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THE KORN INEQUALITY FOR JONES DOMAINS

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Abstract. In this paper we prove the Korn inequality, and its generalization to L^p, $1 < p < \infty$, for bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, satisfying an (ϵ, δ) condition.

1. Introduction

Since the pioneering work of Korn [12, 13] on linear elasticity equations, the inequality named after him, in its different forms, has been the subject of a great number of papers. An interesting review article, where connections with other inequalities and several applications are described, was written by Horgan [7].

Given an open domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, the Korn inequality states that

$$\|\nabla v\|_{L^2(\Omega)^{n\times n}} \leq C\|\varepsilon(v)\|_{L^2(\Omega)^{n\times n}} \tag{1.1}$$

where $\varepsilon(v)$ denotes the symmetric part of ∇v, namely,

$$\varepsilon_{ij}(v) = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)$$

Of course, (1.1) can not be true for arbitrary functions $v \in H^1(\Omega)^n$ since there are functions such that the right hand side vanishes while the left one does not (the so called infinitesimal rigid motions). So, in order to prove the inequality, Korn considered two cases.

The so called first case is to consider $v \in H_0^1(\Omega)^n$. In this case the proof of the inequality (1.1) is simple and was first given by Korn. Moreover, it can be shown that it holds for any open set Ω (not necessarily bounded) and that the constant can be taken as $C = \sqrt{2}$ (see for example [10]).

The situation is quite different in the second case, where now $v \in H^1(\Omega)^n$ satisfies

$$\int_\Omega \Big(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\Big) = 0\,, \quad \text{for} \quad i, j = 1, \dots, n.$$

This case is fundamental for the analysis of the elasticity equations with traction boundary conditions. In this case, the first correct proofs are likely those given by

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Friedrichs, in [4] for $n = 2$ and [5] for $n = 2$ or 3. Indeed, it is not clear whether the original proof of Korn is correct. We have not checked that proof but we can mention that Friedrichs claimed that he has been unable to verify Korn's proof for the second case (see the footnote 3 in page 443 of [5]). Also Nitsche mentioned that the original proof of Korn is doubtful (see the introduction of [14]).

A different way of stating the Korn inequality is to say that, for any $v \in H^1(\Omega)^n$,

$$\|v\|_{H^1(\Omega)^n} \leq C\{\|v\|_{L^2(\Omega)^n} + \|\varepsilon(v)\|_{L^2(\Omega)^{n\times n}}\} \tag{1.2}$$

Indeed, (1.1) in the second case, can be derived from (1.2) by using compactness arguments (see for example [10]), provided that $H^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$, which is true under rather general assumptions on Ω. One the other hand, if (1.1) holds in the second case, then (1.2) also holds (see for example [2]).

An important difference between the first and second case is that, in the last one, (1.1) does not hold for arbitrary domains Ω. Indeed, it is known that the Korn inequality in the second case is not true when the domain has external cusps. The papers [6, 18] presented counter-examples showing this fact. Moreover, in the old paper [4], Friedrichs gave a very nice counter-example for an inequality which can be derived from (1.1). Suppose that Ω is a two dimensional domain and that

$$f(z) = \phi(x, y) + i\psi(x, y)$$

is an analytic function of the variable $z = x+iy$ in Ω with $\int_\Omega \phi = 0$. Then, Friedrichs proved in [4], that under suitable assumptions on Ω, there exists a constant C depending only on Ω, such that

$$\|\phi\|_{L^2(\Omega)} \leq C\|\psi\|_{L^2(\Omega)} \tag{1.3}$$

and he showed that this estimate is not true for some domains which have external cusps (see [4, page 343]). On the other hand, it is not difficult to see that the second case of Korn inequality implies (1.3) whenever the domain Ω is simply connected. This was proved by Horgan and Payne in [8] (they assume that the boundary is smooth but this is not needed for this implication).

