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ON PONTRYAGIN-RODYGIN'S THEOREM FOR CONVERGENCE OF SOLUTIONS OF SLOW AND FAST SYSTEMS

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Abstract. In this paper we study fast and slow systems for which the fast dynamics has limit cycles, for all fixed values of the slow variables. The fundamental tool is the Pontryagin and Rodygin theorem which describes the limiting behavior of the solutions in the continuously differentiable case, when the cycles are exponentially stable. We extend this result to the continuous case, and exponential stability is replaced by asymptotic stability. We give two examples with numerical simulations to illustrate the problem. Our results are formulated in classical mathematics. They are proved using Nonstandard Analysis.

1. Introduction

This paper will focus on slow and fast systems of the form

$$\varepsilon \frac{dx}{dt} = f(x, y, \varepsilon), \quad \frac{dy}{dt} = g(x, y, \varepsilon), \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and ε is a small positive parameter. The variable x is called a *fast variable*, y is called a *slow variable*. The change of time $\tau = t/\varepsilon$ transforms system (1.1) into

$$\frac{dx}{d\tau} = f(x, y, \varepsilon), \quad \frac{dy}{d\tau} = \varepsilon g(x, y, \varepsilon). \tag{1.2}$$

This system is a one parameter deformation of the unperturbed system

$$\frac{dx}{d\tau} = f(x, y, 0), \quad \frac{dy}{d\tau} = 0, \tag{1.3}$$

which is called the *fast equation*.

In the case where solutions of (1.3) tend toward an equilibrium point $\xi(y)$, where $x = \xi(y)$ is a root of equation

$$f(x, y, 0) = 0, \tag{1.4}$$

Tykhonov's theorem [15, 17] gives the limiting behavior of system (1.1). A fast transition brings the solution near the slow manifold (1.4). Then, a slow motion

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takes place near the slow manifold and is approximated by the solution of the reduced equation

$$\frac{dy}{dt} = g(\xi(y), y, 0). \tag{1.5}$$

This result was obtained in [15] for continuous vector fields f and g, under the assumption that $\xi(y)$ is an asymptotically stable equilibrium of the fast equation (1.3), uniformly in y. It was extended in [6] to all systems that belong to a small neighborhood of the unperturbed system. For differentiable systems, if the variational equation has eigenvalues with negative real part for all y in the domain of interest, then the uniform asymptotic stability of the equilibrium $\xi(y)$ holds.

In the case where solutions of (1.3) tend toward a cycle Γ_y, Pontryagin and Rodygin's theorem [11] gives the limiting behavior of system (1.1) : after a fast transition that brings the solutions near the cycles, the solutions of system (1.1) are approximated by the solutions of the average system

$$\frac{dy}{dt} = \frac{1}{T(y)} \int_0^{T(y)} g(x^*(\tau, y), y, 0)\, d\tau, \tag{1.6}$$

where $x^*(\tau, y)$ is a periodic solution of the fast equation (1.3) corresponding to the cycle Γ_y and $T(y)$ is its period. This result was obtained for at least continuously differentiable vector fields f and g, under the assumption that the cycles Γ_y are asymptotically stable in the linear approximation, that is, the variational equation corresponding to the cycle has multipliers with moduli less than 1 with a single exception. To our knowledge the continuous case with asymptotic stability instead of exponential stability was not considered in the literature.

Assume that the equilibrium $\xi(y)$ (resp. the cycle Γ_y) loses its stability, but remains nondegenerate. Neishtadt [8] proved, in analytic systems, that there is a delayed loss of stability of the solutions of (1.1) : the solutions remain for a long time near the unstable equilibrium (resp. the unstable cycle) and the slow variable y remains approximated by the solution of the reduced equation (1.5) (resp. the averaged system (1.6)).

The aim of this work is to extend the result of [11] to continuous vector fields and to define a topology such that the description of the solutions holds for systems that belong to a small neighborhood of the unperturbed system. Following [6], we define in Section 2 a suitable function space of *Initial Value Problems* (IVPs) in order to study small neighborhoods of the unperturbed problem. The main results concerning approximations of solutions on finite and infinite time interval (Theorem 2.2, Theorem 2.3) are stated. In the present work, the results are formulated in classical mathematics and proved within *Internal Set Theory* (IST) [9] which is an axiomatic approach of *Nonstandard Analysis* (NSA) [12]. The idea to use NSA in perturbation theory of differential equations goes back to the seventies with the Reebian school [7]. It has become today a well-established tool in asymptotic theory (see the five-digits classification 34E18 of the 2000 Mathematical Subject Classification). We give in Section 3 a short tutorial on IST in order to characterize the notion of stability and to translate our main results in nonstandard words (Theorem 3.5 and Theorem 3.6). Section 4 consists of presenting some lemmas in view of the proofs of Theorem 3.5 and Theorem 3.6. In Section 5, we apply our result to two examples and we give numerical simulations.

2. Results

Let us consider the differential system

$$\begin{aligned} \varepsilon \dot{x} &= f(x,y), \quad x(0)=\alpha, \\ \dot{y} &= g(x,y), \quad y(0)=\beta, \end{aligned} \tag{2.1}$$

where ε is a positive real number in $]0,\varepsilon_0]$, $f:\Omega\to\mathbb{R}^n$, $g:\Omega\to\mathbb{R}^m$ are continuous on an open subset Ω of $\mathbb{R}^{n+m}$ and $(\alpha,\beta)\in\Omega$. The dot $(\dot{})$ means d/dt. The set

$$\begin{aligned} \mathcal{T} = \{(\Omega,f,g,\alpha,\beta) : \Omega \text{ open subset of } \mathbb{R}^{n+m}, (\alpha,\beta)\in\Omega, \\ f:\Omega\to\mathbb{R}^n,\ g:\Omega\to\mathbb{R}^m \text{ continuous}\} \end{aligned}$$

is provided with the *topology of uniform convergence on compacta* [6]. This topology is the topology for which the neighborhood system of an element $(\Omega_0,f_0,g_0,\alpha_0,\beta_0)$ is generated by the sets

$$\begin{aligned} V(D,a) = \{(\Omega,f,g,\alpha,\beta)\in\mathcal{T} :\ D\subset\Omega,\ \|f-f_0\|_D<a,\ \|g-g_0\|_D<a, \\ \|\alpha-\alpha_0\|<a,\ \|\beta-\beta_0\|<a\}. \end{aligned}$$

Here $\|h\|_D = \sup_{u\in D}\|h(u)\|$, where h is defined on the set D with values in a normed space. The aim is to study the system (2.1) when ε is sufficiently small and $(\Omega,f,g,\alpha,\beta)$ sufficiently close to an element $(\Omega_0,f_0,g_0,\alpha_0,\beta_0)$ of $\mathcal{T}$ in the sense of this topology. The *fast equation* is then defined by

$$x' = f_0(x,y), \tag{2.2}$$

where y is considered as a parameter and $(')$ means the derivative with respect to the fast time $\tau = t/\varepsilon$. We make the following assumptions:

- (A) For all y, the fast equation (2.2) has the uniqueness of the solutions with prescribed initial conditions.
- (B) There exists a family of solutions $x^*(\tau,y)$ depending continuously on $y\in G$, where G is a compact subset of $\mathbb{R}^m$ with a non empty interior, such that $x^*(\tau,y)$ is a periodic solution of the fast equation (2.2) of period $T(y)>0$, the mapping $y\to T(y)$ is continuous, and the cycle Γ_y corresponding to the periodic solution $x^*(\tau,y)$ is asymptotically stable and its basin of attraction is uniform over G.

From Assumption (B) it follows that the cycle Γ_y depends continuously on y and is locally unique, that is, there exists an neighborhood W of Γ_y such that the equation (2.2) has no other cycle in W.

Definition 2.1. The periodic solution $x^*(\tau,y)$ of *(2.2)* is said to be orbitally asymptotically stable (in the sense of Lyapunov) if its orbit Γ_y is :
1. Stable, i.e. for every $\mu>0$, there exists $\eta>0$ such that any solution $\tilde{x}(\tau)$ of *(2.2)* for which dis$(\tilde{x}(0)$,$\Gamma_y)<\eta$ can be continued for all $\tau\geq 0$ and satisfies the inequality dis$(\tilde{x}(\tau)$,$\Gamma_y)<\mu$.
2. and Attractive, i.e. Γ_y admits a neighborhood $\mathcal{V}$ (basin of attraction) such that any solution $\tilde{x}(\tau)$ of *(2.2)* for which $\tilde{x}(0)\in\mathcal{V}$ can be continued for all $\tau\geq 0$ and satisfies $\lim\limits_{\tau\to\infty}$ dis$(\tilde{x}(\tau)$,$\Gamma_y)=0$.
Moreover, the basin of attraction of the orbit Γ_y is uniform over G if there exists a real number $a>0$ such that, for all y in G, the set $\{x\in\mathbb{R}^n : \text{dis}(x,\Gamma_y)\leq a\}$ is in the basin of attraction of Γ_y.

