x|S_T} = 0$, we have

$$\overline{\partial_c \mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon}, \theta_{\sigma\varepsilon}) - \kappa_c (c_{\sigma\varepsilon})_{xx}} = \overline{f'(c_{\sigma\varepsilon}) + h_1'(c_{\sigma\varepsilon})\theta^2_{\sigma\varepsilon} + \varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]}.$$

Thus, using the estimates for $(c_{\sigma\varepsilon})_{xx}$ in Item 2 of Lemma 5.4 and for $\theta_{\sigma\varepsilon}$ in Item 2 of Lemma 5.1 together (3.2) and (3.3), we obtain

$$\|\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})] - \overline{\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]}\|_{L^2(\Omega_T)} \le \widetilde{C}_1 \tag{5.13}$$

We now use the monotonicity of $F'_\sigma(s) - F'_\sigma(1-s)$ and (5.11) to obtain for almost every $t \in [0,T]$:

$$\begin{aligned} &\overline{\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]} \\ &= \varepsilon|\Omega|^{-1} \int_{\Omega^+_{\sigma\delta_1}} [F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})] + \varepsilon|\Omega|^{-1} \int_{[\Omega^+_{\sigma\delta_1}]^c} [F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})] \\ &\le (1-\delta_1)^{1/2}|\Omega|^{-1/2}\|\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]\|_{L^2(\Omega)} \\ &\quad + \varepsilon[F'_\sigma(1-2\delta_1) - F'_\sigma(2\delta_1)]. \end{aligned}$$

In the same way, observing that $F'_\sigma(2\delta_1) - F'_\sigma(1-2\delta_1) < 0$, we have

$$\begin{aligned} \overline{\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]} &\ge -(1-\delta_1)^{1/2}|\Omega|^{-1/2}\|\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) \\ &\quad - F'_\sigma(1-c_{\sigma\varepsilon})]\|_{L^2(\Omega)} + \varepsilon[F'_\sigma(2\delta_1) - F'_\sigma(1-2\delta_1)]. \end{aligned}$$

Therefore, by using that $(a+b)^2 \le a^2(1+\frac{1}{\delta_1}) + b^2(1+\delta_1)$, we have

$$
\begin{aligned}
&\big(\overline{\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]}\big)^2 \\
&\le \varepsilon\big(1+\frac{1}{\delta_1}\big)[F'_\sigma(1-2\delta_1) - F'_\sigma(2\delta_1)]^2(1-\delta_1^2)\frac{1}{|\Omega|}\|\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{5.14}
$$

Multiplying the above estimate by $|\Omega|$, integrating it in t and using (5.12) and (5.13), it results that for σ sufficiently small, we have

$$
\begin{aligned}
&\delta_1^2\|\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]\|^2_{L^2(\Omega_T)} \\
&\le \varepsilon|\Omega_T|(1+\frac{1}{\delta_1})[F'_\sigma(2\delta_1) - F'_\sigma(1-2\delta_1)]^2 \\
&\quad + \Big\|\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})] - \overline{\varepsilon[F'_\sigma(c_{\sigma\varepsilon}) - F'_\sigma(1-c_{\sigma\varepsilon})]}\Big\|^2_{L^2(\Omega_T)} \\
&\le \varepsilon|\Omega_T|(1+\frac{1}{\delta_1})[F'(2\delta_1) - F'(1-2\delta_1)]^2 + \widetilde{C}_1 \le \widetilde{C}_2.
\end{aligned}
$$

□

We define

$$
w_{\sigma\varepsilon} = D(\partial_c\mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon},\theta_{\sigma\varepsilon}) - \kappa_c[c_{\sigma\varepsilon}]_{xx}).
$$