The assumptions made by Friedrichs in [5] to prove the Korn inequality in the second case included domains with a finite number of corners or edges on $\partial\Omega$. After the papers of Friedrichs several proofs have been given under different assumptions on the domain. For example, Payne and Weinberger [15] proved the inequality for the sphere, giving explicit bounds for the constant. Also, they proved that (1.1) is true for domains which can be mapped onto the sphere by a C^2 transformation. In [11], Kondratiev and Oleinik gave a proof for domains which are star shaped with respect to a ball B and gave a bound for the constant in terms of the ratio between the diameters of Ω and B. Nitsche [14] proved the inequality in the form (1.2) for Lipschitz domains. His proof is based on the technique of Stein [16] to prove the extension theorem in Sobolev spaces. A different proof for Lipschitz domains was given by using the Calderón-Zygmund theory of singular integral operators (see [10, 17]). This proof also resembles the extension theorem, but now, the original proof of Calderón (see [1]). This last method also applies for the generalization of (1.2) to the L^p case, $1 < p < \infty$. Finally, we mention that another way of proving (1.2) is by means of the so called "Lion's Lemma" which states that, for a Lipschitz domain, a function $f \in L^2(\Omega)$ if and only if $f \in H^{-1}(\Omega)$ and $\nabla f \in H^{-1}(\Omega)^n$ (see for example [3] for this argument).

An interesting question is whether the Korn inequality in the second case holds for domains more general than Lipschitz. In his paper [9], Jones introduced the notion of (ϵ, δ) domain and prove that the extension theorem in Sobolev spaces holds if and only if the domain is of this class (which is much more general than the Lipschitz class).

As we said above, some of the arguments to prove the Korn inequality are related to those used for the extension theorem. This fact, together with the above mentioned counter-examples which resemble those for the extension theorem, give rise to the following question:

Is the Korn inequality in the second case (or its equivalent form (1.2)) valid for bounded (ϵ, δ) domains?

In this paper we prove that the answer to this question is positive. In order to do that, we modify the arguments of Jones [9], to construct an extension preserving the norm on the right hand side of (1.2). The key point in our construction is the use of the inequality (1.1), in the second case, on cubes or finite union of cubes. Once we have the extension, the argument concludes by applying (1.2) on a ball containing our original domain.

Our arguments apply also for the generalization of the Korn inequality to L^p, $1 < p < \infty$.

2. Proof of the Korn inequality

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open domain satisfying the (ϵ, δ) condition of Jones, namely, for any $x, y \in \Omega$ such that $|x - y| < \delta$ there is a rectifiable arc Γ joining x to y and satisfying

$$l(\Gamma) \leq \frac{1}{\epsilon}|x - y|,$$
$$d(z) \geq \frac{\epsilon|x - z||y - z|}{|x - y|} \quad \forall z \in \Gamma$$

where $l(\Gamma)$ denotes the arclength of Γ and $d(z)$ is the distance from z to Ω^c.

For $1 < p < \infty$, we will prove that there exists a constant $C = C(\Omega, p, \epsilon, \delta, n)$ such that, for any $v \in W^{1,p}(\Omega)^n$,

$$\|v\|_{W^{1,p}(\Omega)^n} \leq C\{\|v\|_{L^p(\Omega)^n} + \|\varepsilon(v)\|_{L^p(\Omega)^{n\times n}}\} \tag{2.1}$$

As mentioned in the introduction, this inequality is equivalent to

$$\|\nabla v\|_{L^p(\Omega)^{n\times n}} \leq C\|\varepsilon(v)\|_{L^p(\Omega)^{n\times n}} \tag{2.2}$$

for any $v \in W^{1,p}(\Omega)^n$ satisfying

$$\int_\Omega \Big(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\Big) = 0\,, \quad \text{for} \quad i, j = 1, \dots, n.$$

which, for $p = 2$, is known as "the second case" of Korn inequality.

Remark 2.1. It is easy to see, by a simple scaling argument, that the constant in the inequality (2.2) depends only on the shape of the domain Ω.

Through the rest of the paper we will use the letter C to denote different constants which may depend on ϵ, δ, n, p and the diameter of Ω (we will indicate this dependence only some times for the sake of clarity).