We define the *slow equation* on the interior G_0 of G by the averaged system

$$\dot{y} = \bar{g}_0(y) := \frac{1}{T(y)} \int_0^{T(y)} g_0(x^*(\tau, y), y)\, d\tau, \tag{2.3}$$

and we add the following two assumptions:

(C) The slow equation (2.3) has the uniqueness of the solutions with prescribed initial conditions.

(D) β_0 is in G_0 and α_0 is in the basin of attraction of Γ_{β_0}.

We refer to the *boundary layer equation* as

$$x' = f_0(x, \beta_0), \quad x(0) = \alpha_0, \tag{2.4}$$

and to the *reduced problem* as

$$\dot{y} = \bar{g}_0(y),\ y(0) = \beta_0. \tag{2.5}$$

We can state the first result.

Theorem 2.2. *Let $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ be in $\mathcal{T}$. Assume that $(A)-(D)$ are satisfied. Let $\tilde{x}(\tau)$ and $\bar{y}(t)$ be the respective solutions of (2.4) and (2.5) and $L \in I$, where I is the positive interval of definition of $\bar{y}(t)$. Then for all $\eta > 0$, there exist $\varepsilon^* > 0$ and a neighborhood $\mathcal{V}$ of $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ in $\mathcal{T}$ such that for all $\varepsilon < \varepsilon^*$ and all $(\Omega, f, g, \alpha, \beta)$ in $\mathcal{V}$, any solution $(x(t), y(t))$ of (2.1) is defined at least on $[0,L]$ and there exists $\omega > 0$ such that $\varepsilon\omega < \eta$, $\|x(\varepsilon\tau) - \tilde{x}(\tau)\| < \eta$ for $0 \le \tau \le \omega$, $\|y(t) - \bar{y}(t)\| < \eta$ for $0 \le t \le L$ and $\operatorname{dis}(x(t), \Gamma_{\bar{y}(t)}) < \eta$ for $\varepsilon\omega \le t \le L$.*

Suppose in addition that there exists a point $\bar{y}_\infty$ such that $\bar{g}_0(\bar{y}_\infty) = 0$. Under the following assumption, the previous theorem holds for all $t \ge 0$.

(E) The point $\bar{y}_\infty$ is an asymptotically stable equilibrium of (2.3) and β_0 is in its basin of attraction.

Theorem 2.3. *Let $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ be in $\mathcal{T}$. Let $\bar{y}_\infty$ be in G_0. Assume that $(A)-(E)$ are satisfied. Let $\tilde{x}(\tau)$ and $\bar{y}(t)$ be the respective solutions of (2.4) and (2.5). Then for all $\eta > 0$, there exist $\varepsilon^* > 0$ and a neighborhood $\mathcal{V}$ of $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ in $\mathcal{T}$ such that for all $\varepsilon < \varepsilon^*$ and all $(\Omega, f, g, \alpha, \beta)$ in $\mathcal{V}$, any solution $(x(t), y(t))$ of (2.1) is defined for all $t \ge 0$ and there exists $\omega > 0$ such that $\varepsilon\omega < \eta$, $\|x(\varepsilon\tau) - \tilde{x}(\tau)\| < \eta$ for $0 \le \tau \le \omega$, $\|y(t) - \bar{y}(t)\| < \eta$ for $t \ge 0$ and $\operatorname{dis}(x(t), \Gamma_{\bar{y}(t)}) < \eta$ for $t \ge \varepsilon\omega$.*

The proofs of the two theorems are postponed to Subsections 4.3 and 4.4.

3. Nonstandard formulations of the results

3.1. **A short tutorial on IST.** As it was outlined in the introduction, *Internal Set Theory* (IST) [9] is an extension of ordinary mathematics, that is, Zermelo-Fraenkel set theory plus axiom of choice (ZFC). The theory IST gives an axiomatic approach of Robinson's *Nonstandard Analysis* [12]. We adjoin to ZFC a new undefined unary predicate *standard* (st) and add to the usual axioms of ZFC three others for governing the use of the new predicate. *All theorems of* ZFC *remain valid in* IST. What is new in IST is an addition, not a change. We call a formula of IST *internal* in the case where it does not involve the new predicate "st" ; otherwise, we call it *external.* The theory IST is a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC. Some of the theorems which are proved in IST are external and can be reformulated so that they become internal. Indeed, there is a *reduction algorithm* due to Nelson which reduces any external formula $F(x_1,...,x_n)$ of IST without other free variables than $x_1,...,x_n$ to an internal formula $F'(x_1\ ,...,x_n)$ with the same free variables, such that $F \equiv F'$, that is, $F \Leftrightarrow F'$ for all standard values of the free variables. We will need the following reduction formula which occurs frequently:

$$\forall x\ (\forall^{\rm st} y\ A \Rightarrow \forall^{\rm st} z\ B) \equiv \forall z\ \exists^{\rm fin} y'\ \forall x\ (\forall y \in y'\ A \Rightarrow B), \tag{3.1}$$

where A (respectively B) is an internal formula with free variable y (respectively z) and standard parameters. The notation $\forall^{\rm st}$ means "for all standard" and $\exists^{\rm fin}$ means "there is a finite".

A real number x is *infinitesimal,* denoted by $x \simeq 0$, if $|x| < a$ for all standard positive real numbers a, *limited* if $|x| < a$ for some standard a, *appreciable* if it is limited and not infinitesimal, and *unlimited* and denoted by $x \simeq \pm\infty$, if it is not limited. Let (E, d) be a standard metric space. Two points x and y in E are *infinitely close*, denoted by $x \simeq y$, if $\mathrm{d}(x,y) \simeq 0$. The element x is *nearstandard* in E if there exists a standard $x_0 \in E$ such that $x \simeq x_0$. Note that a real number is nearstandard in $\mathbb{R}$ if and only if it is limited. The point x_0 is called the *standard part* of x (it is unique) and is also denoted ${}^\circ x$. A vector x in $\mathbb{R}^d$, d standard, is *infinitesimal* (respectively *limited*, *unlimited*) if $|x|$ is infinitesimal (respectively limited, unlimited), where $|.|$ is a standard norm in $\mathbb{R}^d$.

Only internal formulas can be used to define subsets. However, notations as $\{x \in \mathbb{R} : x \text{ limited}\}$ or $\{x \in \mathbb{R} : x \text{ infinitesimal}\}$ can be considered as *external sets.* For instance, we can prove that there do not exist subsets L and I of $\mathbb{R}$ such that*, for all x in $\mathbb{R}$, x is in L if and only if x is limited*, or *x is in I if and only if x is infinitesimal.* This result is frequently used to prove that the validity of a property exceeds the domain where it was established in direct way. Suppose that we have shown that a certain internal property A holds for every limited x, then we know that A holds for some unlimited x, for otherwise we could let $L = \{x \in \mathbb{R} : A\}$. This statement is called the *Cauchy principle.* It has the following consequence [12]

Lemma 3.1 (Robinson's Lemma)**.** *Let r be a real function such that $r(t) \simeq 0$ for all limited $t \geq 0$, then there exists an unlimited ω such that $r(t) \simeq 0$ for all t in $[0, \omega]$.*

3.2. **External characterizations of orbital stability.** We give in this subsection external characterizations of orbital stability.

