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ELLIPTIC REGULARITY AND SOLVABILITY FOR PARTIAL DIFFERENTIAL EQUATIONS WITH COLOMBEAU COEFFICIENTS

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Abstract. This paper addresses questions of existence and regularity of solutions to linear partial differential equations whose coefficients are generalized functions or generalized constants in the sense of Colombeau. We introduce various new notions of ellipticity and hypoellipticity, study their interrelation, and give a number of new examples and counterexamples. Using the concept of $\mathcal{G}^\infty$-regularity of generalized functions, we derive a general global regularity result in the case of operators with constant generalized coefficients, a more specialized result for second order operators, and a microlocal regularity result for certain first order operators with variable generalized coefficients. We also prove a global solvability result for operators with constant generalized coefficients and compactly supported Colombeau generalized functions as right hand sides.

1. Introduction

The purpose of this paper is to clarify a number of foundational issues in the existence and regularity theory for linear partial differential equations with coefficients belonging to the Colombeau algebra of generalized functions. Questions we address are: What is a good notion of ellipticity in the Colombeau setting? In terms of the microlocal point of view - what should be a non-characteristic direction? What sort of regularity results are to be expected? While regularity theory of Colombeau coefficients with classical, constant coefficients was settled in [16], there is current interest in pseudodifferential operators with Colombeau symbols [3] and microlocal analysis [9, 10].

Consider first a classical differential operator of order m on an open set $\Omega \subset \mathbb{R}^n$,

$$P(x, D) = \sum_{|\gamma| \le m} a_\gamma(x) D^\gamma \tag{1.1}$$

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where the coefficient functions a_γ belong to $\mathrm{C}^\infty(\Omega)$ and $D^\gamma = (-\mathrm{i}\,\partial)^\gamma$. Such an operator is called *elliptic*, if its symbol $P(x,\xi)$ satisfies an estimate of the form

$$\begin{gathered}\forall K \Subset \Omega\ \exists r>0\ \exists C>0 \text{ such that } \forall x\in K\ \forall \xi\in\mathbb{R}^n \text{ with } |\xi|\geq r:\\ |P(x,\xi)|\geq C(1+|\xi|)^m\,.\end{gathered} \tag{1.2}$$

In the classical setting, this entails the additional property (with the same r as in (1.2)):

$$\begin{gathered}\forall \alpha,\beta\in\mathbb{N}^n\ \exists C>0 \text{ such that } \forall x\in K\ \forall \xi\in\mathbb{R}^n \text{ with } |\xi|\geq r:\\ |\partial_\xi^\alpha\partial_x^\beta P(x,\xi)|\leq C|P(x,\xi)|(1+|\xi|)^{-|\alpha|}\end{gathered} \tag{1.3}$$

which is related to the notion of hypoellipticity [5, Section 4.1]. Let now $u\in\mathcal{G}(\Omega)$, the Colombeau algebra of generalized functions on Ω, be a generalized solution to the equation

$$P(x,D)\,u = f \tag{1.4}$$

where $f\in\mathcal{G}(\Omega)$. Contrary to the distributional setting, $f\in\mathrm{C}^\infty(\Omega)$ does not entail that $u\in\mathrm{C}^\infty(\Omega)$ for elliptic equations. This follows already from the mere fact that the ring $\widetilde{\mathbb{C}}$ of constants in $\mathcal{G}(\mathbb{R}^n)$ is strictly larger than $\mathbb{C}$. For this reason, a subalgebra $\mathcal{G}^\infty(\Omega)$ of $\mathcal{G}(\Omega)$ was introduced in [16] with the property that its intersection $\mathcal{G}^\infty(\Omega)\cap\mathcal{D}'(\Omega)$ with the space of distributions coincides with $\mathrm{C}^\infty(\Omega)$. It was shown that for constant coefficient hypoelliptic operators and solutions $u\in\mathcal{G}(\Omega)$ of (1.4), $f\in\mathcal{G}^\infty(\Omega)$ implies $u\in\mathcal{G}^\infty(\Omega)$. This regularity result easily carries over to elliptic operators with C^∞-coefficients.

In the Colombeau setting, generalized operators of the form (1.1) with coefficients $a_\gamma\in\mathcal{G}(\Omega)$ arise naturally when one regularizes operators with discontinuous coefficients or studies singular perturbations of operators with constant coefficients. We pose the question of regularity: Under what conditions does $f\in\mathcal{G}^\infty(\Omega)$ entail that the solution $u\in\mathcal{G}(\Omega)$ to (1.4) belongs to $\mathcal{G}^\infty(\Omega)$ as well? It is obvious from the case of multiplication operators that we must require that the coefficients in (1.1) belong to $\mathcal{G}^\infty(\Omega)$ themselves. The Colombeau generalized functions a_γ are defined as equivalence classes of nets $(a_{\gamma\varepsilon})_{\varepsilon\in(0,1]}$ of smooth functions. Inserting these in (1.1) produces a representative $P_\varepsilon(x,\xi)$ of the generalized symbol. The natural generalization of the ellipticity condition (1.2) seems to be

$$\begin{gathered}\forall K\Subset\Omega\ \exists N>0\ \exists\varepsilon_0>0\ \exists r>0\ \exists C>0 \text{ such that}\\ \forall\varepsilon\in(0,\varepsilon_0)\ \forall x\in K\ \forall\xi\in\mathbb{R}^n \text{ with } |\xi|\geq r:\\ |P_\varepsilon(x,\xi)|\geq C\varepsilon^N(1+|\xi|)^m\end{gathered} \tag{1.5}$$

and has been proposed, among others, by [14]. The starting point of this paper has been our observation that this condition does not produce the desired elliptic regularity result. In fact, we shall give examples of operators $P_\varepsilon(D)$ with constant, generalized coefficients satisfying (1.5) such that the homogeneous equation (1.4) with $f\equiv 0$ has solutions which do not belong to $\mathcal{G}^\infty(\Omega)$. We also exhibit a multiplication operator satisfying (1.5), given by an element of $\mathcal{G}^\infty(\mathbb{R})$, whose inverse does not belong to $\mathcal{G}^\infty(\mathbb{R})$ and is not even *equal in the sense of generalized distributions* (see [1]) to an element of $\mathcal{G}^\infty(\mathbb{R})$. The latter notion enters the picture due to results of [3] and [14]. Indeed, in [3] a strong version of (1.3) for generalized symbols $P_\varepsilon(x,\xi)$ is considered, where in addition to property (1.5) it is required (with the same ε_0 and r) that

$$\begin{gathered}\forall\alpha,\beta\in\mathbb{N}^n\ \exists C_{\alpha\beta}>0 \text{ such that}\\ \forall\varepsilon\in(0,\varepsilon_0)\ \forall x\in K\ \forall\xi\in\mathbb{R}^n \text{ with } |\xi|\geq r:\\ |\partial_\xi^\alpha\partial_x^\beta P_\varepsilon(x,\xi)|\leq C_{\alpha\beta}|P_\varepsilon(x,\xi)|(1+|\xi|)^{-|\alpha|}\,.\end{gathered}\tag{1.6}$$

As we show later, this condition does not follow from (1.5). It is proved in [3] that if a generalized operator $P_\varepsilon(x,D)$ satisfies (1.5) and (1.6) and $u\in\mathcal{G}_\tau(\mathbb{R}^n)$ is a solution to (1.4) with $f\in\mathcal{G}_\tau^\infty(\mathbb{R}^n)$, then u is equal in the sense of generalized temperate distributions to an element of $u\in\mathcal{G}_\tau^\infty(\mathbb{R}^n)$; here the notation $\mathcal{G}_\tau$ refers to the space of temperate Colombeau generalized functions [2]. The question immediately arises whether (1.5) and (1.6) together guarantee that u actually *belongs* to $\mathcal{G}^\infty(\Omega)$, if f does. We give a positive answer in the case of operators with constant generalized coefficients. Actually, a weakening of the two conditions suffices: r may depend on ε in a slowly varying fashion. In fact, various refined notions of ellipticity are needed and will be discussed. This carries over to the definition of a non-characteristic direction and microlocal elliptic regularity. Here we obtain a microlocal elliptic regularity result for first order operators with (non-constant) generalized coefficients. Contrary to the classical case, lower order terms do matter. The general case of higher order equations with non-constant generalized coefficients remains open and will be addressed in [11].

Having the tools for studying regularity properties at hand, we are able to answer the question of solvability of the inhomogeneous equation with compactly supported right hand side in the case of operators with constant generalized coefficients. We ask for conditions on the operator $P(D)$ such that

$$\forall f\in\mathcal{G}_c(\Omega):\exists u\in\mathcal{G}(\Omega):\ P(D)u=f\,.\tag{1.7}$$

It was shown in [14, Theorem 2.4] that the solvability property (1.7) holds for operators $P(D)$ such that

$$P_m(\xi_0) \text{ is invertible in } \widetilde{\mathbb{C}} \text{ for some } \xi_0\in\mathbb{R}^n$$

where $P_m(\xi)$ denotes the principal part. Examples show that this is not the most general condition. In fact, we will prove that solvability holds if and only if

$$\widetilde{P}^2(\xi_0) \text{ is invertible in } \widetilde{\mathbb{R}} \text{ for some } \xi_0\in\mathbb{R}^n$$

where $\widetilde{P}^2$ is the associated weight function. We provide an independent and short proof based on the fundamental solution given in [5, Theorem 3.1.1].

The plan of the paper is as follows. Section 2 serves to collect the notions from Colombeau theory needed in the sequel. In Section 3 we introduce various refinements of the ellipticity conditions (1.5) and (1.6) and study their interrelations. In Section 4 we provide a number of examples and counterexamples illustrating the situation. In Section 5 we prove the regularity result for higher order operators with constant generalized coefficients and give additional sufficient conditions for second order operators. Section 6 is devoted to microlocal ellipticity properties of first order generalized symbols and to the corresponding microlocal elliptic regularity result, giving bounds on the wave front set of the solution. Finally, Section 7 addresses the solvability question.