Inequality (2.1) follows from the first case of Korn inequality (or its generalization to L^p) in a larger domain if we can construct an extension Ev of v preserving the norm on the right hand side of (2.1) and such that Ev has compact support. We will construct such an extension by modifying appropriately the extension operator of Jones [9]. Since, for (ϵ, δ) domains, smooth functions are dense in $W^{1,p}(\Omega)$ (see [9]), it is enough to prove inequality (2.1) for $v \in W^{1,\infty}(\Omega)^n$ and so, we will construct the extension for such v.

Given $S \subset \mathbb{R}^n$, a measurable set with $|S| > 0$, we call $\overline{x}$ its barycenter and for $v \in W^{1,\infty}(\Omega)^n$ we associate with S and v the affine vector field

$$P_S(v)(x) = a + B(x - \overline{x}), \tag{2.3}$$

where $a \in \mathbb{R}^n$ and $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ are defined by

$$a = \frac{1}{|S|}\int_S v \quad \text{and} \quad b_{ij} = \frac{1}{2|S|}\int_S \Big(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\Big). \tag{2.4}$$

Observe that, since $b_{ij} = -b_{ji}$, $P_S(v)$ is an "infinitesimal rigid motion", i.e., it satisfies

$$\varepsilon(P_S(v)) = 0\,. \tag{2.5}$$

Moreover, a and B have been chosen in such a way that

$$\int_S \Big(\frac{\partial(v_i - P_S(v)_i)}{\partial x_j} - \frac{\partial(v_j - P_S(v)_j)}{\partial x_i}\Big) = 0\,, \tag{2.6}$$

$$\int_S (v - P_S(v))) = 0\,. \tag{2.7}$$

Assume now that the inequality (2.2) holds in S. Then, in view of (2.5) and (2.6) we have

$$\|\nabla(v - P_S(v)))\|_{L^p(S)^{n\times n}} \le C\|\varepsilon(v)\|_{L^p(S)^{n\times n}} \tag{2.8}$$

where the constant C depends on p and on the shape (but not on the scale!) of S (see Remark 2.1).

Now, from (2.7) and the Friedrichs-Poincaré inequality for functions with vanishing mean value and denoting with $d(S)$ the diameter of S, we have

$$\|v - P_S(v)\|_{L^p(S)^n} \le Cd(S)\|\nabla(v - P_S(v))\|_{L^p(S)^{n\times n}} \tag{2.9}$$

and therefore,

$$\|v - P_S(v)\|_{L^p(S)^n} \le Cd(S)\|\varepsilon(v)\|_{L^p(S)^{n\times n}} \tag{2.10}$$

where, again, the constant C depends only on p and on the shape of S. On the other hand, it is easy to see that

$$\|\nabla P_S(v)\|_{L^\infty(S)^{n\times n}} \le \|\nabla v\|_{L^\infty(S)^{n\times n}} \tag{2.11}$$

and so,

$$\|\nabla(v - P_S(v))\|_{L^\infty(S)^{n\times n}} \le 2\|\nabla v\|_{L^\infty(S)^{n\times n}} \tag{2.12}$$

and therefore, using (2.9) with $p = \infty$, we obtain

$$\|v - P_S(v)\|_{L^\infty(S)^n} \le Cd(S)\|\nabla v\|_{L^\infty(S)^{n\times n}} \tag{2.13}$$

where, also here, the constant C depends only on the shape of S.

Our extension of v will be constructed following the ideas developed in [9] but using now the polynomials $P_S(v)$.