Lemma 3.2. *Assume that f_0 and $x^*(\tau, y)$ are standard. The periodic solution $x^*(\tau, y)$ of the equation (2.2) with orbit Γ_y is*
1. Orbitally stable, if and only if any solution $\tilde{x}(\tau)$ of (2.2) for which $\mathrm{dis}(\tilde{x}(0),\Gamma_y) \simeq 0$ *can be continued for all $\tau \geq 0$ and satisfies* $\mathrm{dis}(\tilde{x}(\tau),\Gamma_y) \simeq 0$.
2. Orbitally attractive if and only if Γ_y admits a standard neighborhood $\mathcal{V}$ (basin of attraction) such that any solution $\tilde{x}(\tau)$ of (2.2) for which $\tilde{x}(0)$ is standard in $\mathcal{V}$ can be continued for all $\tau \geq 0$ and satisfies $\mathrm{dis}(\tilde{x}(\tau),\Gamma_y) \simeq 0$ *for all $\tau \simeq +\infty$.*

Proof. 1. We denote by B the formula "Any solution $\tilde{x}(\tau)$ of *(2.2)* for which $\tilde{x}(0) = \alpha$ can be continued for all $\tau \geq 0$ and satisfies $\mathrm{dis}(\tilde{x}(\tau),\Gamma_y) < \mu$". To say in the lemma $\mathrm{dis}(\alpha,\Gamma_y) \simeq 0$ is the same as to say $\forall^{\rm st}\eta\ \mathrm{dis}(\alpha,\Gamma_y) < \eta$ and to say $\mathrm{dis}(\tilde{x}(\tau),\Gamma_y) \simeq 0$ is the same as to say $\forall^{\rm st}\mu\ \mathrm{dis}(\tilde{x}(\tau), \Gamma_y) < \mu$. Then, the characterization of orbital stability is

$$\forall\alpha\ (\forall^{\rm st}\eta\ \mathrm{dis}(\alpha, \Gamma_y) < \eta \Rightarrow \forall^{\rm st}\mu\ B).$$

In this formula, f_0 and Γ_y are standard parameters and η, μ range over the positive real numbers. By the reduction formula (3.1), this is equivalent to

$$\forall\mu\ \exists^{\mathrm{fin}}\eta'\ \forall\alpha\ (\forall\eta\in\eta'\ \ \mathrm{dis}(\alpha,\Gamma_y)<\eta\Rightarrow B).$$

But η' being a finite set, there exists η such that $\eta=\min\eta'$ and the last formula becomes

$$\forall\mu\ \exists\eta\ \forall\alpha\ (\mathrm{dis}(\alpha,\Gamma_y)<\eta\Rightarrow B).$$

This is exactly the usual definition of orbital stability.
2. By transfer, the orbital attractivity of a solution is equivalent to the existence of a standard basin of attraction. In the lemma, the characterization of the standard basin of attraction $\mathcal{V}$ is that any solution $\tilde{x}(\tau)$ of the equation (2.2) for which $\tilde{x}(0)$ is standard in $\mathcal{V}$ can be continued for all $\tau\ge 0$ and satisfies

$$\forall\tau\ (\forall^{\mathrm{st}}r\ \tau>r)\Rightarrow\forall^{\mathrm{st}}\mu\ \ \mathrm{dis}(\tilde{x}(\tau),\Gamma_y)<\mu\,.$$

In this formula, $\tilde{x}(.)$ and Γ_y are standard parameters while r, μ range over the positive real numbers. By (3.1), this is equivalent to

$$\forall\mu\ \exists^{\mathrm{fin}}r'\ \forall\tau\ (\forall r\in r'\ \tau>r\Rightarrow\mathrm{dis}(\tilde{x}(\tau),\Gamma_y)<\mu).$$

But to say, for r' a finite set, $\forall r\in r'\ \tau>r$ is the same as to say $\tau>r$ for $r=\max r'$ and the formula is equivalent to

$$\forall\mu\ \exists r\ \forall\tau\ (\tau>r\Rightarrow\mathrm{dis}(\tilde{x}(\tau),\Gamma_y)<\mu).$$

Hence, for all standard α in $\mathcal{V}$, any solution $\tilde{x}(\tau)$ of the equation (2.2) for which $\tilde{x}(0)=\alpha$, can be continued for all $\tau\ge 0$ and satisfies $\lim_{\tau\to\infty}\mathrm{dis}(\tilde{x}(\tau),\Gamma_y)=0$. By transfer, this property remains true for all α in $\mathcal{V}$. This is the usual definition of the orbital attractivity. $\square$

The following lemma is needed to reformulate the Assumption (B) and its proof is postponed to subsection 4.1.

Lemma 3.3. *Assume that hypothesis (A) is satisfied and that f_0 and $x^*(\tau,y)$ are standard. Then the periodic solution $x^*(\tau,y)$ of (2.2) is orbitally asymptotically stable if and only if there exists a standard $a>0$ such that any solution $\tilde{x}(\tau)$ of (2.2) for which $\mathrm{dis}(\tilde{x}(0),\ \Gamma_y)<a$ can be continued for all $\tau\ge 0$ and satisfies $\mathrm{dis}(\tilde{x}(\tau),\Gamma_y)\simeq 0$ for all $\tau\simeq+\infty$.*

Finally, according to Lemma 3.3 and assuming that f_0 is standard, the Assumption (B) is equivalent to:

(B') There exists a standard family of solutions $x^*(\tau,y)$ depending continuously on $y\in G$, where G is a standard compact subset of $\mathbb{R}^m$ with a non empty interior, such that $x^*(\tau,y)$ is a periodic solution of the fast equation (2.2) of period $T(y)>0$, the mapping $y\to T(y)$ is continuous, and there exists a standard $a>0$ such that, for all standard y, any solution $\tilde{x}(\tau)$ of (2.2) for which $\mathrm{dis}(\tilde{x}(0),\ \Gamma_y)<a$ can be continued for all $\tau\ge 0$ and satisfies $\mathrm{dis}(\tilde{x}(\tau),\ \Gamma_y)\simeq 0$ for all $\tau\simeq+\infty$.

3.3. **External results.** In classical mathematics we do not have to our disposal a notion of *perturbation* despite of the fact that we have a well established *Perturbation Theory*. In the language of Nonstandard Analysis (NSA), we have a notion of perturbation. Indeed, while this notion is classically described via deformations or neighborhoods, a perturbation of a standard object in NSA is just another object, which is nonstandard and infinitely close to it. Its properties are then investigated directly without using extra-properties with respect to the parameters of deformation.

Definition 3.4. An element $(\Omega, f, g, \alpha, \beta)$ of $\mathcal{T}$ is said to be a perturbation of the standard element $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ of $\mathcal{T}$ if Ω contains all the nearstandard elements in Ω_0, $f(x,y) \simeq f_0(x,y)$ and $g(x,y) \simeq g_0(x,y)$ for all (x,y) nearstandard in Ω_0 and $\alpha \simeq \alpha_0$, $\beta \simeq \beta_0$.

Let us state now the nonstandard version of Theorem 2.2 and Theorem 2.3 where the notation $x(t) \simeq \Gamma_{y(t)}$ means $\operatorname{dis}(x(t), \Gamma_{y(t)}) \simeq 0$.

Theorem 3.5. *Let $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ be a standard element of $\mathcal{T}$. Assume that $(A) - (D)$ are satisfied. Let $\tilde{x}(\tau)$ and $\bar{y}(t)$ be the respective solutions of (2.4) and (2.5) and L be standard in I, where I is the positive interval of definition of $\bar{y}(t)$. Let $\varepsilon > 0$ be infinitesimal and $(\Omega, f, g, \alpha, \beta) \in \mathcal{T}$ be a perturbation of $(\Omega_0, f_0, g_0, \alpha_0, \beta_0) \in \mathcal{T}$. Then any solution $(x(t), y(t))$ of (2.1) is defined at least on $[0,L]$ and there exists $\omega > 0$ such that $\varepsilon\omega \simeq 0$, $x(\varepsilon\tau) \simeq \tilde{x}(\tau)$ for $0 \le \tau \le \omega$, $y(t) \simeq \bar{y}(t)$ for $0 \le t \le L$ and $x(t) \simeq \Gamma_{\bar{y}(t)}$ for $\varepsilon\omega \le t \le L$.*

Theorem 3.6. *Let $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ be a standard element of $\mathcal{T}$. Let $\bar{y}_\infty$ be standard in G_0. Assume that $(A) - (E)$ are satisfied. Let $\tilde{x}(\tau)$ and $\bar{y}(t)$ be the respective solutions of (2.4) and (2.5). Let $\varepsilon > 0$ be infinitesimal and $(\Omega, f, g, \alpha, \beta) \in \mathcal{T}$ be a perturbation of $(\Omega_0, f_0, g_0, \alpha_0, \beta_0) \in \mathcal{T}$. Then any solution $(x(t), y(t))$ of (2.1) is defined for all $t \ge 0$ and there exists $\omega > 0$ such that $\varepsilon\omega \simeq 0$, $x(\varepsilon\tau) \simeq \tilde{x}(\tau)$ for $0 \le \tau \le \omega$, $y(t) \simeq \bar{y}(t)$ for $t \ge 0$ and $x(t) \simeq \Gamma_{\bar{y}(t)}$ for $t \ge \varepsilon\omega$.*

We propose to show that Theorem 3.5, which is external, reduces by Nelson's algorithm to its internal equivalent Theorem 2.2 while we let to the reader the reduction of Theorem 3.6 to Theorem 2.3. We need the following lemma :

Lemma 3.7. *The element $(\Omega, f, g, \alpha, \beta)$ of $\mathcal{T}$ is a perturbation of the standard element $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ of $\mathcal{T}$ if and only if $(\Omega, f, g, \alpha, \beta)$ is infinitely close to $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ for the topology of uniform convergence on compacta, that is, $(\Omega, f, g, \alpha, \beta)$ is in any standard neighborhood of $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$.*

The proof of this lemma can be found in [6, Lemma 2, page 11.].