2. Colombeau algebras

The paper is placed in the framework of algebras of generalized functions introduced by Colombeau in [1, 2]. We shall fix the notation and discuss a number of known as well as new properties pertinent to Colombeau generalized functions here. As a general reference we recommend [4].

Let Ω be an open subset of $\mathbb{R}^n$. The basic objects of the theory as we use it are families $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of smooth functions $u_\varepsilon \in \mathrm{C}^\infty(\Omega)$ for $0 < \varepsilon \leq 1$. To simplify the notation, we shall write $(u_\varepsilon)_\varepsilon$ in place of $(u_\varepsilon)_{\varepsilon\in(0,1]}$ throughout. We single out the following subalgebras:

Moderate families, denoted by $\mathcal{E}_{\mathrm{M}}(\Omega)$, are defined by the property:

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}_0^n\, \exists p \geq 0 : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-p}) \text{ as } \varepsilon \to 0\,. \tag{2.1}$$

Null families, denoted by $\mathcal{N}(\Omega)$, are defined by the property:

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}_0^n\, \forall q \geq 0 : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0\,. \tag{2.2}$$

In words, moderate families satisfy a locally uniform polynomial estimate as $\varepsilon \to 0$, together with all derivatives, while null functionals vanish faster than any power of ε in the same situation. The null families form a differential ideal in the collection of moderate families. The *Colombeau algebra* is the factor algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_{\mathrm{M}}(\Omega)/\mathcal{N}(\Omega)\,.$$

The algebra $\mathcal{G}(\Omega)$ just defined coincides with the *special Colombeau algebra* in [4, Definition 1.2.2], where the notation $\mathcal{G}^s(\Omega)$ has been employed. However, as we will not use other variants of the algebra, we drop the superscript s in the sequel.

Restrictions of the elements of $\mathcal{G}(\Omega)$ to open subsets of Ω are defined on representatives in the obvious way. One can show (see [4, Theorem 1.2.4]) that $\Omega \to \mathcal{G}(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$. Thus the support of a generalized function $u \in \mathcal{G}(\Omega)$ is well defined as the complement of the largest open set on which u vanishes. The subalgebra of compactly supported Colombeau generalized functions will be denoted by $\mathcal{G}_{\mathrm{c}}(\Omega)$.

The space of compactly supported distributions is imbedded in $\mathcal{G}(\Omega)$ by convolution:

$$\iota : \mathcal{E}'(\Omega) \to \mathcal{G}(\Omega),\ \iota(w) = \text{ class of } (w * (\varphi_\varepsilon)|_\Omega)_\varepsilon\,,$$

where

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi\,(x/\varepsilon) \tag{2.3}$$

is obtained by scaling a fixed test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ of integral one with all moments vanishing. By the sheaf property, this can be extended in a unique way to an imbedding of the space of distributions $\mathcal{D}'(\Omega)$.

One of the main features of the Colombeau construction is the fact that this imbedding renders $\mathrm{C}^\infty(\Omega)$ a faithful subalgebra. In fact, given $f \in \mathrm{C}^\infty(\Omega)$, one can define a corresponding element of $\mathcal{G}(\Omega)$ by the constant imbedding $\sigma(f) =$ class of $[(\varepsilon, x) \to f(x)]$. Then the important equality $\iota(f) = \sigma(f)$ holds in $\mathcal{G}(\Omega)$. We summarize the basic properties of the Colombeau algebra in short, referring to the literature for details (e.g. [2, 4, 16, 17]):

(a) $\mathcal{G}(\Omega)$ is a commutative, associative differential algebra with derivations $\partial_1, \ldots, \partial_n$ and multiplication $\diamond$
(b) There is a linear imbedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$

(c) The restriction of each derivation ∂_j to $\mathcal{D}'(\Omega)$ coincides with the usual partial derivative
(d) The restriction of the multiplication $\diamond$ to $\mathrm{C}^\infty(\Omega)$ coincides with the usual product of smooth functions.

One of the achievements of the Colombeau construction is property (d): It is optimal in the sense that in whatever algebra satisfying properties (a) - (c), the multiplication map does not reproduce the product on $\mathrm{C}^k(\Omega)$ for finite k, by Schwartz' impossibility result [18].

We need a couple of further notions from the theory of Colombeau generalized functions. Regularity theory in this setting is based on the subalgebra $\mathcal{G}^\infty(\Omega)$ of *regular generalized functions* in $\mathcal{G}(\Omega)$. It is defined by those elements which have a representative satisfying

$$\forall K \Subset \Omega\, \exists p \geq 0\, \forall \alpha \in \mathbb{N}_0^n : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-p}) \quad \text{as } \varepsilon \to 0\,. \tag{2.4}$$

Observe the change of quantifiers with respect to formula (2.1); locally, all derivatives of a regular generalized function have the same order of growth in $\varepsilon > 0$. One has that (see [16, Theorem 5.2])

$$\mathcal{G}^\infty(\Omega) \cap \mathcal{D}'(\Omega) = \mathrm{C}^\infty(\Omega)\,.$$

For the purpose of describing the regularity of Colombeau generalized functions, $\mathcal{G}^\infty(\Omega)$ plays the same role as $\mathrm{C}^\infty(\Omega)$ does in the setting of distributions.

Let us also recall the *association relation* on the Colombeau algebra $\mathcal{G}(\Omega)$. It identifies elements of $\mathcal{G}(\Omega)$ if they coincide in the weak limit. That is, $u, v \in \mathcal{G}(\Omega)$ are called associated, $u \approx v$, if $\lim_{\varepsilon\to 0} \int \big(u_\varepsilon(x) - v_\varepsilon(x)\big)\psi(x)\, dx = 0$ for all test functions $\psi \in \mathcal{D}(\Omega)$. This may be interpreted as the reduction of the information on the family of regularizations to the usual distributional level.

Next, we need the notion of generalized point values. The ring of Colombeau generalized numbers $\widetilde{\mathbb{C}}$ can be defined as the Colombeau algebra $\mathcal{G}(\mathbb{R}^0)$, or alternatively as the ring of constants in $\mathcal{G}(\mathbb{R}^n)$. More generally, we define generalized points of open subsets Ω of $\mathbb{R}^n$ as follows: On

$$\Omega_M = \{(x_\varepsilon)_\varepsilon \in \Omega^{(0,1]} : \exists p \geq 0 \text{ such that } |x_\varepsilon| = O(\varepsilon^{-p}) \text{ as } \varepsilon \to 0\} \tag{2.5}$$

we introduce an equivalence relation by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \;\Leftrightarrow\; \forall q \geq 0,\ |x_\varepsilon - y_\varepsilon| = O(\varepsilon^q) \text{ as } \varepsilon \to 0$$

and denote by $\widetilde{\Omega} = \Omega_M/\sim$ the set of *generalized points of* Ω. The classes of the nets

$$(x_\varepsilon)_\varepsilon \in \widetilde{\Omega} : \exists K \Subset \Omega \text{ such that } x_\varepsilon \in K \text{ eventually as } \varepsilon \to 0$$

define the subset set of *compactly supported points* $\widetilde{\Omega}_\mathrm{c}$. With this notation, we clearly have that $\widetilde{\mathbb{C}} = \widetilde{\mathbb{R}} + i\widetilde{\mathbb{R}}$.

Given $u \in \mathcal{G}(\Omega)$ and $x \in \widetilde{\Omega}_\mathrm{c}$, the generalized point value $u(x) \in \widetilde{\mathbb{C}}$ is well defined as the class of $(u_\varepsilon(x_\varepsilon))_\varepsilon$. In addition, Colombeau generalized functions are characterized by their point values (see [4, Theorem 1.2.46]):

$$u = 0 \text{ in } \mathcal{G}(\Omega) \;\Leftrightarrow\; u(x) = 0 \text{ in } \widetilde{\mathbb{C}} \text{ for all } x \in \widetilde{\Omega}_\mathrm{c}\,.$$

The generalized numbers $\widetilde{\mathbb{R}}$ and $\widetilde{\mathbb{C}}$ form rings, but not fields. Further, a partial order is defined on $\widetilde{\mathbb{R}}$: $r \leq s$ if there are representatives $(r_\varepsilon)_\varepsilon$, $(s_\varepsilon)_\varepsilon$ with $r_\varepsilon \leq s_\varepsilon$ for all ε. An element $r \in \widetilde{\mathbb{R}}$ such that $0 \leq r$ with respect to this partial order is called *nonnegative*. Concerning invertibility in $\widetilde{\mathbb{R}}$ or $\widetilde{\mathbb{C}}$, we have the following results (see [4, Theorems 1.2.38 and 1.2.39]):

Let r be an element of $\widetilde{\mathbb{R}}$ or $\widetilde{\mathbb{C}}$. Then

> r is invertible if and only if there exists some representative $(r_\varepsilon)_\varepsilon$ and an $m \in \mathbb{N}$ with $|r_\varepsilon| \geq \varepsilon^m$ for sufficiently small $\varepsilon > 0$.

Further,

> r is not invertible if and only if there exists a representative $(\tilde{r}_\varepsilon)_\varepsilon$ of r and a sequence $\varepsilon_k \to 0$ such that $\tilde{r}_{\varepsilon_k} = 0$ for all $k \in \mathbb{N}$, if and only if r is a zero divisor.

Concerning invertibility of Colombeau generalized function, we may state: Let $u \in \mathcal{G}(\Omega)$. Then

> u possesses a multiplicative inverse if and only if there exists some representative $(u_\varepsilon)_\varepsilon$ such that for every compact set $K \subset \Omega$, there is $m \in \mathbb{N}$ with $\inf_{x\in K} |u_\varepsilon(x)| \geq \varepsilon^m$ for sufficiently small $\varepsilon > 0$, if and only if $u(x)$ is invertible in $\widetilde{\mathbb{C}}$ for every $x \in \widetilde{\Omega}_c$.