Then the estimates in Lemmas 5.1, 5.4 and 5.6 and Lemma 5.3 imply that $w_{\sigma\varepsilon}$ converge weakly to w_ε in $L^2(0,T,H^1(\Omega))$, where

$$
w_\varepsilon = D(\partial_c\mathcal{F}_\varepsilon(c_\varepsilon,\theta_\varepsilon) - \kappa_c[c_\varepsilon]_{xx}),
$$

and where $\mathcal{F}_\varepsilon$ is defined as in the next Proposition. To identify the limit of the nonlinear term, we use Lemmas 5.1 and 5.3 to see that $\partial_c\mathcal{F}_\varepsilon(c_\varepsilon,\theta_\varepsilon)$ is the pointwise limit of a subsequence of $\partial_c\mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon},\theta_{\sigma\varepsilon})$. This fact with Lemma 5.6 and Fatou's Lemma imply $\partial_c\mathcal{F}_\varepsilon(c_\varepsilon,\theta_\varepsilon) \in L^2(\Omega_T)$. Finally, we use Lion's Lemma ([7], p. 12) to identify the weak limit of $\partial_c\mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon},\theta_{\sigma\varepsilon})$ as $\partial_c\mathcal{F}_{\sigma\varepsilon}(c_{\sigma\varepsilon},\theta_{\sigma\varepsilon})$. Therefore, arguing as in Proposition 4.4, we can pass to the limit as σ goes to zero to obtain the following result.

Proposition 5.7. *There exists a triplet $(c_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ such that:*

1. $c_\varepsilon, \theta_\varepsilon \in L^\infty(0,T,H^1(\Omega))$,
2. $\partial_t c_\varepsilon \in L^2(0,T,[H^1(\Omega)]')$ *and* $\partial_t\theta_\varepsilon \in L^2(\Omega_T)$,
3. $[c_\varepsilon]_{xx}, [\theta_\varepsilon]_{xx} \in L^2(\Omega_T)$,
4. $|\Omega_T \setminus \mathcal{B}(c_\varepsilon)| = 0$,
5. $\partial_c\mathcal{F}_\varepsilon(c_\varepsilon,\theta_\varepsilon), \partial_\theta\mathcal{F}_\varepsilon(c_\varepsilon,\theta_\varepsilon) \in L^2(\Omega_T)$,
6. $w_\varepsilon \in L^2(0,T,H^1(\Omega))$
7. $c_\varepsilon(0) = c_0(x)$, $\theta_{\varepsilon M}(0) = \theta_0(x)$
8. $[c_\varepsilon]_{x|S_T} = [\theta_{\varepsilon M}]_{x|S_T} = 0$ *in* $L^2(S_T)$
9. $(c_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ *satisfies*

$$
\int_0^T \langle\partial_t c_\varepsilon, \phi\rangle dt = -\iint_{\Omega_T}[w_\varepsilon]_x\phi_x, \quad \forall\phi \in L^2(0,T,H^1(\Omega)) \tag{5.15}
$$

$$
w_\varepsilon = D[\partial_c\mathcal{F}_\varepsilon(c_\varepsilon,\theta_\varepsilon) - \kappa_c(c_\varepsilon)_{xx}] \tag{5.16}
$$

$$
\iint_{\Omega_T}\partial_t\theta_\varepsilon\psi = -\iint_{\Omega_T} L(\partial_\theta\mathcal{F}_\varepsilon(c_\varepsilon,\theta_\varepsilon) - \kappa(\theta_\varepsilon)_{xx})\psi, \quad \forall\psi \in L^2(\Omega_T) \tag{5.17}
$$

where, since $c_\varepsilon \in (0,1)$ *a.e in* Ω_T,

$$\begin{aligned}\mathcal{F}_\varepsilon(c,\theta) &= -\frac{A}{2}(c-c_m)^2 + \frac{B}{4}(c-c_m)^4 + \frac{D_\alpha}{4}(c-c_\alpha)^4 \\ &\quad + \frac{D_\beta}{4}(c-c_\beta)^4 + \frac{\delta}{4}\theta^4 - \frac{\gamma}{2}c^2\theta^2 + \varepsilon[F(c)+F(1-c)]\end{aligned}$$

with $F(s) = s\ln s$.

6. Limit as $\varepsilon \to 0^+$

In this section we finally prove Theorem 2.1 when the spatial dimension is one and the number of crystallographic orientations is $p = 1$. We recall that we treat this simpler case just for simplicity of notation and exposition, since, as we will show in Section 7, the necessary changes to extend the results of Theorem 2.1 to higher spatial dimensions and $p > 1$ are simple ones.