Reduction of Theorem 3.5. We design by F the formula: "Any solution $(x(t), y(t))$ of (2.1) is defined at least on $[0,L]$ and there exists $\omega > 0$ such that $\varepsilon\omega < \eta$, $\|x(\varepsilon\tau) - \tilde{x}(\tau)\| < \eta$ for $0 \le \tau \le \omega$, $\|y(t) - \bar{y}(t)\| < \eta$ for $0 \le t \le L$ and $\operatorname{dis}(x(t), \Gamma_{\bar{y}(t)}) < \eta$ for $\varepsilon\omega \le t \le L$" and respectively by u_0 and u the variables $(\Omega_0, f_0, g_0, \alpha_0, \beta_0)$ and $(\Omega, f, g, \alpha, \beta)$ of $\mathcal{T}$. We also design by F' the formula "any solution $(x(t), y(t))$ of (2.1) is defined at least on $[0, L]$ and there exists $\omega > 0$ such that $\varepsilon\omega \simeq 0$, $x(\varepsilon\tau) \simeq \tilde{x}(\tau)$ for $0 \le \tau \le \omega$, $y(t) \simeq \bar{y}(t)$ for $0 \le t \le L$ and $x(t) \simeq \Gamma_{\bar{y}(t)}$ for $\varepsilon\omega \le t \le L$" On the other hand, to say that "ε is infinitesimal" is the same as to say that "$\forall^{\mathrm{st}}\, \varepsilon^*$, $\varepsilon < \varepsilon^*$", to say that "$u$ is a perturbation of u_0" is the same as to say that "u is in any standard neighborhood $\mathcal{V}$ of u_0". Finally, the formula F' is equivalent to the formula $\forall^{\rm st}\eta\ F$. Then, Theorem 3.5 can be formalized by

$$\forall\varepsilon\ \forall u\ (\forall^{\rm st}\varepsilon^*\ \forall^{\rm st}\mathcal{V}\ K \Rightarrow \forall^{\rm st}\eta\ F), \tag{3.2}$$

where K designates the formula $\varepsilon < \varepsilon^*$ & $u \in \mathcal{V}$. Here, u_0 and L are standard parameters, u ranges over $\mathcal{T}$, while ε and ε^* range over the positive real numbers and $\mathcal{V}$ ranges over the neighborhoods of u_0. Using the reduction formula (3.1), (3.2) is equivalent to

$$\forall\eta\ \exists^{\rm fin}\varepsilon^{*\prime}\ \exists^{\rm fin}\mathcal{V}'\ \forall\varepsilon\ \forall u\ (\forall\varepsilon^* \in \varepsilon^{*\prime}\ \forall\mathcal{V} \in \mathcal{V}'\ K \Rightarrow F).$$

But, $\varepsilon^{*\prime}$ and $\mathcal{V}'$ being finite sets, there exists ε^* and $\mathcal{V}$ such that $\varepsilon^* = \min\varepsilon^{*\prime}$ and $\mathcal{V} = \bigcap_{V\in\mathcal{V}'} V$ and the last formula becomes equivalent to

$$\forall\eta\ \exists\varepsilon^*\ \exists\mathcal{V}\ \forall\varepsilon\ \forall u\ (K \Rightarrow F).$$

Hence, the statement of Theorem 2.2 holds for any standard u_0 and $L \in I$. By transfer, it holds for any u_0 and $L \in I$. $\square$

4. Proofs of Theorems 3.5 and 3.6

4.1. **Fundamental lemmas.** We present in this subsection two fundamental lemmas of the *nonstandard perturbation theory of differential equations*. The *stroboscopic method* was proposed by J. L. Callot and G. Reeb and improved by R. Lutz and T. Sari (see [2], [7], [13], [14]).

Let $\mathcal{O}$ be a standard open subset of $\mathbb{R}^n$, $F : \mathcal{O} \to \mathbb{R}^n$ a standard continuous function. Let J be an interval of $\mathbb{R}$ containing 0 and $\phi : J \to \mathbb{R}^n$ a function such that $\phi(0)$ is nearstandard in $\mathcal{O}$, that is, there exists a standard $x_0 \in \mathcal{O}$ such that $\phi(0) \simeq x_0$. Let I be a connected subset of J, eventually external, such that $0 \in I$.

Definition 4.1 (Stroboscopic property)**.** *Let t and t' be in I. The function ϕ is said to satisfy the stroboscopic property $\mathcal{S}(t, t')$ if $t' \simeq t$, and $\phi(s) \simeq \phi(t)$ for all s in $[t, t']$ and*

$$\frac{\phi(t) - \phi(t')}{t - t'} \simeq F(\phi(t)).$$

Under suitable conditions, the Stroboscopy Lemma asserts that the function ϕ is approximated by the solution of the initial value problem

$$\frac{dx}{dt} = F(x), \quad x(0) = x_0. \tag{4.1}$$

Theorem 4.2 (Stroboscopy Lemma)**.** *Suppose that*

- (i) *There exists $\mu > 0$ such that, whenever $t \in I$ is limited and $\phi(t)$ is nearstandard in $\mathcal{O}$, there is $t' \in I$ such that $t' - t \geq \mu$ and the function ϕ satisfies the stroboscopic property $\mathcal{S}(t, t')$.*
- (ii) *The initial value problem (4.1) has a unique solution $x(t)$.*

Then, for any standard L in the maximal positive interval of definition of $x(t)$, we have $[0, L] \subset I$ and $\phi(t) \simeq x(t)$ for all $t \in [0, L]$.

An other tool which is related to the theory of regular perturbations is needed. Let us define the two initial value problems

$$\frac{dx}{dt} = F_0(x),\ x(0) = a_0 \in \mathcal{O}, \tag{4.2}$$

$$\frac{dx}{dt} = F(x),\ x(0) = a \in \mathcal{O}. \tag{4.3}$$

The so-called *Short Shadow Lemma* answers to the problem of comparing the solutions of (4.2) and (4.3) when F is close to F_0 and the initial condition a is close to a_0 (see [14]).

Theorem 4.3 (Short Shadow Lemma). *Let $\mathcal{O}$ be a standard open subset of $\mathbb{R}^n$ and let $F_0 : \mathcal{O} \to \mathbb{R}^n$ be standard and continuous. Let $a_0 \in \mathcal{O}$ be standard. Assume that the initial value problem (4.2) has a unique solution $x_0(t)$ and let $J = [0, \omega)$, $0 < \omega \le +\infty$, be its maximal positive interval of definition. Let $F : \mathcal{O} \to \mathbb{R}^n$ be continuous such that $F(x) \simeq F_0(x)$ for all x nearstandard in $\mathcal{O}$. Then, every solution $x(t)$ of the initial value problem (4.3) with $a \simeq a_0$, is defined for all t nearstandard in J and satisfies $x(t) \simeq x_0(t)$.*

With the help of the last theorem, we give now the proof of Lemma 3.3.