We briefly touch upon the subject of linear algebra in $\widetilde{\mathbb{R}}^n$. Let A be an $(n \times n)$-matrix with coefficients in $\widetilde{\mathbb{R}}$. It defines an $\widetilde{\mathbb{R}}$-linear map from $\widetilde{\mathbb{R}}^n$ to $\widetilde{\mathbb{R}}^n$. We have (see [4, Lemma 1.2.41]):

> $A : \widetilde{\mathbb{R}}^n \to \widetilde{\mathbb{R}}^n$ is bijective if and only if $\det(A)$ is an invertible element of $\widetilde{\mathbb{R}}$, if and only if all eigenvalues of A are invertible elements of $\widetilde{\mathbb{C}}$.

The last equivalence follows from the characterization of invertibility in $\widetilde{\mathbb{C}}$ above. Finally, a symmetric matrix A will be called *positive definite*, if all its eigenvalues are nonnegative and invertible elements of $\widetilde{\mathbb{R}}$.

To be able to speak about symbols of differential operators, we shall need the notion of a polynomial with generalized coefficients. The most straightforward definition is to consider a generalized polynomial of degree m as a member

$$\sum_{|\gamma|\leq m} a_\gamma \xi^\gamma \in \mathcal{G}_m[\xi]$$

of the space of polynomials of degree m in the indeterminate $\xi = (\xi_1, \dots, \xi_n)$, with coefficients in $\mathcal{G} = \mathcal{G}(\Omega)$. Alternatively, we can and will view $\mathcal{G}_m[\xi]$ as the factor space

$$\mathcal{G}_m[\xi] = \mathcal{E}_{\mathrm{M},m}[\xi]/\mathcal{N}_m[\xi] \tag{2.6}$$

of families of polynomials of degree m with moderate coefficients modulo those with null coefficients. In this interpretation, generalized polynomials $P(x,\xi)$ are represented by families

$$(P_\varepsilon(x,\xi))_\varepsilon = \Big(\sum_{|\gamma|\leq m} a_{\varepsilon\gamma}(x)\xi^\gamma\Big)_\varepsilon .$$

Sometimes it will also be useful to regard polynomials as polynomial functions and hence as elements of $\mathcal{G}(\Omega \times \mathbb{R}^n)$. Important special cases are the polynomials with regular coefficients, $\mathcal{G}^\infty_m[\xi]$, and with constant generalized coefficients, $\widetilde{\mathbb{C}}_m[\xi]$. The union of the spaces of polynomials of arbitrary degree are the rings of polynomials $\mathcal{G}[\xi], \mathcal{G}^\infty[\xi]$, and $\widetilde{\mathbb{C}}[\xi]$. Letting $D = (-i\partial_1, \dots, -i\partial_n)$, a differential operator $P(x,D)$ with coefficients in $\mathcal{G}(\Omega)$ simply is an element of $\mathcal{G}[D]$.

We now turn to a new notion which will be essential for the paper, the notion of *slow scale nets.* Consider a moderate net of complex numbers $r = (r_\varepsilon)_\varepsilon \in \widetilde{\mathbb{C}}_M$; it satisfies an estimate as exhibited in (2.5). The *order of* r is defined as

$$\kappa(r) = \sup\{q \in \mathbb{R} : \exists \varepsilon_q \, \exists C_q > 0 \text{ such that } |r_\varepsilon| \le C_q \varepsilon^q \, , \forall \varepsilon \in (0, \varepsilon_q)\} \, .$$

Lemma 2.1. *Let* $r = (r_\varepsilon)_\varepsilon \in \widetilde{\mathbb{C}}_M$. *The following statements are equivalent:*

(a) *The net* r *has order* $\kappa(r) \ge 0$
(b) $\forall t \ge 0 \, \exists \varepsilon_t > 0$ *such that* $|r_\varepsilon|^t \le \varepsilon^{-1}$, $\forall \varepsilon \in (0, \varepsilon_t)$
(c) $\exists N \ge 0 \, \forall t \ge 0 \, \exists \varepsilon_t > 0$ *such that* $|r_\varepsilon|^t \le \varepsilon^{-N}$, $\forall \varepsilon \in (0, \varepsilon_t)$
(d) $\exists N \ge 0 \, \forall t \ge 0 \, \exists \varepsilon_t > 0 \, \exists C_t > 0$ *such that* $|r_\varepsilon|^t \le C_t \varepsilon^{-N}$, $\forall \varepsilon \in (0, \varepsilon_t)$
(e) $\exists N \ge 0 \, \exists \varepsilon_0 > 0 \, \forall t \ge 0 \, \exists C_t > 0$ *such that* $|r_\varepsilon|^t \le C_t \varepsilon^{-N}$, $\forall \varepsilon \in (0, \varepsilon_0)$
(f) $\exists \varepsilon_0 > 0 \, \forall t \ge 0 \, \exists C_t > 0$ *such that* $|r_\varepsilon|^t \le C_t \varepsilon^{-1}$, $\forall \varepsilon \in (0, \varepsilon_0)$.

Proof. (a) $\Leftrightarrow$ (b): If r has order zero, we have that for all $t \ge 0$ there is $\varepsilon_t > 0$ and $C_t > 0$ such that $|r_\varepsilon| \le C_t \varepsilon^{-1/2t}$, $\forall \varepsilon \in (0, \varepsilon_t)$. Since $C_t \le \varepsilon^{-1/2t}$ for sufficiently small $\varepsilon > 0$, assertion (b) follows. The converse direction is obvious.
(b) $\Rightarrow$ (c) $\Rightarrow$ (d) is clear.
(d) $\Rightarrow$ (b): Diminishing ε_t so that $C_t \le \varepsilon^{-1}$ for $\varepsilon \in (0, \varepsilon_t)$ we first achieve: $\exists N \ge 0 \, \forall t \ge 0 \, \exists \varepsilon_t > 0$ such that $|r_\varepsilon|^t \le \varepsilon^{-(N+1)}, \forall \varepsilon \in (0, \varepsilon_t)$. ¿From here we conclude that $|r_\varepsilon|^{t/(N+1)} \le \varepsilon^{-1}, \forall \varepsilon \in (0, \varepsilon_t)$. Since t is arbitrary, the assertion (b) follows.
(d) $\Rightarrow$ (e): The net r being moderate, there is $p \ge 0$ and $0 < \varepsilon_0 \le 1$ such that $|r_\varepsilon| \le \varepsilon^{-p}$ for $\varepsilon \in (0, \varepsilon_0)$. Let $t \ge 0$. If $\varepsilon_t \ge \varepsilon_0$ there is nothing to prove. Otherwise, we observe that the net $(r_\varepsilon)_\varepsilon$ is bounded on $[\varepsilon_t, \varepsilon_0]$ by a constant D_t, say. Then

$$|r_\varepsilon|^t \le \max(C_t, D_t^t) \varepsilon^{-N}, \forall \varepsilon \in (0, \varepsilon_0) \, .$$

(e) $\Rightarrow$ (f) follows again by taking the N-th root of the inequality. Finally, that (f) $\Rightarrow$ (d) is clear, and the proof of the lemma is complete. ☐

Definition 2.2. Nets satisfying the equivalent properties of Lemma 2.1 are termed *slow scale nets.*

The name derives from the crucial property that for all $t \ge 0$

$$|r_\varepsilon|^t = O\Big(\frac{1}{\varepsilon}\Big) \quad \text{as } \varepsilon \to 0 \, . \tag{2.7}$$

Another important class of nets are the *log-type nets*, which are defined by the condition

$$\exists \varepsilon_0 > 0 \ \text{ such that } |r_\varepsilon| \le \log \frac{1}{\varepsilon} \, , \ \forall \varepsilon \in (0, \varepsilon_0) \, . \tag{2.8}$$

In the same way as in Lemma 2.1 one can prove that $(r_\varepsilon)_\varepsilon$ is of log-type if and only if:

$$\exists \varepsilon_0 > 0 \, \exists C > 0 \ \text{ such that } |r_\varepsilon| \le C \log \frac{1}{\varepsilon} \, , \ \forall \varepsilon \in (0, \varepsilon_0) \, .$$

Every log-type net is a slow scale net. In fact, every net which satisfies

$$\exists q \ge 0 \, \exists \varepsilon_0 > 0 \ \text{ such that } |r_\varepsilon| \le \Big(\log \frac{1}{\varepsilon}\Big)^q \, , \ \forall \varepsilon \in (0, \varepsilon_0) \, .$$

is slow scale. Note, however, that $\big(\log \frac{1}{\varepsilon}\big)^q$ does not define a log-type net if $q > 1$, so slow scale nets need not be log-type. The following result explores how far slow scale nets are apart from being of log-type.

Lemma 2.3. *Let* $(r_\varepsilon)_\varepsilon$ *be a moderate net,* $r_\varepsilon \ge 0$ *for all* ε. *The following statements are equivalent:*

(a) $\big(\exp(r_\varepsilon)\big)_\varepsilon$ *is moderate*
(b) $\big(r_\varepsilon\big)_\varepsilon$ *is log-type*
(c) $\big(r_\varepsilon\big)_\varepsilon$ *is slow scale and* $\exists N \geq 0\,\exists \varepsilon_0 > 0\,\exists C > 0$ *and families* $\big(c_{\varepsilon t}\big)_{\varepsilon>0,t\in\mathbb{N}}$ *such that* $\sum_{t=0}^\infty c_{\varepsilon t} \leq 1$ *and* $r_\varepsilon^t \leq Ct! c_{\varepsilon t}\varepsilon^{-N}$, $\forall \varepsilon \in (0,\varepsilon_0)$.