We start by observing that, as before, we have the mean value of c_ε in Ω given by

$$\overline{c_\varepsilon} = \overline{c_0} \in (0,1), \tag{6.1}$$

Since the estimates obtained for σ in Lemmas 5.1, 5.4 and 5.6 do not depend on ε, we have

Lemma 6.1. *There exists a constant* C_1 *independent of* ε *such that*

(1) $\|c_\varepsilon\|_{L^\infty(0,T,H^1(\Omega))} \le C_1$
(2) $\|\theta_\varepsilon\|_{L^\infty(0,T,H^1(\Omega))} \le C_1$
(3) $\|(w_\varepsilon)_x\|_{L^2(\Omega_T)} \le C_1$
(4) $\|\partial_\theta\mathcal{F}_\varepsilon - \kappa(\theta_\varepsilon)_{xx}\|_{L^2(\Omega_T)} \le C_1$
(5) $\|\partial_t c_\varepsilon\|_{L^\infty(0,T,[H^1(\Omega)]')} \le C_1$
(6) $\|\partial_t\theta_\varepsilon\|_{L^2(\Omega_T)} \le C_1$
(7) $\|\partial_c\mathcal{F}_\varepsilon(c_\varepsilon,\theta_\varepsilon)\|_{L^2(\Omega_T)} \le C_1$,
(8) $\|\partial_\theta\mathcal{F}_\varepsilon(c_\varepsilon,\theta_\varepsilon)\|_{L^2(\Omega_T)} \le C_1$,
(9) $\|[c_\varepsilon]_{xx}\|_{L^2(\Omega_T)} \le C_1$,
(10) $\|[\theta_\varepsilon]_{xx}\|_{L^2(\Omega_T)} \le C_1$

Now, we complete the proof of Theorem 2.1.

Proof of the case $\mathbf{d}=\mathbf{1}$ *and* $\mathbf{p}=\mathbf{1}$. We recall

$$\begin{aligned}\mathcal{F}(c,\theta) &= -\frac{A}{2}(c-c_m)^2 + \frac{B}{4}(c-c_m)^4 + \frac{D_\alpha}{4}(c-c_\alpha)^4 \\ &\quad + \frac{D_\beta}{4}(c-c_\beta)^4 - \frac{\gamma}{2}(c-c_\alpha)^2\theta^2 + \frac{\delta}{4}\theta^4\end{aligned}$$

Then, we argue as Elliott and Luckhaus [5], p. 35. For this, let

$$\phi^\rho \in K^+ = \{\phi \in H^1(\Omega),\ \ 0<\phi<1\} \quad \text{and} \quad \rho < \phi^\rho < 1-\rho$$

for some small positive ρ. We have $[F'(\phi^\rho) - F'(1-\phi^\rho)] \in L^2(\Omega)$ because $\rho < \phi^\rho < 1-\rho$. Hence it follows from (5.16) that

$$\begin{aligned}&\int_0^T \xi(t)(w_\varepsilon, \phi^\rho - c_\varepsilon)dt \\ &= D\int_0^T \xi(t)(\partial_c\mathcal{F}(c_\varepsilon,\theta_\varepsilon) + \varepsilon[F'(c_\varepsilon) - F'(1-c_\varepsilon)] - \kappa_c(c_\varepsilon)_{xx}, \phi^\rho - c_\varepsilon)dt.\end{aligned}$$

Integrating by parts and rewriting, we obtain

$$\begin{aligned}
&\int_0^T \xi(t)\left\{\kappa_c D(\nabla c_\varepsilon, \nabla\phi^\rho) - (w_\varepsilon - D\partial_c\mathcal{F}(c_\varepsilon,\theta_\varepsilon), \phi^\rho - c_\varepsilon)\right\} dt \\
&= \int_0^T \xi(t)\kappa_c D(\nabla c_\varepsilon, \nabla c_\varepsilon) dt \\
&\quad + \varepsilon D \int_0^T \xi(t)([F'(\phi^\rho) - F'(1-\phi^\rho)] - [F'(c_\varepsilon) - F'(1-c_\varepsilon)], \phi^\rho - c_\varepsilon) dt \\
&\quad - \varepsilon D \int_0^T \xi(t)([F'(\phi^\rho) - F'(1-\phi^\rho)], \phi^\rho - c_\varepsilon) dt.
\end{aligned}$$