Recall that any open set $\Omega \subset \mathbb{R}^n$ admits a Whitney decomposition into closed dyadic cubes S_k (see [19, 16]) , i.e., $\Omega = \cup_k S_k$, such that, if $\ell(S)$ denotes the edgelength of a cube S,

$$1 \le \frac{\operatorname{dist}(S_k, \partial\Omega)}{\ell(S_k)} \le 4\sqrt{n} \quad \forall k\,, \tag{2.14}$$

$$S_j^0 \cap S_k^0 = \emptyset \quad \text{if } j \ne k\,, \tag{2.15}$$

$$\frac{1}{4} \le \frac{\ell(S_j)}{\ell(S_k)} \le 4 \quad \text{if } S_j \cap S_k \ne \emptyset\,. \tag{2.16}$$

Let $W_1 = \{S_k\}$ be a Whitney decomposition of Ω and $W_2 = \{Q_j\}$ one of $(\Omega^c)^0$. We define

$$W_3 = \big\{Q_j \in W_2 \,:\, \ell(Q_j) \le \frac{\epsilon\delta}{16n}\big\}$$

It was shown by Jones (see Lemmas 2.4 and 2.8 of [9]) that, for each $Q_j \in W_3$, it is possible to choose a "reflected" cube $Q_j^* = S_k \in W_1$ such that

$$1 \le \frac{\ell(S_k)}{\ell(Q_j)} \le 4 \quad \text{and} \quad d(Q_j, S_k) \le C\ell(Q_j)$$

and moreover, if $Q_j, Q_k \in W_3$ and $Q_j \cap Q_k \ne \emptyset$, there is a chain $F_{j,k} = \{Q_j^* = S_1, S_2, \cdots, S_m = Q_k^*\}$ (i.e, $S_j \cap S_{j+1} \ne \emptyset$) of cubes in W_1, connecting Q_j^* to Q_k^* and with $m \le C(\epsilon, \delta)$.

It is known that, associated with a Whitney decomposition, there exists a partition of unity $\{\phi_j\}$ such that $\phi_j \in C^\infty(\mathbb{R}^n)$, *supp* $\phi_j \subset \frac{17}{16}Q_j$, $0 \le \phi_j \le 1$,

$$\sum_{Q_j \in W_3} \phi_j \equiv 1 \quad \text{on} \quad \bigcup_{Q_j \in W_3} Q_j$$

and

$$|\nabla\phi_j| \le C\ell(Q_j)^{-1} \quad \forall j$$

(see [16, 19]). Now, given $v \in W^{1,\infty}(\Omega)^n$, let $P_j = P_{Q_j^*}(v)$ defined as in (2.3) and (2.4) with $S = Q_j^*$. Then, we define Ev, the extension to $\mathbb{R}^n$ of v, in the following way,

$$Ev = \sum_{Q_j \in W_3} P_j\phi_j \quad \text{in } (\Omega^c)^0,$$
$$Ev = v \quad \text{in } \Omega.$$

Since $|\partial\Omega| = 0$ (see Lemma 2.3 in [9]), it follows that Ev is defined *p.p.* in $\mathbb{R}^n$.

The arguments of the following lemmas are similar to those in [9]. In particular we will make repeated use of Lemma 2.1 of [9] which says,

Lemma 2.2. *Let Q be a cube and $F, G \subset Q$ be two measurable subsets such that $|F|, |G| \ge \gamma|Q|$ for some $\gamma > 0$. If P is a polynomial of degree 1 then,*

$$\|P\|_{L^p(F)} \le C(\gamma)\|P\|_{L^p(G)}.$$

Lemma 2.3. *Let $F = \{S_1, \cdots, S_m\}$ be a chain of cubes in W_1. Then,*

$$\|P_{S_1}(v) - P_{S_m}(v)\|_{L^p(S_1)^n} \le Cc(m)\ell(S_1)\|\varepsilon(v)\|_{L^p(\cup_j S_j)^{n\times n}}\,, \tag{2.17}$$

and

$$\|P_{S_1}(v) - P_{S_m}(v)\|_{L^\infty(S_1)^n} \le Cc(m)\ell(S_1)\|\nabla v\|_{L^\infty(\cup_j S_j)^{n\times n}}\,. \tag{2.18}$$

Proof. We will use (2.10) with S being a cube or a union of two neighboring cubes. In view of (2.16) there are a finite number, depending only on the dimension n, of possible shapes for the union of two neighboring cubes, and so, we can take a uniform constant in (2.10).