Proof. (a)$\Leftrightarrow$ (b): If (a) holds, there is $p \geq 0, \varepsilon_0 > 0$ such that $\exp(r_\varepsilon) \leq \varepsilon^{-p}, \forall \varepsilon \in (0,\varepsilon_0)$. Thus $r_\varepsilon \leq p\log(1/\varepsilon)$ and this means that $r_\varepsilon \geq 0$ is log-type, as was observed above. The converse is clear anyway.
(a) $\Rightarrow$ (c): By assumption, there is $\varepsilon > 0, N \geq 0$ such that

$$\sum_{t=0}^\infty \frac{r_\varepsilon^t}{t!} \leq \varepsilon^{-N},\ \forall \varepsilon \in (0,\varepsilon_0)\,.$$

Put $c_{\varepsilon t} = \varepsilon^N r_\varepsilon^t/t!$. Then $\sum_{t=0}^\infty c_{\varepsilon t} \leq 1$ and $r_\varepsilon^t = t! c_{\varepsilon t}\varepsilon^{-N},\ \forall \varepsilon \in (0,\varepsilon_0)\,.$
(c) $\Rightarrow$ (a): We have that

$$\exp(r_\varepsilon) = \sum_{t=0}^\infty \frac{r_\varepsilon^t}{t!} \leq C\sum_{t=0}^\infty c_{\varepsilon t}\varepsilon^{-N} \leq C\varepsilon^{-N},\ \forall \varepsilon \in (0,\varepsilon_0)\,.$$

This shows that $\exp(r_\varepsilon)$ forms a moderate net and completes the proof. ☐

Remark 2.4. If $(r_\varepsilon)_\varepsilon$ is slow scale and the constants C_t in Lemma 2.1(e) satisfy $\sum_{t=0}^\infty C_t/t! = C < \infty$, then condition (c) of Lemma 2.3 is satisfied with $c_{\varepsilon t} = C_t/(Ct!)$, hence $(\exp(r_\varepsilon))_\varepsilon$ is moderate. If in addition the constants C_t are actually bounded by a constant C', say, then $(r_\varepsilon)_\varepsilon$ is bounded. This follows from the fact that at each fixed $\varepsilon \in (0,\varepsilon_0)$, $\lim_{t\to\infty}(C'/\varepsilon^N)^{1/t} = 1$.

3. Notions of ellipticity in the Colombeau setting

In this section, we study linear differential operators of the form (1.1) with coefficients $a_\gamma \in \mathcal{G}^\infty(\Omega)$; throughout, Ω is an open subset of $\mathbb{R}^n$ and the order of $P(x,D)$ is $m \geq 0$. The symbol and the principal symbol, respectively,

$$P(x,\xi) = \sum_{|\gamma|\leq m} a_\gamma(x)\xi^\gamma\,, \quad P_m(x,\xi) = \sum_{|\gamma|=m} a_\gamma(x)\xi^\gamma$$

are elements of $\mathcal{G}_m^\infty[\xi]$. Due to the $\mathcal{G}^\infty$-property, every representative $(P_\varepsilon(x,\xi))_\varepsilon$ satisfies an estimate from above of the following form:

$$\begin{gathered}\forall K \Subset \Omega\,\exists p \geq 0\,\forall \alpha,\beta \in \mathbb{N}_0^n\,\exists \varepsilon_0 > 0\,\exists C > 0 \text{ such that}\\ \forall \varepsilon \in (0,\varepsilon_0)\,\forall x \in K\,\forall \xi \in \mathbb{R}^n :\\ \big|\partial_\xi^\alpha \partial_x^\beta P_\varepsilon(x,\xi)\big| \leq C\varepsilon^{-p}\big(1+|\xi|\big)^{m-|\alpha|}.\end{gathered}$$

Various estimates from below will lead to various notions of ellipticity, which - due to the presence of the additional parameter $\varepsilon \in (0,1]$ - are more involved than in the classical case. In this section, we introduce these concepts and relate them to properties of the principal symbol. Examples distinguishing these notions and their relevance for regularity theory will be given in the subsequent sections.

Definition 3.1. Let $P = P(x,\xi) \in \mathcal{G}_m^\infty[\xi]$.

(a) P is called *S-elliptic*, if for some representative $P_\varepsilon(x,\xi)$ the following condition holds:

$$\begin{gathered}\forall K \Subset \Omega\ \exists N > 0\ \exists \varepsilon_0 > 0\ \exists r > 0 \text{ such that}\\ \forall \varepsilon \in (0,\varepsilon_0)\ \forall x \in K\ \forall \xi \in \mathbb{R}^n \text{ with } |\xi| \geq r :\\ |P_\varepsilon(x,\xi)| \geq \varepsilon^N (1+|\xi|)^m.\end{gathered} \tag{3.1}$$

(b) P is called *W-elliptic*, if

$$\begin{gathered}\forall K \Subset \Omega\ \exists N > 0\ \exists \varepsilon_0 > 0\ \forall \varepsilon \in (0,\varepsilon_0)\ \exists r_\varepsilon > 0 \text{ such that}\\ \forall \varepsilon \in (0,\varepsilon_0)\ \forall x \in K\ \forall \xi \in \mathbb{R}^n \text{ with } |\xi| \geq r_\varepsilon :\\ |P_\varepsilon(x,\xi)| \geq \varepsilon^N (1+|\xi|)^m.\end{gathered} \tag{3.2}$$

(c) P is called *SH-elliptic*, if it is S-elliptic and in addition (with the same ε_0 and r as in (a))

$$\begin{gathered}\forall \alpha,\beta \in \mathbb{N}_0^n\ \exists C_{\alpha\beta} > 0 \text{ such that}\\ \forall \varepsilon \in (0,\varepsilon_0)\ \forall x \in K\ \forall \xi \in \mathbb{R}^n \text{ with } |\xi| \geq r :\\ |\partial_\xi^\alpha \partial_x^\beta P_\varepsilon(x,\xi)| \leq C_{\alpha\beta} |P_\varepsilon(x,\xi)| (1+|\xi|)^{-|\alpha|}\,.\end{gathered} \tag{3.3}$$

(d) P is called *WH-elliptic*, if it is W-elliptic and in addition (with the same ε_0 and r_ε as in (b))

$$\begin{gathered}\forall \alpha,\beta \in \mathbb{N}_0^n\ \exists C_{\alpha\beta} > 0 \text{ such that}\\ \forall \varepsilon \in (0,\varepsilon_0)\ \forall x \in K\ \forall \xi \in \mathbb{R}^n \text{ with } |\xi| \geq r_\varepsilon :\\ |\partial_\xi^\alpha \partial_x^\beta P_\varepsilon(x,\xi)| \leq C_{\alpha\beta} |P_\varepsilon(x,\xi)| (1+|\xi|)^{-|\alpha|}\,.\end{gathered} \tag{3.4}$$

Remark 3.2. (i) It is clear that if any of these conditions holds for one representative (in the sense of (2.6)) of $P(x,\xi)$, then it holds for all representatives. Indeed, if $(Q_\varepsilon(x,\xi))_\varepsilon$ belongs to $\mathcal{N}_m[\xi]$, then

$$\begin{gathered}\forall K \Subset \Omega\, \forall \alpha,\beta\, \forall q \geq 0\, \exists \varepsilon_0 > 0 \text{ such that}\\ \forall \varepsilon \in (0,\varepsilon_0)\, \forall x \in K\, \forall \xi \in \mathbb{R}^n :\\ \big|\partial_\xi^\alpha \partial_x^\beta Q_\varepsilon(x,\xi)\big| \leq \varepsilon^q \big(1+|\xi|\big)^{m-|\alpha|}\end{gathered}$$

and this entails the assertion.
(ii) The letters "S" and "W" should be reminiscent of "strong" and "weak", respectively, while the "H" is intended to invoke an association with "hypo-". The "weak" conditions differ from the "strong" ones by the fact that the radius r from which on the basic estimate is required to hold may grow as $\varepsilon \to 0$.
(iii) The implications (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d) as well as (c) $\Rightarrow$ (a) and (d) $\Rightarrow$ (b) are obvious. None of the reverse implications hold, as will be seen by the examples in Section 4.

Proposition 3.3. *Let $P(x,\xi) \in \mathcal{G}_m^\infty[\xi]$ be an operator of order m. Then $P(x,\xi)$ is W-elliptic if and only if its principal part $P_m(x,\xi)$ is S-elliptic.*

Proof. Assume that P_m is S-elliptic. Separating the homogeneous terms, we may write

$$P_\varepsilon(x,\xi) = P_{m,\varepsilon}(x,\xi) + P_{m-1,\varepsilon}(x,\xi) + \cdots + P_{0,\varepsilon}(x,\xi)\,.$$

By assumption and the fact that each coefficient $a_{\gamma\varepsilon}$ is moderate, we have

$$\begin{aligned}|P_\varepsilon(x,\xi)| &\geq \varepsilon^N(1+|\xi|)^m - C'\varepsilon^{-N'}(1+|\xi|)^{m-1}\\ &= (1+|\xi|)^m\varepsilon^N\big(1-C'\varepsilon^{-N-N'}(1+|\xi|)^{-1}\big)\end{aligned}$$

for certain constants $C, N' > 0$, when x varies in a relatively compact set $K, |\xi| \geq r$ and $\varepsilon \in (0,\varepsilon_0)$. Defining r_ε by the property that

$$\big(1-C'\varepsilon^{-N-N'}(1+|r_\varepsilon|)^{-1}\big) \geq \frac{1}{2}\,,$$

we get

$$|P_\varepsilon(x,\xi)| \geq \frac{1}{2}(1+|\xi|)^m\varepsilon^N \geq (1+|\xi|)^m\varepsilon^{N+1}$$

if $|\xi| \geq r_\varepsilon$, as desired.