Proof of Lemma 3.3. Assume that the periodic solution $x^*(\tau, y)$ is orbitally asymptotically stable. By attractivity, its orbit Γ_y has a standard basin of attraction $\mathcal{V}$. Let $a > 0$ be standard such that the closure of the set $\mathcal{A} = \{x \in \mathbb{R}^n : \operatorname{dis}(x, \Gamma_y) < a\}$ is included in $\mathcal{V}$. Let $\alpha \in \mathcal{A}$ and $\tilde{x}(\tau)$ be the solution of (2.2) such that $\tilde{x}(0) = \alpha$. Let α_0 be standard in $\mathcal{V}$ such that $\alpha \simeq \alpha_0$. By attractivity of Γ_y, the solution $\tilde{x}_0(\tau)$ of (2.2) starting by α_0 is defined for all $\tau \ge 0$ and satisfies $\operatorname{dis}(\tilde{x}_0(\tau), \Gamma_y) \simeq 0$ for all $\tau \simeq +\infty$. By the Short Shadow Lemma, $\tilde{x}(\tau) \simeq \tilde{x}_0(\tau)$ for all limited $\tau > 0$. By Robinson's Lemma, there exists $\upsilon \simeq +\infty$ such that $\tilde{x}(\tau) \simeq \tilde{x}_0(\tau)$ for all τ in $[0, \upsilon]$. Thus, $\operatorname{dis}(\tilde{x}(\tau), \Gamma_y) \simeq 0$ for all unlimited $\tau \le \upsilon$. By stability of the closed orbit, this approximation still holds for all $\tau > \upsilon$. Hence, $\operatorname{dis}(\tilde{x}(\tau), \Gamma_y) \simeq 0$ for all $\tau \ge 0$. Conversely, if the orbit Γ_y is assumed to satisfy the property in the lemma, then by definition, the standard set $\mathcal{A}$ is in the basin of attraction of Γ_y. Hence, the considered closed orbit is attractive. Let $\tilde{x}(\tau)$ be a solution of (2.2) such that $\tilde{x}(0) = \alpha$, where α is infinitely close to a standard $\alpha_0 \in \Gamma_y$. Since $\alpha \in \mathcal{A}$, by hypothesis, $\tilde{x}(\tau)$ can be continued for all $\tau \ge 0$ and satisfies $\operatorname{dis}(\tilde{x}(\tau), \Gamma_y) \simeq 0$ for all $\tau \simeq +\infty$. On the other hand, if $\tilde{x}_0(\tau)$ is the maximal solution of (2.2) such that $\tilde{x}_0(0) = \alpha_0$, its trajectory is the closed orbit Γ_y. Hence, by the Short Shadow Lemma, $\operatorname{dis}(\tilde{x}(\tau), \Gamma_y) \simeq 0$ for all limited $\tau \ge 0$. Finally, Γ_y is stable. □

4.2. **Preparatory lemmas.** Let $\mathcal{C} = \bigcup_{y \in G} (\Gamma_y \times \{y\})$ and consider the system

$$\begin{aligned} \varepsilon\,\dot{x} &= f(x, y),\\ \dot{y} &= g(x, y). \end{aligned} \tag{4.4}$$

The following lemma asserts that a trajectory of (4.4) which comes infinitely close to $\mathcal{C}$ remains close to it as long as y is not infinitely close to the boundary of G.

Lemma 4.4. *Let Assumptions (A) and (B′) be satisfied. Let $(x(t), y(t))$ be a solution of (4.4) such that $y(t)$ is nearstandard in G_0 for $t \in [t_0, t_1]$ and $x(t_0) \simeq \Gamma_{y(t_0)}$, then $x(t) \simeq \Gamma_{y(t)}$ for all t in $[t_0, t_1]$.*

Proof. Let y_0 be standard in G_0 and let x_0 be standard in Γ_{y_0} such that $x(t_0) \simeq x_0$ and $y(t_0) \simeq y_0$. As a function of τ, $(x(t_0+\varepsilon\tau), y(t_0+\varepsilon\tau))$ is the solution of system

$$\begin{aligned} x' &= f(x,y), \\ y' &= \varepsilon g(x,y), \end{aligned} \tag{4.5}$$

with initial condition $(x(t_0), y(t_0))$. This IVP is a regular perturbation of system

$$\begin{aligned} x' &= f_0(x,y), \\ y' &= 0, \end{aligned} \tag{4.6}$$

with initial condition (x_0,y_0). According to the Short Shadow Lemma, we obtain

$$x(t_0+\varepsilon\tau) \simeq \Gamma_{y_0},\ y(t_0+\varepsilon\tau) \simeq y_0 \text{ for limited } \tau \geq 0. \tag{4.7}$$

Assume that there exists $s \in]t_0, t_1]$ such that $\mathrm{dis}(x(s),\Gamma_{y(s)}) = \gamma_0$ is not infinitesimal. Since the asymptotic stability of the cycles Γ_y is uniform over G, there exists $a > 0$ satisfying the property stated in Assumption (B'). Let $\gamma < \gamma_0$ be such that $0 < \gamma \leq a$ and $\gamma \not\simeq 0$ and let chose $s \in]t_0, t_1]$ such that $\mathrm{dis}(x(s),\Gamma_{y(s)}) = \gamma$. Since $\mathrm{dis}(x(t_0),\Gamma_{y(t_0)}) \simeq 0$ and $y(t)$ is nearstandard in G_0 for all $t \in [t_0, s]$, there exists a smallest $m \in]t_0, t_1]$ such that $\mathrm{dis}(x(m), \Gamma_{y(m)}) = \gamma$ and a standard (x_1,y_1) such that $y_1 \in G_0$ and $(x_1,y_1) \simeq (x(m), y(m))$. If $\tau_0 = (m - t_0)/\varepsilon$ was limited, by (4.7) one will have $x(m) \simeq \Gamma_{y_0}$ and $y(m) \simeq y_0$, thus $x(m) \simeq \Gamma_{y(m)}$. This contradicts $\mathrm{dis}(x(m), \Gamma_{y(m)}) = \gamma$. The value τ_0 is then unlimited. Let us consider the solution $(x(m + \varepsilon\tau), y(m + \varepsilon\tau))$ of (4.5) with initial condition $(x(m), y(m))$. This problem is a regular perturbation of (4.6) with initial condition (x_1,y_1), of maximal solution $(\tilde{x}(\tau), y_1)$. According to the Short Shadow Lemma, $x(m + \varepsilon\tau) \simeq \tilde{x}(\tau)$ and $y(m+\varepsilon\tau) \simeq y_1$ for all limited $\tau \leq 0$. By Robinson's Lemma, there exists $\tau_1 < 0$ unlimited, which can be chosen such that $-\tau_0 < \tau_1$, still satisfying $x(m+\varepsilon\tau_1) \simeq \tilde{x}(\tau_1)$. By noting that $\mathrm{dis}(x(m + \varepsilon\tau), \Gamma_{y(m+\varepsilon\tau)}) < \gamma$ for all $\tau \in [-\tau_0, 0[$, we will have in particular $\mathrm{dis}(\tilde{x}(\tau_1), \Gamma_{y_1}) < \gamma \leq a$. According to Assumption (B'), $\tilde{x}(\tau_1 + \tau)$ is defined for all $\tau \geq 0$ and satisfies $\tilde{x}(\tau_1 + \tau) \simeq \Gamma_{y_1}$ for all τ positive and unlimited. In particular, for $\tau = -\tau_1$, $\tilde{x}(0) \simeq \Gamma_{y_1}$ i.e. $x(m) \simeq \Gamma_{y_1} \simeq \Gamma_{y(m)}$. This contradicts the fact that $\mathrm{dis}(x(m), \Gamma_{y(m)}) = \gamma$. □

The following lemma asserts that the y-component of a trajectory of (4.4) which is infinitely close to $\mathcal{C}$ is approximated by a solution of the slow equation (2.3).

Lemma 4.5. *Let Assumptions (A), (B') and (C) be satisfied. Let $(x(t), y(t))$ be a solution of (4.4) such that $y(t_0)$ is nearstandard in G_0. Let y_0 be standard in G_0 such that $y(t_0) \simeq y_0$. Let $\bar{y}(t)$ be the solution of (2.3) with initial condition y_0 and defined on the standard interval $[0, L]$. Let $t_1 \geq t_0$ such that $t_1 \leq t_0 + L$ and $x(t) \simeq \Gamma_{y(t)}$ for $t \in [t_0,t_1]$. Then $y(t_0 + s) \simeq \bar{y}(s)$ for all $0 \leq s \leq L$ such that $t_0 + s \leq t_1$.*

Proof. Let $(x(t), y(t))$ be a solution of (4.4) such that $y(t_0)$ is nearstandard in G_0. Let us consider the external set

$$I = \{t \geq t_0 : (x(s), y(s)) \text{ is defined and } x(s) \simeq \Gamma_{y(s)} \text{ for all } s \in [t_0, t]\}$$

which contains, by hypothesis, the interval $[t_0,t_1]$. Let us show that $y(t)$ satisfies the hypothesis (i) of the Stroboscopy Lemma (Theorem 4.2). Let $\mu = \varepsilon \min_{y\in G} T(y)$.