Using Lemma 2.2 we have

$$\begin{aligned}
&\|P_{S_1}(v) - P_{S_m}(v)\|_{L^p(S_1)^n} \\
&\le \sum_{r=1}^{m-1} \|P_{S_r}(v) - P_{S_{r+1}}(v)\|_{L^p(S_1)^n} \\
&\le c(m) \sum_{r=1}^{m-1} \|P_{S_r}(v) - P_{S_{r+1}}(v)\|_{L^p(S_r)^n} \\
&\le c(m) \sum_{r=1}^{m-1} \{\|P_{S_r}(v) - P_{S_r \cup S_{r+1}}(v)\|_{L^p(S_r)^n} \\
&\quad + \|P_{S_r \cup S_{r+1}}(v) - P_{S_{r+1}}(v)\|_{L^p(S_{r+1})^n}\} \\
&\le c(m) \sum_{r=1}^{m-1} \{\|v - P_{S_r}(v)\|_{L^p(S_r)^n} + \|v - P_{S_{r+1}}(v)\|_{L^p(S_{r+1})^n} \\
&\quad + \|v - P_{S_r \cup S_{r+1}}(v)\|_{L^p(S_r \cup S_{r+1})^n}\} \\
&\le Cc(m)\ell(S_1)\|\varepsilon(v)\|_{L^p(\cup_j S_j)^{n\times n}}
\end{aligned}$$

where we have used (2.10). The proof of (2.18) is analogous using now (2.13). $\square$

Now, for each $Q_j, Q_k \in W_3$ such that $Q_j \cap Q_k \neq \emptyset$, we choose a chain $F_{j,k}$ connecting Q_j^* to Q_k^* and with $m \le C(\epsilon, \delta)$ and define

$$F(Q_j) = \bigcup_{Q_k \in W_3,\, Q_j \cap Q_k \neq \emptyset} F_{j,k}$$

then,

$$\Big\| \sum_{Q_k,\, Q_j \cap Q_k \neq \emptyset} \chi_{\cup F_{j,k}} \Big\|_{L^\infty(\mathbb{R}^n)} \le C \quad \forall Q_j \in W_3 \tag{2.19}$$

The following lemmas will allow us to control the norms of Ev, $\varepsilon(Ev)$ and $\nabla(Ev)$ in $(\Omega^c)^0$.

Lemma 2.4. *For $Q_0 \in W_3$ we have*

$$\|Ev\|_{L^p(Q_0)^n} \le C\{\|v\|_{L^p(Q_0^*)^n} + \ell(Q_0)\|\varepsilon(v)\|_{L^p(\cup F(Q_0))^{n\times n}}\}, \tag{2.20}$$

$$\|\varepsilon(Ev)\|_{L^p(Q_0)^{n\times n}} \le C\|\varepsilon(v)\|_{L^p(\cup F(Q_0))^{n\times n}}, \tag{2.21}$$

$$\|Ev\|_{L^\infty(Q_0)^n} \le C\{\|v\|_{L^\infty(Q_0^*)^n} + \ell(Q_0)\|\nabla v\|_{L^\infty(\cup F(Q_0))^{n\times n}}\}, \tag{2.22}$$

$$\|\nabla(Ev)\|_{L^\infty(Q_0)^{n\times n}} \le C\|\nabla v\|_{L^\infty(\cup F(Q_0))^{n\times n}}\}. \tag{2.23}$$

Proof. On Q_0 we have

$$Ev = \sum_{Q_j \in W_3} P_j \phi_j$$

Now, since $\sum_{Q_j \in W_3} \phi_j \equiv 1$ on $\cup_{Q_j \in W_3} Q_j$, then

$$\| \sum_{Q_j \in W_3} P_j \phi_j \|_{L^p(Q_0)^n} \le \|P_0\|_{L^p(Q_0)^n} + \| \sum_{Q_j \in W_3} (P_j - P_0)\phi_j \|_{L^p(Q_0)^n} = I + II$$