Conversely, assume that P is W-elliptic. Let $\eta \in \mathbb{R}^n$ with $|\eta| = 1$ and choose $\xi \in \mathbb{R}^n$ such that $|\xi| \geq r_\varepsilon$ and $\eta = \xi/|\xi|$. Then

$$\big|P_{m,\varepsilon}(x,\eta) + P_{m-1,\varepsilon}(x,\eta)\frac{1}{|\xi|} + \dots + P_{0,\varepsilon}(x,\eta)\frac{1}{|\xi|^m}\big| \geq \varepsilon^N\big(1+\frac{1}{|\xi|}\big)^m$$

by hypothesis. By the moderation property of each term, we obtain for $x \in K$ that

$$P_{m-j,\varepsilon}(x,\eta)\frac{1}{|\xi|^j} \leq \varepsilon^{-N_j}\frac{1}{|\xi|^j} \leq \varepsilon^{N+1}$$

if $|\xi| \geq \max\big(r_\varepsilon, \varepsilon^{-N_j-N-1}\big)$. Thus

$$\big|P_{m,\varepsilon}(x,\eta)\big| \geq \frac{1}{2}\varepsilon^N$$

for sufficiently small $\varepsilon > 0$ and all η with $|\eta| = 1$. We conclude that

$$\big|P_{m,\varepsilon}(x,\xi)\big| \geq \frac{1}{2}\varepsilon^N|\xi|^m \geq \varepsilon^{N+1}(1+|\xi|)^m$$

for $\xi \in \mathbb{R}^n, |\xi| \geq 1$ and sufficiently small $\varepsilon > 0$, as required. □

For operators with constant (generalized) coefficients, the S-ellipticity of the principal part can be characterized by the usual pointwise conditions, using generalized points.

Proposition 3.4. *Let $P(\xi) \in \widetilde{\mathbb{C}}_m[\xi]$ be an operator with constant coefficients in $\widetilde{\mathbb{C}}$. The following are equivalent:*

(a) *P_m is S-elliptic*
(b) *for each representative $P_\varepsilon(x,\xi)$ it holds that*

$$\exists N > 0\ \exists\varepsilon_0 > 0 \text{ such that } \forall\varepsilon \in (0,\varepsilon_0)\ \ \forall\xi \in \mathbb{R}^n : |P_{m,\varepsilon}(\xi)| \geq \varepsilon^N|\xi|^m$$

(c) $\forall\tilde{\xi} \in \widetilde{\mathbb{R}}^n : P_m(\tilde{\xi}) = 0 \Leftrightarrow \tilde{\xi} = 0$.

Proof. (a) $\Rightarrow$ (b): If P_m is S-elliptic and $\eta = \xi/|\xi|$, there is $N \geq 0, \varepsilon_0 > 0$ and $r > 0$ such that

$$|P_{m,\varepsilon}(\eta)| \geq \varepsilon^N\big(1+\frac{1}{|\xi|}\big)^m$$

for $\varepsilon \in (0,\varepsilon_0)$ and $|\xi| \geq r$. Letting $|\xi| \to \infty$ we obtain $|P_{m,\varepsilon}(\eta)| \geq \varepsilon^N$ whenever $\varepsilon \in (0,\varepsilon_0)$ and $|\eta| = 1$. This in turn implies (b).

(b) $\Rightarrow$ (a): If $|\xi| \geq 1$, we have that

$$|P_{m,\varepsilon}(\xi)| \geq \varepsilon^N |\xi|^m \geq \frac{\varepsilon^N}{2^m}(1+|\xi|)^m \geq \varepsilon^{N+1}(1+|\xi|)^m$$

whenever $0 < \varepsilon < \min(\varepsilon_0, 1/2^m)$.

(b) $\Rightarrow$ (c): If $\tilde{\xi} \neq 0$ in $\widetilde{\mathbb{R}}^n$ then there is $q \geq 0$ and a sequence $\varepsilon_k \to 0$ such that $|\xi_{\varepsilon_k}| \geq \varepsilon_k^q$, where $(\xi_\varepsilon)_\varepsilon$ is a representative of $\tilde{\xi}$. But then (b) implies that $|P_{m,\varepsilon_k}(\xi_{\varepsilon_k}) \geq \varepsilon_k^{N+mq}$ for sufficiently large $k \in \mathbb{N}$, so $P_m(\tilde{\xi}) \neq 0$ in $\widetilde{\mathbb{C}}$.

(c) $\Rightarrow$ (b): The negation of (b) is:

$$\forall N > 0\ \forall \varepsilon_0 > 0\ \exists \varepsilon < \varepsilon_0\ \exists \eta \in \mathbb{R}^n, |\eta| = 1 \text{ such that } |P_{m,\varepsilon}(\eta)| < \varepsilon^N.$$

In particular, choosing $\varepsilon_0 = 1/N$ we obtain $\varepsilon_N \in (0, 1/N)$ and η_N with $|\eta_N| = 1$ such that $|P_{m,\varepsilon_N}(\eta_N)| < \varepsilon_N^N$. Note that $P_{m,\varepsilon}(0) = 0$. Define $\tilde{\eta} \in \widetilde{\mathbb{R}}^n$ as the class of

$$\eta_\varepsilon = \begin{cases} 0, & \text{if } \varepsilon \notin \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots\} \\ \eta_N, & \text{if } \varepsilon = \varepsilon_N \text{ for some } N \in \mathbb{N}. \end{cases}$$

Then clearly $\tilde{\eta} \neq 0$ in $\widetilde{\mathbb{R}}^n$, but $P_m(\tilde{\eta}) = 0$ in $\widetilde{\mathbb{C}}$. □

4. Examples

In this section we present examples that distinguish the notions of ellipticity defined in the previous sections and relate them to generalized hypoellipticity. In particular, we shall scrutinize operators $P(x, D)$ with coefficients in $\mathcal{G}^\infty(\Omega)$ or in $\widetilde{\mathbb{C}}$ with respect to the regularity property:

$$\big[u \in \mathcal{G}(\Omega),\ f \in \mathcal{G}^\infty(\Omega) \text{ and } P(x, D)u = f \text{ in } \mathcal{G}(\Omega)\big] \Longrightarrow u \in \mathcal{G}^\infty(\Omega). \tag{4.1}$$

Operators that enjoy property (4.1) on every open subset $\Omega \subset \mathbb{R}^n$ are called $\mathcal{G}^\infty$-*hypoelliptic*. The examples given here will also illuminate the range of validity of the hypoellipticity results in the subsequent sections.

Example 4.1. The operator $P(\xi) \in \widetilde{\mathbb{C}}[\xi_1, \xi_2]$ on $\mathbb{R}^2$ defined by the representative $P_\varepsilon(\xi) = \varepsilon\xi_1 + i\xi_2$: It is S-elliptic, but not WH-elliptic (hence also W-elliptic, but not SH-elliptic). Indeed, it is homogeneous of degree one and

$$|\varepsilon\xi_1 + i\xi_2| = \sqrt{\varepsilon^2\xi_1^2 + \xi_2^2} \geq \varepsilon|\xi|$$

for $\varepsilon \in (0, 1]$ and all $\xi \in \mathbb{R}^2$, hence $P(\xi)$ is S-elliptic by Proposition 3.4. On the other hand, the inequality

$$\big|\partial_{\xi_2} P_\varepsilon(\xi)\big| \leq C|\varepsilon\xi_1 + i\xi_2|(1 + |\xi|)^{-1}$$

entails for $\xi = (\xi_1, 0)$ that

$$1 \leq C\varepsilon|\xi_1|(1 + |\xi_1|)^{-1}$$

and thus does not hold for whatever C when $|\xi_1| \to \infty$. Thus there is no family of radii r_ε which could produce the WH-ellipticity estimate (3.4). The corresponding homogeneous differential equation on $\Omega = \mathbb{R} \times (0, \infty)$,

$$\big(-i\varepsilon\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\big)\, u_\varepsilon(x_1, x_2) = 0$$

is solved by the moderate family $u_\varepsilon(x_1, x_2) = \exp(ix_1/\varepsilon - x_2)$ which does not define an element of $\mathcal{G}^\infty(\Omega)$, so $P(D)$ is not $\mathcal{G}^\infty$-hypoelliptic.

Example 4.2. The operator $P(\xi) = a^2\xi_1^2 + \xi_2^2 \in \widetilde{\mathbb{C}}[\xi_1, \xi_2]$ where a is a nonnegative element of $\widetilde{\mathbb{R}}$: We first consider the case when a is not invertible. Then P is not S-elliptic. Indeed, a is a zero divisor (Section 2), so there is $b \in \widetilde{\mathbb{R}}, b \neq 0$ such that $ab = 0$. Taking $\tilde{\xi} = (b, 0) \in \widetilde{\mathbb{R}}^2$ we have that $\tilde{\xi} \neq 0$, but $P(\tilde{\xi}) = 0$ (see Proposition 3.4). Also, if $u = u(\xi_1)$ is any element of $\mathcal{G}(\mathbb{R}^2)$ not depending on ξ_2, then $P(D)\,bu = 0$, so $P(D)$ is not $\mathcal{G}^\infty$-hypoelliptic.

Second, assume that a is invertible. Then $P(\xi)$ is S-elliptic. Indeed, by the invertibility of a, its representatives satisfy an estimate from below of the form $a_\varepsilon \geq \varepsilon^m$ for some m and sufficiently small $\varepsilon > 0$. Hence

$$|P_\varepsilon(\xi)| \geq \min(1, a_\varepsilon^2)(\xi_1^2 + \xi_2^2) \geq \varepsilon^{2m}|\xi|^2\,,$$

demonstrating the S-ellipticity by Proposition 3.4. On the other hand, if a_ε is not bounded away from zero by a positive, real constant, then $P(\xi)$ is not WH-elliptic. Indeed, an inequality of the form

$$\left|\partial_{\xi_2}^2 P_\varepsilon(\xi)\right| \leq C|a_\varepsilon^2\xi_1^2 + \xi_2^2|(1 + |\xi|)^{-2}$$

means for $\xi = (\xi_1, 0)$ that

$$2 \leq C a_\varepsilon \xi_1^2 (1 + |\xi_1|)^{-2}\,.$$

Letting $|\xi_1| \to \infty$ produces the lower bound $a_\varepsilon \geq 2/C$.

Finally, the moderate family $u_\varepsilon(x_1, x_2) = \sin(x_1/a_\varepsilon)\sinh(x_2)$ defines a solution of the homogeneous equation $P(D)u = 0$ in $\mathcal{G}(\mathbb{R}^2)$. When $(1/a_\varepsilon)_\varepsilon$ is not slow scale, u does not belong to $\mathcal{G}^\infty(\mathbb{R}^2)$, in which case $P(D)$ is not $\mathcal{G}^\infty$-hypoelliptic.