Since T is continuous and G is compact, $\mu > 0$. Let t_λ limited be in I such that $y(t_\lambda)$ is nearstandard in G_0. The change of variables

$$\tau = \frac{t - t_\lambda}{\varepsilon}, \quad X(\tau) = x(t_\lambda + \varepsilon\tau), \quad Y(\tau) = \frac{y(t_\lambda + \varepsilon\tau) - y(t_\lambda)}{\varepsilon}, \tag{4.8}$$

transforms the problem (4.4) with initial condition $(x(t_\lambda), y(t_\lambda))$ into

$$X' = f(X, y(t_\lambda) + \varepsilon Y), \quad X(0) = x(t_\lambda), \tag{4.9}$$

$$Y' = g(X, y(t_\lambda) + \varepsilon Y), \quad Y(0) = 0. \tag{4.10}$$

For τ and Y limited, this problem is a regular perturbation of

$$\begin{gathered} X' = f_0(X, y_\lambda), \quad X(0) = x_\lambda, \\ Y' = g_0(X, y_\lambda), \quad Y(0) = 0, \end{gathered} \tag{4.11}$$

where x_λ and y_λ are standard and such that $x_\lambda \simeq x(t_\lambda)$, $y_\lambda \simeq y(t_\lambda)$. The Short Shadow Lemma gives

$$X(\tau) \simeq X_0(\tau), \quad Y(\tau) \simeq Y_0(\tau), \tag{4.12}$$

for all limited τ, where $(X_0(\tau), Y_0(\tau))$ is the solution of (4.11). Knowing that $x(t_\lambda) \simeq \Gamma_{y(t_\lambda)} \simeq \Gamma_{y_\lambda}$ and that x_λ and Γ_{y_λ} are standard, we obtain that $x_\lambda \in \Gamma_{y_\lambda}$. The first equation of (4.11) is nothing else than the fast equation (2.2) with initial condition x_λ and parameter $y = y_\lambda$. There exists $\tau_\lambda \in [0, T(y_\lambda)]$ such that $x^*(\tau_\lambda, y_\lambda) = x_\lambda$. Hence,

$$X_0(\tau) = x^*(\tau + \tau_\lambda, y_\lambda), \quad Y_0(\tau) = \int_0^\tau g_0(x^*(s + \tau_\lambda, y_\lambda), y_\lambda) ds. \tag{4.13}$$

Using (4.12) and (4.13) and the periodicity of x^*, we obtain

$$Y(T(y_\lambda)) \simeq \int_0^{T(y_\lambda)} g_0(x^*(s, y_\lambda), y_\lambda) ds. \tag{4.14}$$

Let us consider now the instant $t_\nu := t_\lambda + \varepsilon T(y_\lambda)$. We claim that $t_\nu \in I$, that is, $x(s) \simeq \Gamma_{y(s)}$ for all s in $[t_0, t_\nu]$. Since t_λ is in I, this property holds for all s in $[t_0, t_\lambda]$. Its remains to show that its holds also for all s in $[t_\lambda, t_\nu]$. Indeed, let $s = t_\lambda + \varepsilon\tau$. We have $y(s) = y(t_\lambda) + \varepsilon Y(\tau) \simeq y(t_\lambda)$, for all τ in $[0, T(y_\lambda)]$. Since $y(t_\lambda)$ is nearstandard in G_0 and $x(t_\lambda) \simeq \Gamma_{y(t_\lambda)}$, by Lemma 4.4 we have $x(s) \simeq \Gamma_{y(s)}$ for all s in $[t_\lambda, t_\nu]$.

We have proved that, for t_λ limited in I and $y(t_\lambda)$ nearstandard in G_0, there exists t_ν such that $0 \simeq t_\nu - t_\lambda \geq \mu$, $[t_\lambda, t_\nu] \subset I$, $y(s) \simeq y(t_\lambda)$ for all s in $[t_\lambda, t_\nu]$. By (4.14), we have

$$\frac{y(t_\nu) - y(t_\lambda)}{t_\nu - t_\lambda} = \frac{Y(T(y_\lambda))}{T(y_\lambda)} \simeq \bar{g}_0(y_\lambda) \simeq \bar{g}_0(y(t_\lambda)).$$

By the Stroboscopy Lemma, $[t_0, t_0 + L] \subset I$ and $y(t_0 + s) \simeq \bar{y}(s)$ for all $0 \leq s \leq L$. Therefore, this approximation holds for all s such that $t_0 + s \leq t_1$. □

4.3. **Proof of Theorem 3.5.** Let $(x(t), y(t))$ be a solution of the system (2.1). Then $(x(\varepsilon\tau), y(\varepsilon\tau))$ is a solution of (4.5) with initial condition (α, β). This problem is a regular perturbation of (4.6) with initial conditions $(\alpha_0,\ \beta_0)$. By the Short Shadow Lemma, $x(\varepsilon\tau) \simeq \tilde{x}(\tau)$ and $y(\varepsilon\tau) \simeq \beta_0$ for all limited $\tau \geq 0$. By Robinson's Lemma, there exists ω positive unlimited such that those approximations still hold for all $\tau \in [0,\ \omega]$. One can chose ω such that $\varepsilon\omega \simeq 0$. On the other hand, by

Assumptions (B') and (D), $\tilde{x}(\tau)$ is defined for all $\tau \geq 0$ and satisfies $\tilde{x}(\tau) \simeq \Gamma_{\beta_0}$ for all τ positive and unlimited. This last property is true in particular for $\tau = \omega$, which means that after a time $t_0 := \varepsilon\omega$ the solution of (2.1) is infinitely close to $\Gamma_{\beta_0} \subset \mathcal{C}$. Assume that there exists $s \in]t_0, L]$ such that $y(s) \not\simeq \bar{y}(s)$. Let $r > 0$ be standard such that $\|y(s) - \bar{y}(s)\| \geq r$. Since $\bar{y}(t)$ is nearstandard in G_0, we can chose r small enough such that the tubular neighborhood

$$\mathcal{B} = \{(t, y) : t \in [0, L], y \in G_0 \text{ and } \|\bar{y}(t) - y\| < r\}$$

satisfies the property that y is nearstandard in G_0 for every $(t, y) \in \mathcal{B}$. Let $t_1 < L$ be the smallest value of t for which $y(t_1)$ is on the boundary of $\mathcal{B}$. Lemma 4.4 ensures that the solution stays infinitely close to $\mathcal{C}$ for $t \in [t_0, t_1]$. Lemma 4.5 permits to assert that $y(t) \simeq \bar{y}(t)$ for $t \leq t_1$. In particular, $y(t_1) \simeq \bar{y}(t_1)$ which contradicts $\|y(t_1) - \bar{y}(t_1)\| = r$. Therefore, $y(t)$ is defined at least on $[0,\, L]$ and satisfies $y(t) \simeq \bar{y}(t)$. Hence $x(t) \simeq \Gamma_{y(t)} \simeq \Gamma_{\bar{y}(t)}$ for all t in $[\varepsilon\omega, L]$.

Remark 4.6. It is useful to add that a trajectory which is infinitely close to $\mathcal{C}$ at a time $\bar{t}$ stays near the cycle $\Gamma_{\bar{y}(t)}$ while performing rapid oscillations along it of period approximately $\varepsilon T(\bar{y}(t))$. More exactly, for all $\bar{t} \in [\varepsilon\omega, L]$, there exists $\delta(\bar{t}) \in [0,\, T(\bar{y}(\bar{t}))]$ such that for τ limited,

$$x(t + \varepsilon\tau) \simeq x^*\left(\tau + \delta(\bar{t}), \bar{y}(\bar{t})\right).$$

Indeed, let $\bar{t} \in [\varepsilon\omega, L]$. By what precedes, $x(\bar{t}) \simeq \Gamma_{y(\bar{t})}$. Let x° be standard such that $x^\circ \simeq x(\bar{t})$ and $x^\circ \in \Gamma_{y(\bar{t})}$. Then there exists $\delta(\bar{t}) \in [0, T(y(\bar{t}))]$ such that $x^*(\delta(\bar{t}), y(\bar{t})) = x^\circ$. By setting $\tau = (t - \bar{t})/\varepsilon$ in (4.4), the Short Shadow Lemma gives the approximation $x(\bar{t} + \varepsilon\tau) \simeq x^*(\tau + \delta(\bar{t}), y(\bar{t}))$ for all limited t. Finally, the assertion is proved knowing that $y(\bar{t}) \simeq \bar{y}(\bar{t})$.