By using the monotonicity of $F'(\cdot) - F'(1-\cdot)$ and the convergence properties of $(c_\varepsilon, w_\varepsilon, \theta_\varepsilon)$ we may pass to the limit and obtain for $\xi \in C[0,T]$, $\xi \geq 0$, that

$$\begin{aligned}
&\int_0^T \xi(t)\left\{\kappa_c D(\nabla c, \nabla\phi^\rho) - (w - D\partial_c\mathcal{F}(c,\theta), \phi^\rho - c)\right\} dt \\
&\geq \limsup_{\varepsilon\to 0^+} \int_0^T \xi(t)\left\{\kappa_c D(\nabla c_\varepsilon, \nabla\phi^\rho) - (w - D\partial_c\mathcal{F}(c_\varepsilon,\theta_\varepsilon), \phi^\rho - c_\varepsilon)\right\} dt \\
&\geq \liminf_{\varepsilon\to 0^+} \int_0^T \xi(t)\kappa_c D(\nabla c_\varepsilon, \nabla c_\varepsilon) dt \\
&\quad - \lim_{\varepsilon\to 0^+} \varepsilon D \int_0^T \xi(t)([F'(\phi^\rho) - F'(1-\phi^\rho)], \phi^\rho - c_\varepsilon) dt \\
&\geq \int_0^T \xi(t)\kappa_c D(\nabla c, \nabla c) dt.
\end{aligned}$$

Furthermore, since any $\phi \in K = \{\phi \in H^1(\Omega),\ 0 \leq \phi \leq 1, \overline{\phi} = \overline{c_0}\}$ can be approximated by $\phi^\rho \in K^+$, for small ρ with $\rho < \phi^\rho < 1-\rho$, we may pass to the limit as ρ goes to zero in the left hand side of the above inequality and obtain

$$\int_0^T \xi(t)\left\{\kappa_c(\nabla c, \nabla\phi - \nabla c) - (w - \partial_c\mathcal{F}(c,\theta), \phi - c)\right\} dt \geq 0 \tag{6.2}$$

for $\xi \in C[0,T)$, $\xi \geq 0$, and $\phi \in K$. Arguing as in the previous sections we also obtain

$$\int_0^T \langle\partial_t c, \phi\rangle dt = -\iint_{\Omega_T} w_x\phi_x, \quad \forall\phi \in L^2(0,T,H^1(\Omega)), \tag{6.3}$$

$$\iint_{\Omega_T} \partial_t\theta\psi = -\iint_{\Omega_T} L(\partial_\theta\mathcal{F}(c,\theta) - \kappa\theta_{xx})\psi, \quad \forall\psi \in L^2(\Omega_T). \tag{6.4}$$

Thus, for spatial dimension one and $p = 1$, Theorem 2.1 is a direct consequence of Lemma 6.1, (6.2), (6.3) and (6.4). $\square$

7. Proof of the Case of Higher Spatial Dimensions and $p > 1$

In the case of higher spatial dimensions and $p > 1$, we have to slightly change the previously presented arguments. Firstly, we show the changes to be done when the spatial dimension satisfies $2 \leq d \leq 3$. We start by remarking that, as observed by Passo et al. [3], Proposition 3.1 is valid for any dimension. Also, in higher dimensions, we use an argument of elliptic regularity of the Laplacian to obtain estimates in $L^2(0,T,H^2(\Omega))$ and in $L^2(0,T,H^3(\Omega))$. Furthermore, all of our previous arguments hold for dimensions $2 \le d \le 3$, except the result of Corollary 5.2, where the fact that dimension was one was essential. This result was only used in the proof of Lemma 5.3 to extract an uniformly convergent subsequence and conclude that the measure of the set $\Omega_T \setminus \mathcal{B}(c_\varepsilon)$ is zero (where $\mathcal{B}(c) = \{(x,t) \in \operatorname{cl}\Omega_T,\ c(x,t) \in I\}$). Once Lemma 5.3 is stated for $2 \le d \le 3$, all results but Corollary 5.2 are also valid for $2 \le d \le 3$. On the next Lemma, we show how to state the same result of Lemma 5.3 when the dimension d satisfies $2 \le d \le 3$. As mentioned before, all the results stated before Corollary 5.1 does not depend upon dimension one and can be extended for dimensions $2 \le d \le 3$. We will use this fact on the proof of the next Lemma.

Lemma 7.1. $|\Omega_T \setminus \mathcal{B}(c_\varepsilon)| = 0$ *with* $\mathcal{B}(c) = \{(x,t) \in \operatorname{cl}\Omega_T,\ c(x,t) \in I\}$.