Ordinary differential operators may serve to illustrate a few further points:

Example 4.3. The operator $P(\xi) = -a^2\xi^2 + 1 \in \widetilde{\mathbb{C}}[\xi]$ on $\mathbb{R}$ where a is a nonnegative, invertible element of $\widetilde{\mathbb{R}}$: Its principal symbol $P_m(\xi)$ is clearly S-elliptic, so $P(\xi)$ is W-elliptic. We will show that $P(\xi)$ is not S-elliptic if the representatives $(a_\varepsilon)_\varepsilon$ are not bounded away from zero by a positive, real constant. This demonstrates that the full symbol of an operator need not inherit the S-ellipticity property from the principal symbol (contrary to the classical situation). Indeed, the fact that $P(1/a_\varepsilon) = 0$ shows that an estimate

$$|P_\varepsilon(\xi)| \geq \varepsilon^N(1 + |\xi|)^2$$

cannot hold for $|\xi| \geq r$ with r independent of ε, if a_ε has a subsequence converging to zero.

However, $P(\xi)$ is WH-elliptic, and we shall now estimate the required radius; in fact, $r_\varepsilon = s/a_\varepsilon$ for arbitrary $s > 1$ does the job. Indeed, if $r_\varepsilon = s/a_\varepsilon$ with $s > 1$ and $|\xi| \geq r_\varepsilon$, we have $|P_\varepsilon(\xi)| = a_\varepsilon^2\xi^2 - 1 > 0$ and

$$\Big|\frac{\partial^2}{\partial\xi^2}P_\varepsilon(\xi)\Big| = 2a_\varepsilon^2 = 2\frac{|P_\varepsilon(\xi)|}{\xi^2 - 1/a_\varepsilon^2} \leq 2\frac{|P_\varepsilon(\xi)|}{\xi^2(1 - 1/s^2)} \leq c|P_\varepsilon(\xi)|(1 + |\xi|)^{-2}$$

where $c = 8s^2/(s^2 - 1)$ and $|\xi| \geq \max(r_\varepsilon, 1)$. A similar estimate holds for the first derivative $\partial_\xi P_\varepsilon(\xi)$, thus the second condition in the definition of WH-ellipticity holds. What concerns the first, we must come up with $N \geq 0$ such that

$$a_\varepsilon^2\xi^2 - 1 \geq \varepsilon^N(1 + |\xi|)^2 \tag{4.2}$$

for $\varepsilon \in (0, \varepsilon_0)$ and $|\xi| \geq r_\varepsilon$. If we prove that

$$a_\varepsilon^2\xi^2 - 1 \geq \varepsilon^N\xi^2$$

in this range, then (4.2) follows by possibly enlarging N. Since a is invertible, there is $q \geq 0$ such that $a_\varepsilon \geq \varepsilon^q$ for sufficiently small ε. We may choose $N = 2q+1$, as is seen from the calculation

$$a_\varepsilon^2 \xi^2 - \varepsilon^N \xi^2 \geq (a_\varepsilon^2 - \varepsilon^{2q+1}) \frac{s^2}{a_\varepsilon^2} = s^2 (1 - \varepsilon (\frac{\varepsilon^q}{a_\varepsilon})^2) \geq s^2 (1-\varepsilon) \geq 1$$

for sufficiently small ε and $|\xi| \geq r_\varepsilon$. The corresponding homogeneous equation

$$(a_\varepsilon^2 \frac{d^2}{dx^2} + 1) u_\varepsilon = 0$$

has the solution $u_\varepsilon(x) = \sin(x/a_\varepsilon)$ which does not define an element of $\mathcal{G}^\infty(\Omega)$, unless $1/a_\varepsilon$ is slow scale. This shows that the WH-property alone does not guarantee $\mathcal{G}^\infty$-hypoellipticity of the operator and suggests that conditions on the radius r_ε have to enter the picture (as will be expounded in Section 5).

Remark 4.4. The case $a_\varepsilon = \varepsilon$ deserves some more attention. In this case the class $u \in \mathcal{G}(\mathbb{R})$ of $u_\varepsilon(x) = \sin(x/\varepsilon)$, while being a non-regular solution to the homogeneous equation $a^2 u''(x) + 1 = 0$, is actually equal to zero in the sense of generalized distributions, that is,

$$\int u(x) \psi(x)\, dx = 0 \text{ in } \widetilde{\mathbb{C}}$$

for all $\psi \in \mathcal{D}(\mathbb{R})$. Indeed,

$$\Big| \int \sin(\frac{x}{\varepsilon}) \psi(x)\, dx \Big| = \Big| \varepsilon^{4q} \int \sin(\frac{x}{\varepsilon}) \psi^{(4q)}(x)\, dx \Big| = O(\varepsilon^{4q})$$

for every $q \geq 0$. This shows, in particular, that an element of $\mathcal{G}(\mathbb{R})$ which equals a function in $\mathrm{C}^\infty(\mathbb{R})$ in the sense of generalized distributions need not belong to $\mathcal{G}^\infty(\mathbb{R})$ (compare with [14, Proposition 3.17]).

Remark 4.5. The operator $P(\xi) = a^2 \xi^2 + 1$, with a nonnegative and invertible, is SH-elliptic with radius 1, since one easily verifies the following inequalities, valid when $\varepsilon \in (0, 1/4)$:

$$|P_\varepsilon(\xi)| \geq \varepsilon^{N+1} (1+|\xi|)^2, \quad \frac{|\partial_\xi P_\varepsilon(\xi)|}{|P_\varepsilon(\xi)|} \leq \frac{4}{1+|\xi|}, \quad \frac{|\partial_\xi^2 P_\varepsilon(\xi)|}{|P_\varepsilon(\xi)|} \leq \frac{8}{(1+|\xi|)^2}.$$

We will see in the next section that SH-ellipticity implies $\mathcal{G}^\infty$-hypoellipticity. This shows that lower order terms do matter.

Example 4.6. The multiplication operator $P(x)$ on $\mathbb{R}$ given by $P_\varepsilon(x) = 1 + x^2/\varepsilon$: The zero order operator P is obviously S-elliptic. It is not WH-elliptic, because an estimate of the form

$$\big| \partial_x^2 P_\varepsilon(x) \big| = \frac{2}{\varepsilon} \leq C |P_\varepsilon(x)| = 1 + x^2/\varepsilon$$

as $\varepsilon \to 0$ does not hold at $x = 0$. Further, P is not $\mathcal{G}^\infty$-hypoelliptic; the solution u to the equation $Pu = 1$ does not belong to $\mathcal{G}^\infty(\mathbb{R})$. In fact, it is given by

$$u_\varepsilon(x) = \big(1 + \frac{x^2}{\varepsilon}\big)^{-1} = \sum_{k=0}^\infty \big(-\frac{1}{\varepsilon}\big)^k x^{2k}$$

where the series representation holds for $|x| < \sqrt{\varepsilon}$. At $x = 0$ the derivatives are

$$\partial_x^{2\alpha} u_\varepsilon(0) = \frac{1}{2\alpha!} \big(-\frac{1}{\varepsilon}\big)^\alpha$$

and hence do not have a uniform, finite order independently of α.

Remark 4.7. Contrary to the non-regular solution discussed in Remark 4.4, u is not even equal to an element of $\mathcal{G}^\infty(\mathbb{R})$ in the sense of generalized distributions. In fact, for $\psi \in \mathcal{D}(\mathbb{R})$,

$$\int \big(1 + \frac{x^2}{\varepsilon}\big)^{-1}\psi(x)\,dx = \sqrt{\varepsilon}\int (1+y^2)^{-1}\psi(\sqrt{\varepsilon}x)\,dx \to 0 \tag{4.3}$$

as $\varepsilon \to 0$, showing that u is associated with zero. But u is not zero in the sense of generalized distributions, because if it were, the right hand side of (4.3) should be $O(\varepsilon^q)$ for every $q \geq 0$. This is not the case if $\psi(0) \neq 0$; then it actually has the precise order $\kappa = \sqrt{\varepsilon}$, since the integral converges to the finite limit $\pi\psi(0)$.

In addition, Example 4.6 exhibits an invertible element of $\mathcal{G}(\mathbb{R})$, namely the class of $(P_\varepsilon(x))_\varepsilon$, which is a member of $\mathcal{G}^\infty(\mathbb{R})$ but whose multiplicative inverse does not belong to $\mathcal{G}^\infty(\mathbb{R})$.

5. The elliptic regularity result for WH-elliptic operators with constant coefficients

We start this section with a general regularity result for the constant coefficient case. Consider an operator P with symbol $P(\xi) = \sum_{|\gamma|\leq m} a_\gamma \xi^\gamma$, with coefficients $a_\gamma \in \widetilde{\mathbb{C}}$, which is assumed to be WH-elliptic. Thus it satisfies the condition

$$\exists N > 0\ \exists \varepsilon_0 > 0\ \forall \varepsilon \in (0,\varepsilon_0)\ \exists r_\varepsilon > 0 \text{ such that } |P_\varepsilon(\xi)| \geq \varepsilon^N(1+|\xi|)^m$$

for all $\varepsilon \in (0,\varepsilon_0)$ and for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq r_\varepsilon$, as well as an estimate ($\alpha \in \mathbb{N}_0^n$)

$$|\partial^\alpha P_\varepsilon(\xi)| \leq C_\alpha |P_\varepsilon(\xi)|(1+|\xi|)^{-|\alpha|}$$

for ε and ξ in the same range.