Using Lemma 2.2 and (2.10), we have

$$\begin{aligned} I = \|P_0\|_{L^p(Q_0)^n} &\le C\|P_0\|_{L^p(Q_0^*)^n} \\ &\le C\|P_0 - v\|_{L^p(Q_0^*)^n} + \|v\|_{L^p(Q_0^*)^n} \\ &\le C\{\|v\|_{L^p(Q_0^*)^n} + \ell(Q_0)\|\varepsilon(v)\|_{L^p(Q_0^*)^{n\times n}}\} \end{aligned}$$

Now, since on Q_0 there are a finite number (depending only on n) of non-vanishing ϕ_j and $0 \le \phi_j \le 1$, to bound II it is enough to bound $\|(P_j - P_0)\|_{L^p(Q_0)^n}$. But, using (2.17) and again Lemma 2.2, we have

$$\|(P_j - P_0)\|_{L^p(Q_0)^n} \le C\|(P_j - P_0)\|_{L^p(Q_0^*)^n} \le C\ell(Q_0)\|\varepsilon(v)\|_{L^p(\cup F_{0,j})^{n\times n}}$$

and therefore, summing up and using (2.19) we obtain (2.20). Analogously, we can prove (2.22) using now (2.13) and (2.18).

Now, calling P_j^r the components of P_j and recalling that $\varepsilon(P_j) = 0$ we have

$$\varepsilon_{rs}(P_j\phi_j) = \frac{1}{2}P_j^r\frac{\partial\phi_j}{\partial x_s} + \frac{1}{2}P_j^s\frac{\partial\phi_j}{\partial x_r} \tag{2.24}$$

On Q_0,

$$Ev = P_0 + \sum_{Q_j\in W_3}(P_j - P_0)\phi_j$$

and therefore, since $\varepsilon(P_0) = 0$ we have,

$$\varepsilon(Ev) = \sum_{Q_j\in W_3}\varepsilon((P_j - P_0)\phi_j)$$

but, there are at most C cubes Q_j such that ϕ_j does not vanishes in Q_0 and these Q_j intersect Q_0 and therefore, $\ell(Q_j) \ge \frac{1}{4}\ell(Q_0)$. Thus,

$$|\nabla\phi_j| \le C\ell(Q_0)^{-1}$$

whenever $\phi_j \ne 0$ for some $x \in Q_0$. Then, for these values of j, it follows from (2.24) and (2.17) that

$$\begin{aligned} \|\varepsilon((P_j - P_0)\phi_j)\|_{L^p(Q_0)^{n\times n}} &\le C\ell(Q_0)^{-1}\|P_j - P_0\|_{L^p(Q_0)^n} \\ &\le C\ell(Q_0)^{-1}\|P_j - P_0\|_{L^p(Q_0^*)^n} \le C\|\varepsilon(v)\|_{L^p(\cup F_{0,j})^{n\times n}} \end{aligned}$$

Summing up in j, we obtain (2.21).

The proof of (2.23) is similar to that of (2.21) but using now (2.11). Indeed, we have

$$\nabla(Ev) = \nabla P_0 + \sum_{Q_j\in W_3}\nabla((P_j - P_0)\phi_j)$$

and we have to estimate also the terms ∇P_0 and $\nabla(P_j - P_0)$. But,

$$\|\nabla P_0\|_{L^\infty(Q_0)^{n\times n}} \le C\|\nabla P_0\|_{L^\infty(Q_0^*)^{n\times n}} \le C\|\nabla v\|_{L^\infty(Q_0)^{n\times n}},$$

and

$$\begin{aligned} \|\nabla(P_j - P_0)\|_{L^\infty(Q_0)^{n\times n}} &\le C\|\nabla(P_j - P_0)\|_{L^\infty(Q_0^*)^{n\times n}} \\ &\le C\|\nabla(P_j - P_0)\|_{L^\infty(Q_0^*\cup Q_j^*)^{n\times n}} \le C\|\nabla v\|_{L^\infty(Q_0^*\cup Q_j^*)^{n\times n}} \\ &\le C\|\nabla v\|_{L^\infty(\cup F_{j,0})^{n\times n}}\} \end{aligned}$$