Proof. Arguing exactly as in the proof of Lemma 5.3, we obtain

$$-\varepsilon \iint_{\Omega_T} [F'_\sigma(c_{\sigma\varepsilon}) + F'_\sigma(1 - c_{\sigma\varepsilon})] \le C_4 \tag{7.1}$$

To complete the proof of the lemma, suppose by contradiction that the set $\Omega_T \setminus \mathcal{B}(c_\varepsilon)$ has a positive measure. Suppose by instance that

$$A = \{(x,t) \in \Omega_T,\ c_\varepsilon \le 0\}$$

has positive measure. Using Item 1 of Lemma 5.1, we can extract a subsequence of $(c_{\sigma\varepsilon})$ which converges almost everywhere to c_ε on Ω_T. Now, by using Egoroff's Theorem, we may conclude that such subsequence also converges almost uniformly on Ω_T. Thus, there exists a set $B \subset \Omega_T$ such that $|B| \le \frac{1}{2}|A|$ and $c_{\sigma\varepsilon}$ converges uniformly to c_ε on $\Omega_T \setminus B$. Let $C = A \cap (\Omega_T \setminus B)$. We can see that $|C| > 0$. Since $F'_\sigma \le 1$, the estimate (5.8) gives

$$-\varepsilon \iint_C F'_\sigma(c_{\sigma\varepsilon}) \le C_4.$$

Note, however, that the uniform convergence of $c_{\sigma\varepsilon}$ implies that

$$\forall \lambda > 0,\ \exists \sigma_\lambda, \quad c_{\sigma\varepsilon} \le \lambda, \quad \forall (x,t) \in C,\ \ \sigma < \sigma_\lambda$$

therefore, due to the convexity of F_σ, we have $F'_\sigma(c_{\sigma\varepsilon}) \le F'_\sigma(\lambda)$. Hence

$$-\varepsilon|C|(\ln\lambda + 1) = -\varepsilon \lim_{\sigma \to 0^+} \iint_C F'_\sigma(\lambda) \le C_4$$

which leads to a contradiction for $\lambda \in (0,1)$ sufficiently small. The same argument shows that $B = \{(x,t) \in \Omega_T,\ c_\varepsilon \ge 1\}$ has zero measure. $\square$

We remark that In this last proof we have just repeated the contradiction argument presented in the proof of Lemma 5.3 with the only difference that now we have supposed by contradiction that there exists a subset of $\Omega_T \setminus \mathcal{B}(c_\varepsilon)$ that has positive measure and where the convergence is uniform.

Now we explain the necessary modifications to be done when the number of crystallographic orientations is larger than one. In this case, the local free energy density is

$$\begin{aligned}\mathcal{F}(c,\theta_1,\dots,\theta_p) = &-\frac{A}{2}(c-c_m)^2+\frac{B}{4}(c-c_m)^4+\frac{D_\alpha}{4}(c-c_\alpha)^4+\frac{D_\beta}{4}(c-c_\beta)^4\\ &+\sum_{i=1}^{p}[-\frac{\gamma}{2}(c-c_\alpha)^2\theta_i^2+\frac{\delta}{4}\theta_i^4]+\sum_{i\neq j=1}^{p}\frac{\varepsilon_{ij}}{2}\theta_i^2\theta_j^2.\end{aligned}$$

The introduction of the mixed terms depending only on the θ_i's (the last terms) will not change greatly the arguments presented in the case when p was equal to one. In fact, in the following we will point out how our previous estimates can be extended for this case.

The main feature of the perturbed systems in Section 3 is that their corresponding local free energy density have lower bounds that do not depend on the truncation parameter M. Since the extended local free energy just introduces nonnegative terms, we can define a similar truncation that maintains the same property, with such perturbed systems it is then possible to similarly establish Lemma 4.1.

As for Lemma 4.3, we treat the new terms by using the immersion of $H^1(\Omega)$ in $L^4(\Omega)$ and the estimates for the orientation field variables given in Lemma 4.1.

After we have extended the results of Lemmas 4.1 and 4.3, all the other lemmas are their direct consequence without any significant change due to the introduction of the new terms.