We begin by constructing a generalized parametrix for the operator P. Let $\chi \in \mathrm{C}^\infty(\mathbb{R}^n)$, $\chi(\xi) \equiv 0$ for $|\xi| \leq 1$ and $\chi(\xi) \equiv 1$ for $|\xi| \geq 2$ and put

$$\chi_\varepsilon(\xi) = \chi(\xi/r_\varepsilon)$$

and

$$Q_\varepsilon = \mathcal{F}^{-1}\big(\frac{\chi_\varepsilon}{P_\varepsilon}\big), \quad h_\varepsilon = \mathcal{F}^{-1}(1-\chi_\varepsilon).$$

It is clear that, for fixed $\varepsilon \in (0,\varepsilon_0)$, $Q_\varepsilon \in \mathcal{S}'(\mathbb{R}^n)$ and $h_\varepsilon \in \mathcal{S}(\mathbb{R}^n)$. We also have

$$P_\varepsilon(D)Q_\varepsilon = \mathcal{F}^{-1}(\chi_\varepsilon) = \delta - h_\varepsilon \tag{5.1}$$

where δ denotes the Dirac measure. The family $(Q_\varepsilon)_{\varepsilon\in(0,1]}$ of tempered distributions will serve as the generalized parametrix.

Lemma 5.1. *There is $N \geq 0$ such that for all $\alpha \in \mathbb{N}_0^n$ there is $C_\alpha > 0$ such that*

$$|\partial^\alpha\big(\frac{\chi_\varepsilon(\xi)}{P_\varepsilon(\xi)}\big)| \leq C_\alpha \varepsilon^{-N}(1+|\xi|)^{-m-|\alpha|} \tag{5.2}$$

for all $\xi \in \mathbb{R}^n$ and $\varepsilon \in (0,\varepsilon_0)$; C_0 can be chosen to be 1.

Proof. If $|\alpha| = 0$ the assertion is obvious from the hypothesis. We use induction over $|\alpha|$. From the Leibniz rule we obtain, for $|\alpha| \geq 1$,

$$\partial^\alpha(\chi_\varepsilon/P_\varepsilon) = \partial^\alpha\chi_\varepsilon/P_\varepsilon - \sum_{\beta<\alpha}\binom{\alpha}{\beta}\partial^\beta(\chi_\varepsilon/P_\varepsilon)\partial^{\alpha-\beta}P_\varepsilon/P_\varepsilon.$$

We have that $\partial^\alpha \chi_\varepsilon(\xi) \neq 0$ only if $|\xi| \le 2r_\varepsilon$. In this range, $1 \le (1+2r_\varepsilon)^{|\alpha|}(1+|\xi|)^{-|\alpha|}$, hence

$$\begin{aligned}|\partial^\alpha \chi_\varepsilon / P_\varepsilon| &\le \|\partial^\alpha \chi_\varepsilon\|_{\mathrm{L}^\infty} \varepsilon^{-N}(1+2r_\varepsilon)^{|\alpha|}(1+|\xi|)^{-m-|\alpha|}\\ &= r_\varepsilon^{-|\alpha|}\|\partial^\alpha \chi\|_{\mathrm{L}^\infty} \varepsilon^{-N}(1+2r_\varepsilon)^{|\alpha|}(1+|\xi|)^{-m-|\alpha|}\\ &\le C'_\alpha \varepsilon^{-N}(1+|\xi|)^{-m-|\alpha|}.\end{aligned}$$

Here, N is chosen as in the ellipticity condition (and is independent of α). Furthermore, $|\partial^{\alpha-\beta}P_\varepsilon(\xi)/P_\varepsilon(\xi)| \le C''_{\alpha-\beta}(1+|\xi|)^{|\beta|-|\alpha|}$ and $|\partial^\beta(\chi_\varepsilon(\xi)/P_\varepsilon(\xi))| \le C_\beta \varepsilon^{-N}(1+|\xi|)^{-m-|\beta|}$ when $|\beta| < |\alpha|$ by assumption. Thus the conclusion follows. ☐

Lemma 5.2. $\|\partial^\alpha h_\varepsilon\|_{\mathrm{L}^\infty} \le 2^{|\alpha|} r_\varepsilon^{|\alpha|+n}\|1-\chi\|_{\mathrm{L}^1}$, *for all* $\varepsilon \in (0,\varepsilon_0)$. *In particular,* $(h_\varepsilon)_{\varepsilon\in(0,1)} \in \mathcal{E}^\infty_{\mathrm{M}}(\mathbb{R}^n)$ *if* r_ε *is of slow scale.*

Proof.

$$\begin{aligned}\|\partial^\alpha h_\varepsilon\|_{\mathrm{L}^\infty} &\le \int |\xi|^{|\alpha|}|1-\chi_\varepsilon(\xi)|\,d\xi\\ &\le (2r_\varepsilon)^{|\alpha|}\int |1-\chi(\frac{\xi}{r_\varepsilon})|\,d\xi = 2^{|\alpha|} r_\varepsilon^{|\alpha|+n}\|1-\chi\|_{\mathrm{L}^1}.\end{aligned}$$

The second inequality follows from the fact that $1-\chi_\varepsilon(\xi)) \equiv 0$ when $|\xi| \ge 2r_\varepsilon$. ☐

Lemma 5.3. *For every* $K \Subset \Omega$ *and* $s > n/2$ *there is a constant* $C > 0$ *such that the Sobolev estimate*

$$\|Q_\varepsilon * \varphi\|_{\mathrm{L}^\infty} \le C\varepsilon^{-N}\|\varphi\|_{W^{s,\infty}}$$

holds for all $\varphi \in \mathcal{D}(K)$ *and all* $\varepsilon \in (0,\varepsilon_0)$.

Proof.

$$\|Q_\varepsilon * \varphi\|_{\mathrm{L}^\infty} \le \|\chi_\varepsilon/P_\varepsilon\|_{\mathrm{L}^\infty}\|\widehat{\varphi}\|_{\mathrm{L}^1} \le C\varepsilon^{-N}\|\varphi\|_{H^s} \le C\varepsilon^{-N}\|\varphi\|_{W^{s,\infty}}$$

with $C^2 = \int(1+|\xi|)^{-2s}\,d\xi$ by usual Sobolev space arguments. The second inequality uses the fact that $\|\chi_\varepsilon/P_\varepsilon\|_{\mathrm{L}^\infty} \le \varepsilon^{-N}$ by Lemma 5.1. ☐

Proposition 5.4. $\big(Q_\varepsilon|_{\mathbb{R}^n\setminus\{0\}}\big)_{\varepsilon\in(0,1]}$ *defines an element of* $\mathcal{E}^\infty_{\mathrm{M}}(\mathbb{R}^n\setminus\{0\})$.

Proof. Take $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varphi \equiv 1$ in a neighborhood of zero, and put $\psi(x) = \varphi(\sigma x) - \varphi(x/\sigma)$ with $\sigma > 0$. By taking σ sufficiently small, every compact subset of $\mathbb{R}^n\setminus\{0\}$ eventually lies in the region where $\psi \equiv 1$. Thus it suffices to establish the $\mathcal{E}^\infty_{\mathrm{M}}$-estimates (2.4) for ψQ_ε. Its Fourier transform equals

$$\mathcal{F}(\psi Q_\varepsilon) = \int \frac{\chi_\varepsilon(\xi-\eta)}{P_\varepsilon(\xi-\eta)}\widehat{\psi}(\eta)\,d\eta = \sum_{|\beta|=q}\frac{1}{\beta!}\int \partial^\beta\Big(\frac{\chi_\varepsilon}{P_\varepsilon}\Big)(\theta)\,\eta^\beta\widehat{\psi}(\eta)\,d\eta$$

for every $q \ge 1$, where θ lies between ξ and $\xi-\eta$. This follows by Taylor expansion and observing that $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^n)$ has all its moments vanishing. By Lemma 5.1 and Peetre's inequality we have for $|\beta| = q$ that

$$\big|\partial^\beta\Big(\frac{\chi_\varepsilon}{P_\varepsilon}\Big)(\theta)\big| \le C_q\,\varepsilon^{-N}\big(1+|\theta|\big)^{-m-q} \le C'_q\,\varepsilon^{-N}\big(1+|\xi|\big)^{-m-q}\big(1+|\eta|\big)^{m+q}.$$

Let $\alpha \in \mathbb{N}_0^n$ and choose q large enough so that $|\alpha| - m - q < -n$. Then

$$\xi^\alpha\mathcal{F}\big(\psi Q_\varepsilon\big)(\xi) \in L^1(\mathbb{R}^n), \quad \partial^\alpha\big(\psi Q_\varepsilon\big)(x) \in L^\infty(\mathbb{R}^n)$$

and

$$\|\partial^\alpha(\psi Q_\varepsilon)\|_{\mathrm{L}^\infty} \le C'_q\,\varepsilon^{-N}\int \big(1+|\eta|\big)^{m+q}|\eta|^q|\widehat{\psi}(\eta)|\,d\eta \int |\xi|^{|\alpha|-m-q}\,d\xi\,.$$

This proves that $\psi Q_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n)$ with all derivatives satisfying a bound of order ε^{-N}. □

Theorem 5.5. *Let the operator $P(D) = \sum_{|\gamma|\le m} a_\gamma D^\gamma$, with $a_\gamma \in \widetilde{\mathbb{C}}$, be WH-elliptic with radius r_ε of slow scale. Let $f \in \mathcal{G}^\infty(\Omega)$ and $u \in \mathcal{G}(\Omega)$ be a solution to $P(D)u = f$. Then $u \in \mathcal{G}^\infty(\Omega)$.*

Proof. Let $\omega \Subset \Omega$ and choose $\varphi \in \mathcal{D}(\Omega)$, $\varphi \equiv 1$ on ω. Then $P(D)(\varphi u) = \varphi f + v$, where $v \equiv 0$ on ω and v has compact support in Ω. It suffices to show that φu enjoys the $\mathcal{G}^\infty$-property on every compact set $K \subset \omega$. We have that (see (5.1))

$$\begin{aligned}
\varphi u_\varepsilon &= P_\varepsilon(D)Q_\varepsilon * (\varphi u_\varepsilon) + h_\varepsilon * (\varphi u_\varepsilon)\\
&= Q_\varepsilon * (\varphi f_\varepsilon + v_\varepsilon) + h_\varepsilon * (\varphi u_\varepsilon)\\
&= Q_\varepsilon * (\varphi f_\varepsilon) + (\psi Q_\varepsilon) * v_\varepsilon + \big((1-\psi)Q_\varepsilon\big) * v_\varepsilon + h_\varepsilon * (\varphi u_\varepsilon)\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}
\end{aligned}$$

where ψ is a cut-off supported in a small neighborhood of zero, $\psi(0) = 1$. We shall prove the $\mathcal{G}^\infty$-property of each term on K.