4.4. **Proof of Theorem 3.6.** According to Theorem 3.5 and Assumption (E), one has

$$\begin{aligned} y(t) &\simeq \bar{y}(t) \quad \text{for all } t \in [0,\, L], \\ x(t) &\simeq \Gamma_{\bar{y}(t)} \quad \text{for all } t \in [\varepsilon\omega, L], \end{aligned}$$

for all limited $L > 0$. By Robinson's Lemma, those approximations still hold for a certain $L \simeq +\infty$. Thus, using Assumption (E), $y(L) \simeq \bar{y}(L) \simeq \bar{y}_\infty$ and $x(L) \simeq \Gamma_{\bar{y}_\infty}$. Applying again Theorem 3.5 to the solution starting from $(x(L),\, y(L))$ gives

$$y(L + k) \simeq \bar{y}_\infty, \quad x(L + k) \simeq \Gamma_{\bar{y}_\infty} \text{ for all limited } k \geq 0. \tag{4.15}$$

Suppose that there exists $s \geq L$ such that $y(s)$ is not infinitely close to $\bar{y}_\infty$ and let us find a contradiction. Then we can suppose that $\|y(s) - \bar{y}_\infty\| = \mu$ standard. The value s is chosen such that the ball $\mathcal{B}$ of center $\bar{y}_\infty$ and radius μ is contained in the basin of attraction of $\bar{y}_\infty$. Let m be the smallest value of such numbers s (this m exists by compactness of $\partial\mathcal{B}$ and $\|y(m) - \bar{y}_\infty\| = \mu$). It can be seen from (4.15) that $k_0 := m - L$ is positive unlimited. The solution starting by $(x(m),\, y(m))$ satisfies $y(m + k) \in \mathcal{B}$ for all k in $[-k_0,\, 0]$. Let $\bar{y}(k)$ be the solution of the slow equation (2.3) with initial condition $\bar{y}(0) = y_0(m)$, where $y_0(m)$ is a standard verifying $y_0(m) \simeq y(m)$. Lemma 4.5 asserts that $y(m + k) \simeq \bar{y}(k)$ for all limited $k \leq 0$ provided $x(m + k) \simeq \Gamma_{y(m+k)}$, which can be established by contradiction as in the proof of Lemma 4.4. By Robinson's Lemma, there exists $k_1 < 0$ unlimited such that $y(m + k_1) \simeq \bar{y}(k_1)$ which may be chosen such that $-k_0 \leq k_1$. This means that $\bar{y}(k_1)$ is in $\mathcal{B}$, thus in the basin of attraction of $\bar{y}_\infty$. Assumption (E) then gives $\bar{y}(k_1+k) \simeq \bar{y}_\infty$ for all unlimited $k>0$. In particular, for $k=-k_1$, one has $\bar{y}(0) \simeq \bar{y}_\infty$. But $\bar{y}(0) = y_0(m)$ and $y_0(m) \simeq y(m)$, then $y(m) \simeq \bar{y}_\infty$, which is absurd.

5. Applications

5.1. **A system with delayed loss of stability.** The aim of this example is to illustrate both the result of Theorem 2.2 and the delayed loss of stability phenomenon pointed out in the introduction. Let us consider the three dimensional slow-fast system

$$\begin{aligned} \varepsilon\dot{x}_1 &= x_2 - yx_1(1-x_1^2-x_2^2)^3,\\ \varepsilon\dot{x}_2 &= -x_1 - yx_2(1-x_1^2-x_2^2)^3,\\ \dot{y} &= x_1^2. \end{aligned} \tag{5.1}$$

The fast equation is

$$\begin{aligned} x_1' &= x_2 - yx_1(1-x_1^2-x_2^2)^3,\\ x_2' &= -x_1 - yx_2(1-x_1^2-x_2^2)^3, \end{aligned} \tag{5.2}$$

where y is a parameter. In terms of the polar coordinates ($x_1 = r\cos\theta, x_2 = r\sin\theta$), the equation (5.2) is written as

$$\begin{aligned} r' &= -ry(1-r^2)^3,\\ \theta' &= -1. \end{aligned} \tag{5.3}$$

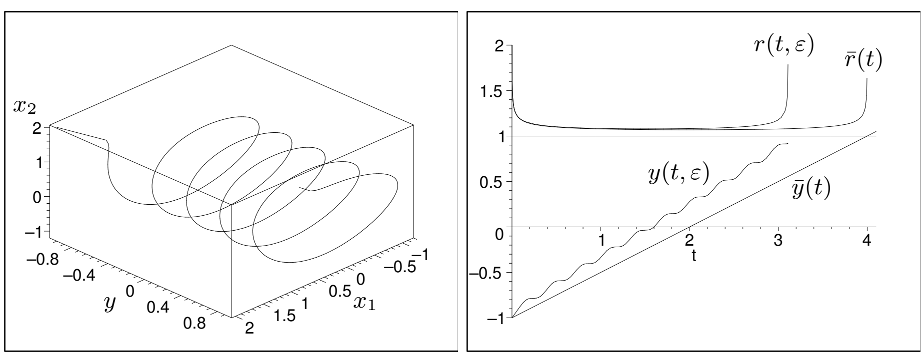

Figure 1. A solution of (5.1) with $\varepsilon = 0.1$, $x_1^0 = 2$, $x_2^0 = 2, y^0 = -1$ in the phase space (x_1, x_2, y). The functions $r(t,\varepsilon)$ and $y(t,\varepsilon)$ are approximated respectively by the functions $\bar{r}(t)$, and $\bar{y}(t)$ even after $t = 2$ where the cycles become unstable.

From (5.3) it is seen that the fast equation (5.2) admits a unique cycle Γ_y for all $y \neq 0$, namely the circle of center the origin and radius 1. This cycle corresponds for instance to the 2π-periodic solution $x^*(\tau, y) = (\cos\tau, -\sin\tau)$. The cycles are asymptotically stable for all $y<0$, while they are unstable for $y>0$. If $y=0$, the origin of (5.2) is a center. Notice that the cycles Γ_y are not exponentially stable, so that the result of Pontryagin and Rodygin does not apply. Note that the basin of attraction of Γ_y is the whole plan (x_1, x_2) for all $y<0$, except the origin. The asymptotic stability is therefore uniform over any interval G of $]-\infty, 0[$. We consider the IVP consisting of the system (5.1) together with the initial condition (x_1^0, x_2^0, y^0), such that $y^0 < 0$. The reduced problem is defined by

$$\dot{y} = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\tau d\tau = \frac{1}{2}, \quad y(0) = y^0.$$

Its solution is $\bar{y}(t) = y^0 + t/2$. According to Theorem 2.2, the solution of (5.1) satisfies $\lim_{\varepsilon\to 0} y(t,\varepsilon) = \bar{y}(t)$ as long as $0 \le t \le L < -2y^0$. By Remark 4.6, $(x_1(t,\varepsilon), x_2(t,\varepsilon))$ stays near the cycle $\Gamma_{\bar{y}(t)}$ while performing rapid oscillations along it of period approximately $2\pi\varepsilon$, that is, $r(t)$ is approximated by the solution of the averaged equation

$$\varepsilon\dot{r} = -r\bar{y}(t)(1-r^2)^3, \quad r(0) = \sqrt{(x_1^0)^2 + (x_2^0)^2}. \tag{5.4}$$

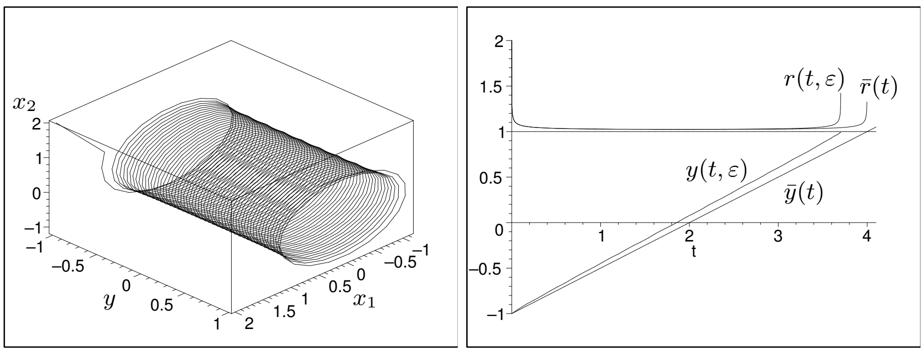

FIGURE 2. A solution of (5.1) with $\varepsilon = 0.01$, $x_1^0 = 2$, $x_2^0 = 2, y^0 = -1$ in the phase space (x_1, x_2, y). The functions $r(t,\varepsilon)$ and $y(t,\varepsilon)$ are approximated respectively by the functions $\bar{r}(t)$, and $\bar{y}(t)$ asymptotically until the exit-time $t = 4$ of the averaged system.