8. Final Remarks

Remark 8.1. Observe that (2.3) implies that for almost all $t\in(0,T]$ there holds

$$\begin{aligned}&\kappa_c D(\nabla c(\cdot,t),\nabla\phi-\nabla c(\cdot,t))\\ &-(w-D\partial_c\mathcal{F}(c(\cdot,t),\theta_1(\cdot,t),\dots,\theta_p(\cdot,t)),\phi-c(\cdot,t))\geq 0,\end{aligned}$$

for all $\phi\in K=\{\eta\in H^1(\Omega),\ 0\leq\eta\leq 1,\ \overline{\eta}=\overline{c_0}\}$.

Moreover, since $c\in L^2(0,T,H^2(\Omega))$ and $1\leq d\leq 3$, by standard Sobolev imbeddings, we can assume that $c(x,t)$ is a continuous function of x for the same times t as above. Fix such a t and assume that in a neighborhood $\mathcal{V}$ of a point $\overline{x}\in\Omega$ we have that $0<c(x,t)<1$ for x in the closure of $\mathcal{V}$. Thus, for a given C^∞–function φ of compact support in $\mathcal{V}$ satisfying $\overline{\varphi}=0$, $c(\cdot,t)+\lambda\varphi(\cdot)\in K$ for small enough real λ. By taking $\phi=c+\lambda\varphi$ back in the last inequality, and observing that λ assumes positive and negative values, we conclude that for any C^∞–function φ of compact support in $\mathcal{V}$ such that $\overline{\varphi}=0$,

$$\kappa_c D(\nabla c(\cdot,t),\nabla\varphi)-(w-D\partial_c\mathcal{F}(c(\cdot,t),\theta_1(\cdot,t),\dots,\theta_p(\cdot,t)),\varphi)=0.$$

We conclude that in regions such $0<c(x,t)<1$, for almost all times $t\in(0,T]$, $w=D\nabla(\partial_c\mathcal{F}-\kappa_c\Delta c)$, up to a function of time. Substituting back in the first equation of (1.1), we obtain that $\partial_t c=\nabla\cdot[D\nabla(\partial_c\mathcal{F}-\kappa_c\Delta c)]$ in such regions. In this sense, the obtained solution is a generalized solution of the original problem.

Remark 8.2. When $\gamma=\delta=\varepsilon_{ij}=0$, we obtain c that solves a generalized formulation of the Cahn–Hilliard equation, and for which the estimate $0\leq c\leq 1$ is still valid. But, we are able to guarantee that the classical Cahn-Hilliard equation is satisfied only in regions where $0<c<1$. Thus, unfortunately we are not able to reach this kind of estimate for the classical Cahn-Hilliard equation with polynomial free-energy.

The physical model we are considering in this paper assumes that the coefficients appearing in the free-energy functions are positive. In mathematical terms, we could have a more general situation. For instance, $\varepsilon_{ij} \geq 0$ is enough to guarantee that, in the case when $p > 1$ and the cross terms $\theta_i^2\theta_j^2$ are present, the free energy functional $\mathcal{F}_{\sigma\varepsilon M}$ is still bounded below by a constant that does not depend on M, as indicated in formula (3.5), and the results in the paper are true. The coefficient γ could have any sign. When $\gamma \leq 0$, the problem in fact is simpler because in this case the corresponding term in the free-energy functional has the "right sign" In the case that $\gamma > 0$, to get the first inequality in (3.5), the coefficient δ must be positive.

Remark 8.3. In our first attempts to study the problem with the free-energy functional presented in the paper, we tried to use Galerkin method. However, we were not able to get the necessary estimates (basically due to presence of the term $c^2\theta_i^2$) except in the special case in which the coefficients are such that the original free-energy functional is bounded from below. In this special case it is possible to solve the problem using Galerkin method. Even in such special case, however, we cannot identify such solution with the generalized solution presented in this paper. This is due to the following facts: our uniqueness result is based upon the fact that the c presented in the paper is an element of the set $K = \{\eta \in H^1(\Omega),\ 0 \leq \eta \leq 1, \overline{\eta} = \overline{c_0}\}$; however, we are not able to get a L^∞–bound for the solution obtained by the Galerkin method in the special case, and thus we do not know whether such solution belongs to K. Therefore, even in the special case we are not able to compare the generalized solution and the solution obtained by the Galerkin method.