and so (2.23) holds. □

Lemma 2.5. *For $Q_0 \in W_2 \setminus W_3$ we have*

$$\|Ev\|_{L^p(Q_0)^n} \le C \sum_{Q_j \in W_3,\, Q_j \cap Q_0 \neq \emptyset} \{\|v\|_{L^p(Q_j^*)^n} + \|\varepsilon(v)\|_{L^p(Q_j^*)^{n\times n}}\}, \tag{2.25}$$

$$\|\varepsilon(Ev)\|_{L^p(Q_0)^{n\times n}} \le C \sum_{Q_j \in W_3,\, Q_j \cap Q_0 \neq \emptyset} \{\|v\|_{L^p(Q_j^*)^n} + \|\varepsilon(v)\|_{L^p(Q_j^*)^{n\times n}}\}, \tag{2.26}$$

$$\|Ev\|_{L^\infty(Q_0)^n} \le C \sum_{Q_j \in W_3,\, Q_j \cap Q_0 \neq \emptyset} \{\|v\|_{L^\infty(Q_j^*)^n} + \|\nabla v\|_{L^\infty(Q_j^*)^{n\times n}}\}, \tag{2.27}$$

$$\|\nabla(Ev)\|_{L^\infty(Q_0)^{n\times n}} \le C \sum_{Q_j \in W_3,\, Q_j \cap Q_0 \neq \emptyset} \{\|v\|_{L^\infty(Q_j^*)^n} + \|\nabla v\|_{L^\infty(Q_j^*)^{n\times n}}\} \tag{2.28}$$

Proof. If ϕ_j does not vanish on Q_0 then $Q_j \cap Q_0 \neq \emptyset$ and so,

$$\ell(Q_j) \ge \frac{1}{4}\ell(Q_0) \ge \frac{\epsilon\delta}{64n}$$

therefore, on Q_0, we have

$$|Ev| = |\sum_{Q_j \in W_3,\, Q_j \cap Q_0 \neq \emptyset} \phi_j P_j| \le C \sum_{Q_j \in W_3,\, Q_j \cap Q_0 \neq \emptyset} |P_j|,$$

but

$$\|P_j\|_{L^p(Q_0)^n} \le C\|P_j\|_{L^p(Q_0^*)^n} \le C\{\|v - P_j\|_{L^p(Q_0^*)^n} + \|v\|_{L^p(Q_0^*)^n}\}.$$

Now, since Ω is bounded, $\ell(Q_j^*)$ is bounded by a constant depending only on Ω and therefore, using (2.10), we obtain

$$\|P_j\|_{L^p(Q_0)^n} \le C\{\|\varepsilon(v)\|_{L^p(Q_0^*)^{n\times n}} + \|v\|_{L^p(Q_0^*)^n}\} \tag{2.29}$$

and therefore, (2.25) is proved.

On the other hand, on Q_0 we have

$$\varepsilon_{rs}(Ev) = \frac{1}{2} \sum_{Q_j \in W_3,\, Q_j \cap Q_0 \neq \emptyset} \{P_j^r \frac{\partial \phi_j}{\partial x_s} + P_j^s \frac{\partial \phi_j}{\partial x_r}\}$$

but, $\ell(Q_j) \ge \frac{\epsilon\delta}{64n}$, and so $|\nabla\phi_j| \le C$ and (2.26) follows using again (2.29). Finally, (2.27) and (2.28) are obtained by similar arguments, using now (2.11). $\square$

Corollary 2.6. *If $v \in W^{1,\infty}(\Omega)$ then*

$$\|Ev\|_{L^p((\Omega^c)^0)^n} + \|\varepsilon(Ev)\|_{L^p((\Omega^c)^0)^{n\times n}} \le C\{\|v\|_{L^p(\Omega)^n} + \|\varepsilon(v)\|_{L^p(\Omega)^{n\times n}}\} \tag{2.30}$$

and

$$\|Ev\|_{W^{1,\infty}((\Omega^c)^0)^n} \le C\|v\|_{W^{1,\infty}(\Omega)^n} \tag{2.31}$$

Proof. It follows immediately from Lemmas 2.4 and 2.5 by summing up over all $Q_0 \in W_2$. $\square$

We can now state the main theorem which follows from the results above and arguments given in [9].