The solution of (5.4) is denoted by $\bar{r}(t)$. Its satisfies the property $\bar{r}(-4y^0 - t) = \bar{r}(t)$, hence $\bar{r}(-4y^0) = \bar{r}(0)$. Since $\bar{y}(-4y^0) = -y^0$, if a trajectory of the averaged system approaches the cycles of radius 1 for some value $y^0 < 0$, then it remains near the cycles as long as $y^0 < \bar{y}(t) < -y^0$. Notice that for $0 < \bar{y}(t) < -y^0$ the cycles are unstable : there is a delayed loss of stability for the averaged system and the *entrance-exit* function near the cycles is defined by $y^0 \mapsto -y^0$. According to Theorem 2.2, the solution of (5.1) is approximated by the averaged solution as long $0 \le t < -2y^0$, that is, as long as $y^0 \le \bar{y}(t) < 0$. The numerical simulations in Figures 1 and 2 show that the actual solution $(r(t,\varepsilon), y(t,\varepsilon))$ is approximated by the averaged solution $(\bar{r}(t), \bar{y}(t))$ even after time $t = -2y^0$ where the cycles become unstable. This approximation holds asymptotically until the *exit-time* $t = -4y_0$ of the averaged system. The rolling up of the trajectory $(x_1(t,\varepsilon),\, x_2(t,\varepsilon),\, y(t,\varepsilon))$ around the cycles Γ_y still holds for positive values of y, although the cycles became unstable. If we consider y as a dynamical bifurcation parameter, the delayed loss of stability phenomenon established by Neishtadt [8] turns to be still available. Recall that in this example the stability of the cycles is just asymptotic and not exponential, so that the result of Neishtadt does not apply. This problem deserves a special investigation.

5.2. **A model from population ecology.** Let us consider the following three trophic levels food chain model

$$
\begin{aligned}
\varepsilon \dot{x}_1 &= U(x_1) - x_2 V_1(x_1),\\
\varepsilon \dot{x}_2 &= \alpha_1 x_2 V_1(x_1) - D_2 x_2 - y V_2(x_2),\\
\dot{y} &= \alpha_2 y V_2(x_2) - D y,
\end{aligned}
\tag{5.5}
$$

where ε is a small positive parameter. The nonnegative variables x_1, x_2 and y are the respective densities of the prey, the predator and the superpredator. The function $U(x_1)$ is the *growth function* of the prey. The functions $V_1(x_1)$ and $V_2(x_2)$ are the *functional responses* of the predator and the superpredator respectively. The parameters D_2 and D are the respective death rates of the predator and the superpredator and α_1 and α_2 are conversion coefficients of the biomass respectively from the prey to the predator and from the predator to the superpredator. For more details on this kind of models and these biological constants, all positive, see for example [1, 10]. Note that the presence of the small parameter ε emphasizes the fact that the multiplications of the prey and the predator are of same order and much faster than the growth of the superpredator.

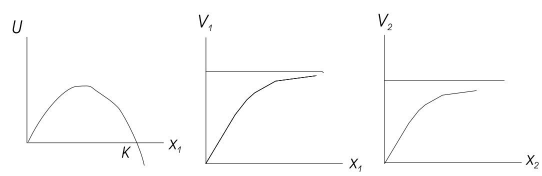

FIGURE 3. The growth function U and the functional responses V_1 and V_2 of the three trophic levels food chain model (5.5).

We assume that the functions U, V_1 and V_2 are continuous. Nonsmooth righthand side of differential equations (and even discontinuous righthand sides) are of interest in the biological literature (see for example [3]). We assume also that these functions satisfy the following properties (see Figure 3):

- $U(0) = V_1(0) = V_2(0) = 0$,
- there exists $K > 0$ such that $U(K) = 0$ and U is positive on $]0, K[$ and negative on $]K, +\infty[$,
- The functions V_1 and V_2 are strictly increasing and $\lim_{x_1 \to +\infty} V_1(x_1)$ and $\lim_{x_2 \to +\infty} V_2(x_2)$ are finite.

Such properties are satisfied by the model with *logistic growth* of the prey and *Holling type II* functional responses of the predator and the superpredator :

$$
U(x_1) = r x_1 (1 - x_1/K), \quad V_1(x_1) = \frac{a_1 x_1}{b_1 + x_1}, \quad V_2(x_2) = \frac{a_2 x_2}{b_2 + x_2}, \tag{5.6}
$$

where r, K, a_1, a_2, b_1 and b_2 are biological positive parameters.

When $z = 0$, the fast equation associated to (5.5-5.6) is the classical prey-predator system. For this model, under suitable conditions on the parameters, the uniqueness and the exponential stability of a limit cycle have been proved in [5]. For $z > 0$, numerical simulations [4] give strong evidence that the model still have a limit cycle for certain values of the parameters, but there is no theoretical results on the existence of a cycle, nor on its stability. More details are given in [16].

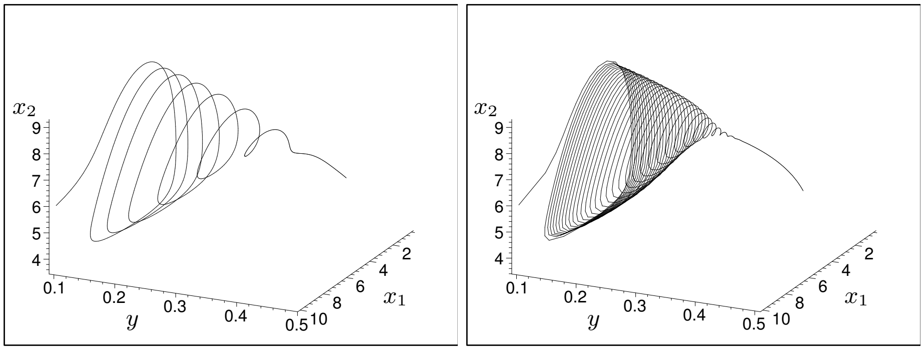

FIGURE 4. A numerical simulation of model (5.5-5.6) with the following values of the parameters : $r = K = 10$, $\alpha_1 = 0.4$, $a_1 = 5$, $b_1 = 2.5$, $D_2 = 1$, $\alpha_2 = 0.5$, $a_2 = 10$, $b_2 = 5$ and $D = 2$. The figure at left corresponds to $\varepsilon = 0.05$ and the figure at right corresponds to $\varepsilon = 0.01$. The initial condition is $x_1^0 = 10$, $x_2^0 = 6$, $y^0 = 0.1$.

We return to the general model (5.5) and we assume that the fast equation

$$\begin{aligned} x_1' &= U(x_1) - x_2 V_1(x_1), \\ x_2' &= \alpha_1 x_2 V_1(x_1) - D_2 x_2 - y V_2(x_2), \end{aligned} \tag{5.7}$$

satisfies Assumption (B). More precisely we assume that there exist α_1, α_2, D, D_2 and a compact interval G of $[0, +\infty[$ with a non empty interior such that, for all $y \in G$, the fast system (5.7) has a unique cycle Γ_y which is asymptotically stable with a uniform basin of attraction over G. Let $(x_1^*(\tau, y), x_2^*(\tau, y))$ be a $T(y)$-periodic solution of orbit Γ_y and define on G_0 the function

$$M(y) = \frac{1}{T(y)} \int_0^{T(y)} (\alpha_2 y V_2(x_2^*(\tau, y)) - Dy) d\tau.$$

According to Theorem 2.2, it follows that for every initial condition (x_1^0, x_2^0, y^0) such that $y^0 \in G_0$ and (x_1^0, x_2^0) is in the basin of attraction of Γ_{y^0}, the evolution of the superpredator in the model (5.5) is approximated by the solution $\bar{y}(t)$ of the reduced problem

$$\dot{y} = M(y), \quad y(0) = y^0.$$

More exactly, if $((x_1(t, \varepsilon), x_2(t, \varepsilon), y(t, \varepsilon))$ is the solution of the IVP, we have

$$\lim_{\varepsilon \to 0} y(t, \varepsilon) = \bar{y}(t) \text{ for all } 0 \le t \le T$$

$$\lim_{\varepsilon \to 0} \operatorname{dis}((x_1(t, \varepsilon), x_2(t, \varepsilon)), \Gamma_{\bar{y}(t)}) = 0 \text{ for all } 0 < t \le T,$$

where T is in the positive interval of definition of $\bar{y}(t)$. An illustration of the results is presented through numerical simulations of the model (5.5-5.6). Figure 4 shows how the high-frequency oscillations of the prey-predator subsystem are damped by increasing the density of the superpredator.

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