Remark 8.4. The physical meaning of c (concentration of one of the two materials in the mixture) requires that $0 \leq c \leq 1$. On the other hand, from physical arguments, it is expected that asymptotically in time the concentration approach certain values of the minimizers of the function f appearing in $\mathcal{F}$ (maybe different ones in different regions of Ω.) In terms of mathematical possibilities, it may occur that in certain situations two of these minimizer (exactly two in the special case of two-wells potentials) correspond to pure materials ($c = 0$ or $c = 1$.) However, in physical terms, these minimizers may be associated in fact to values of c corresponding to mixtures of materials (say, c_1 and c_2, $0 < c_1 < c_2 < 1$,) although, maybe, with clear predominance of one of them (even in cases with two-wells potentials.) In these cases, it is perfectly possible to have a initial condition for the concentration, c_0 such that the values of $c_0(x)$ are not confined to the interval $[c_1, c_2]$. As before, one expects that asymptotically in time the concentration evolves in such way to approach either c_1 or c_2, depending on the region of Ω. But, if it does so, it cannot not satisfy $c_1 \leq c(x,t) \leq c_2$ for all time t and x, although on physical grounds it should satisfy $0 \leq c(x,t) \leq 1$. Thus, the location of the minimizers of the potential is in fact independent of the required physical range of c. Here, we considered the problem at this level of possibilities, and our generalized solutions satisfy this physical requirement.

A more precise and difficult question related to this situation is the following: suppose c_0 satisfies $c_1 \leq c_0(x) \leq c_2$ for all x, does c satisfy $c_1 \leq c(x,t) \leq c_2$ for all x and t? For certain parabolic (scalar) equations, one expect this to be true. However, for general systems, where interactions between the unknown variables play a significant role, this may be not so. Even in homogeneous situations, when spatial variables play no significant role, the trajectories could be spirals approaching the equilibrium points of the corresponding ordinary differential system. We do not know whether this is the case of the system considered in this article. Certainly this is an interesting point to investigate.

Another interesting point to consider is the following. We placed the singularities of the logarithmic perturbation at 0 and 1 to comply with the required physical restriction $0 \leq c \leq 1$ on the concentration. However, suppose that we have an initial condition for the concentration satisfying $1/3 \leq c_0 \leq 2/3$; if we repeat the argument of the paper, but with the logarithmic singularities placed at $1/3$ and $2/3$, we obtain a solution satisfying $1/3 \leq c \leq 2/3$. The same sort of reasoning would imply that it is always possible to construct a solution satisfying $\min c_0 \leq c \leq \max c_0$. Moreover, if we consider that we have uniqueness, this would be the solution, taking us to the conclusion that every solutions satisfy $\min c_0 \leq c \leq \max c_0$. But this is strange because, as we said above, on physical grounds one expects the values of c approach either c_1 or c_2. This raises the question whether there is something wrong.

The key to ease this discomfort is the observation that, when we change where we place the logarithmic singularities, in fact we are changing the problem to be solved. For instance, in the previous example of changing the placement of the singularities, the differential inequality (5) would require that c and the test functions ϕ belong to modified $\widetilde{K} = \{\eta \in H^1(\Omega),\ 1/3 \leq \eta \leq 2/3,\ \overline{\eta} = \overline{c_0}\}$. Moreover, the solutions of these two different problems are not comparable, since it is clear that the uniqueness stated in Lemma 2.5 holds for exactly the same problem. Thus, we cannot reason as in the previous paragraph and its "conclusions" do not hold.

A final point must be considered. Since different placements of the logarithmic singularities introduce different generalized problems, and therefore different solutions, which is the "right" one to be picked? We argue that the one in this paper is the "right" one based on two reasons. First, the only reasonable physical restriction to the concentration is $0 \leq c \leq 1$, which requires the placement of the singularities as we chose, and not in different places. Second, as we explained in a previous remark, our generalized solutions satisfy the classical Cahn-Hilliard equation in regions described basically by $0 < c(x,t) < 1$, leaving out, maybe, only the regions of pure materials, where the physical mechanisms for the mixture no longer apply. The same sort of reasoning, when applied for a problem obtained with perturbation with singularities placed for instance at $1/3$ and $2/3$ would guarantee the Cahn-Hilliard equation only in regions such that $1/3 < c(x,t) < 2/3$, leaving out, maybe, regions where in fact the mechanisms for mixtures still apply, which is not physically reasonable.

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