Theorem 2.7. *If Ω is a bounded (ϵ, δ) domain and $1 < p < \infty$, there exists a constant C depending on ϵ, δ, n, p and the diameter of Ω such that, for any $v \in W^{1,p}(\Omega)^n$,*

$$\|v\|_{W^{1,p}(\Omega)^n} \le C\{\|v\|_{L^p(\Omega)^n} + \|\varepsilon(v)\|_{L^p(\Omega)^{n\times n}}\} \tag{2.32}$$

Proof. By density, it is enough to prove the inequality for smooth v. So, we assume that $v \in W^{1,\infty}(\Omega)^n$. As we already said, the extension Ev is defined almost everywhere in $\mathbb{R}^n$ because $|\partial\Omega| = 0$. Moreover, the support of Ev is contained in a ball B.

In view of (2.31) and the fact that $|\partial\Omega| = 0$, if Ev is a continuous function then, it is Lipschitz. But, the continuity of Ev can be proved exactly as in Lemma 3.5 of [9]. Therefore, $Ev \in W^{1,p}(B)^n$ and so, from (2.30), it follows that

$$\|Ev\|_{L^p(B)^n} + \|\varepsilon(Ev)\|_{L^p(B)^{n\times n}} \le C\{\|v\|_{L^p(\Omega)^n} + \|\varepsilon(v)\|_{L^p(\Omega)^{n\times n}}\} \tag{2.33}$$

but, using the Korn inequality for smooth domains, we have

$$\|Ev\|_{W^{1,p}(B)^n} \le C\{\|Ev\|_{L^p(B)^n} + \|\varepsilon(Ev)\|_{L^p(B)^{n\times n}}\}$$

which together with (2.33) concludes the proof. □

We conclude the paper stating two consequences of our main theorem which follows by known arguments. The first result is the second case of the Korn inequality and the second one is the Friedrichs inequality for complex analytic functions (and their generalizations to L^p).

Corollary 2.8. *If $\Omega \subset \mathbb{R}^n$ is a bounded (ϵ, δ) domain and $1 < p < \infty$, there exists a constant C depending on ϵ, δ, n, p such that, for any $v \in W^{1,p}(\Omega)^n$ which satisfies*

$$\int_\Omega \Big(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\Big) = 0,$$
$$\|\nabla v\|_{L^p(\Omega)^{n\times n}} \le C\|\varepsilon(v)\|_{L^p(\Omega)^{n\times n}}.$$

Corollary 2.9. *If $\Omega \subset \mathbb{R}^2$ is a bounded simply connected (ϵ, δ) domain and $1 < p < \infty$, there exists a constant C depending on ϵ, δ, n, p such that for any analytic function of the variable $z = x + iy$ in Ω, $f(z) = \phi(x, y) + i\psi(x, y)$, with $\int_\Omega \phi = 0$,*

$$\|\phi\|_{L^p(\Omega)} \le C\|\psi\|_{L^p(\Omega)}.$$

Conclusions. We have proved that, if Ω is an (ϵ, δ) domain, the Korn inequality holds on Ω. It is interesting to observe that the converse is not true. Indeed, consider the two dimensional domain

$$\Omega = (-1, 1)^2 \setminus \{(x, 0) : x \in [0, 1)\}.$$

Since Ω can be written as the union of two Lipschitz domains, the Korn inequality is valid on Ω. On the other hand, it is not difficult to see that Ω is not an (ϵ, δ) domain. Therefore, to give a characterization of the domains satisfying the Korn inequality is an interesting problem which remains open.

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