For term (I) we have

$$\|\partial^\alpha\big(Q_\varepsilon * (\varphi f_\varepsilon)\big)\|_{\mathrm{L}^\infty} \le C\varepsilon^{-N}\|\partial^\alpha(\varphi f_\varepsilon)\|_{W^{s,\infty}}$$

by Lemma 5.3, when $s > n/2$. But $f \in \mathcal{G}^\infty(\Omega)$, so the latter term has a bound of order $\varepsilon^{-N'}$ independently of α. Concerning term (II), we may choose the support of the cut-off ψ so small that this term actually vanishes on K.

Coming to term (III), we have by Proposition 5.4 that $(1-\psi)Q$ belongs to $\mathcal{G}^\infty(\mathbb{R}^n)$. To estimate $\partial^\alpha\big((1-\psi)Q_\varepsilon\big) * v_\varepsilon$, we let all the derivatives fall on the first factor, observe that $v_\varepsilon \in \mathrm{C}(\overline{\omega})$ and evoke the continuity of the convolution map $\mathrm{C}(\mathbb{R}^n) \times \mathrm{C}(\overline{\omega}) \to \mathrm{C}(\mathbb{R}^n)$ to conclude that $\partial^\alpha\big((1-\psi)Q_\varepsilon\big) * v_\varepsilon$ has a bound of order ε^{-N} independently of α, uniformly on each compact subset of $\mathbb{R}^n$. Finally, term (IV) is treated by the same argument, observing that $h \in \mathcal{G}^\infty(\mathbb{R}^n)$ by Lemma 5.2. □

Remark 5.6. The operator $P(\xi) = -a\xi^2 + 1$ from Example 4.3 was shown to be WH-elliptic with radius $r_\varepsilon = s/a_\varepsilon, s > 1$. From Theorem 5.5 and the explicit solution of the homogeneous equation we may now assert that it is $\mathcal{G}^\infty$-hypoelliptic if and only if r_ε is slow scale. This once again emphasizes the importance of the slow scale property.

Remark 5.7. If $P(D)$ is W-elliptic with radius r_ε, then all real roots of $P_\varepsilon(\xi) = 0$ lie within the radius r_ε. Let $m_\varepsilon = \max\{|\xi| : P_\varepsilon(\xi) = 0, \xi \in \mathbb{R}^n\}$. Since $m_\varepsilon \le r_\varepsilon$, a necessary requirement for the conditions of Theorem 5.5 to hold is that m_ε is slow scale. If m_ε is moderate, the slow scale property is also necessary for the solutions of $P(D)u = 0$ to belong to $\mathcal{G}^\infty$, as is seen from the solutions $u_\varepsilon(x) = \exp(ix\xi_\varepsilon)$ where $P_\varepsilon(\xi_\varepsilon) = 0$.

The remainder of this section is devoted to second order operators

$$P(D) = \sum_{i,j=1}^n a_{ij}D_iD_j + \sum_{j=1}^n b_jD_j + c\,,$$

in which case we have a regularity result under somewhat different assumptions than in Theorem 5.5. For its generalized constant coefficients, we assume that $a_{ij} \in \widetilde{\mathbb{R}}$, $b_j, c \in \widetilde{\mathbb{C}}$. Such an operator is called *G-elliptic* (for *generalized elliptic*), if the matrix $A = (a_{ij})_{i,j} \in \widetilde{\mathbb{R}}^{n^2}$ is symmetric and positive definite. Thus all eigenvalues $\lambda_1, \dots, \lambda_n \in \widetilde{\mathbb{R}}$ of A are invertible and nonnegative. Employing $\mathbb{R}$-linear algebra on representatives at fixed $\varepsilon > 0$, one sees that there is an orthogonal matrix Q with coefficients in $\widetilde{\mathbb{R}}$ such that

$$A = Q^T \Lambda Q, \quad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n).$$

Define

$$\beta_i = \sum_{j=1}^{n} Q_{ij} b_j,$$

and, putting $\gamma = c - \sum_{j=1}^{n} \beta_j^2 / 4\lambda_j$, let

$$\tilde{\lambda}_i = \begin{cases} \lambda_i & \text{if } \gamma = 0, \\ \lambda_i / |\gamma| & \text{if } \gamma \text{ is invertible.} \end{cases}$$

The case of non-zero, non-invertible γ is outside the scope of the following result.

Proposition 5.8. *Let $P(D)$ be a second order G-elliptic operator and assume that γ is either equal to zero or invertible. Let $f \in \mathcal{G}^\infty(\mathbb{R}^n)$ and let $u \in \mathcal{G}(\mathbb{R}^n)$ be a solution to $P(D)u = f$. If β_j/λ_j is of log-type and $\tilde{\lambda}_i$, $\tilde{\lambda}_i^{-1}$ is slow scale, $i, j = 1, \dots, n$, then $u \in \mathcal{G}^\infty(\mathbb{R}^n)$.*

Proof. The proof proceeds by stepwise reduction to the Laplace- or Helmholtz equation. First, we change the independent variable to $y = Qx$ and define $v(y) = u(Q^T y)$, $g(y) = f(Q^T y)$, that is, we define representatives $v_\varepsilon(y) = u_\varepsilon(Q_\varepsilon^T y)$ and $g_\varepsilon(y) = f\varepsilon(Q_\varepsilon^T y)$. The columns of Q and Q^T have length 1, so v is moderate if and only if u is moderate, and $u \in \mathcal{G}^\infty(\mathbb{R}^n)$ if and only if $v \in \mathcal{G}^\infty(\mathbb{R}^n)$; the same holds of f and g. Further,

$$P(D)u = f$$

is equivalent to

$$\sum_{i=1}^{n} \lambda_i D_i^2 v + \sum_{j=1}^{n} \beta_j D_j v + cv = g \tag{5.3}$$

in $\mathcal{G}(\mathbb{R}^n)$ with respect to the new variables $y = Qx$. We may rewrite (5.3) to

$$\sum_{i=1}^{n} \lambda_i \big(D_i + \frac{\beta_i}{2\lambda_i}\big)^2 v + \big(c - \sum_{j=1}^{n} \frac{\beta_j^2}{4\lambda_j}\big) v = g \tag{5.4}$$

and put

$$w(y) = \prod_{j=1}^{n} \exp\big(-i \frac{\beta_j}{2\lambda_j} y_j\big) v(y).$$

If β_j/λ_j is of log-type, this transformation respects moderateness, and so (5.4) is equivalent to

$$\sum_{i=1}^{n} \lambda_i D_i^2 w(y) + \gamma w(y) = h(y)$$

in $\mathcal{G}(\mathbb{R}^n)$, where $h(y) = \prod_{j=1}^{n} \exp(-i \frac{\beta_j}{2\lambda_j} y_j) g(y)$. Further, if β_j/λ_j is of log-type, then $v \in \mathcal{G}^\infty(\mathbb{R}^n)$ iff $w \in \mathcal{G}^\infty(\mathbb{R}^n)$, and similarly for g and h.

Let $\tilde{h}(y) = h(y)/|\gamma|$ if γ is invertible, and $\tilde{h}(y) = h(y)$ if $\gamma = 0$. According to the hypotheses of the proposition, we may rewrite the latter equation as

$$\sum_{i=1}^{n} \tilde{\lambda}_i D_i^2 w(y) + \sigma w(y) = \tilde{h}(y)$$

where $\sigma = 0$ or else $|\sigma| = 1$. Finally, we put

$$\tilde{w}(y_1, \dots, y_n) = w(\sqrt{\tilde{\lambda}_1} y_1, \dots, \sqrt{\tilde{\lambda}_n} y_n),$$
$$\tilde{\tilde{h}}(y_1, \dots, y_n) = \tilde{h}(\sqrt{\tilde{\lambda}_1} y_1, \dots, \sqrt{\tilde{\lambda}_n} y_n),$$

and arrive at the equation

$$-\Delta \tilde{w}(y) + \sigma \tilde{w}(y) = \tilde{\tilde{h}}(y).$$

As above, the hypotheses imply that $w \in \mathcal{G}^\infty(\mathbb{R}^n)$ iff $\tilde{w} \in \mathcal{G}^\infty(\mathbb{R}^n)$, and the same holds for $\tilde{h}$ and $\tilde{\tilde{h}}$.

Collecting everything, we see that $f \in \mathcal{G}^\infty(\mathbb{R}^n)$ iff $\tilde{\tilde{h}} \in \mathcal{G}^\infty(\mathbb{R}^n)$. But the operator $|\xi|^2 + \sigma$ is clearly SH-elliptic (with radius r = 1), so Theorem 5.5 shows that $\tilde{w} \in \mathcal{G}^\infty(\mathbb{R}^n)$ which in turn implies that $u \in \mathcal{G}^\infty(\mathbb{R}^n)$ as desired. □

Example 5.9. The second order homogeneous ODE

$$\big(\lambda \frac{d^2}{dx^2} + b \frac{d}{dx} + c\big) u(x) = 0$$

has the solution $u(x) = C_1 \exp(\mu_+ x) + C_2 \exp(\mu_- x)$, where

$$\mu_\pm = -\frac{b}{2\lambda} \pm \frac{1}{\sqrt{\lambda}} \sqrt{\frac{b^2}{4\lambda} - c}\,.$$

$(u_\varepsilon)_\varepsilon$ is moderate and thus defines an element of $\mathcal{G}(\mathbb{R})$ if either $\mu_\pm$ has nonzero real part and is of log-type or else if its real part is zero and it is slow scale. In both cases, the solution belongs to $\mathcal{G}^\infty(\mathbb{R})$. This illustrates the hypotheses required in Proposition 5.8.

Remark 5.10. The conditions stated in Theorem 5.5 and Proposition 5.8 are independent (for second order G-elliptic operators) as can be seen from the following examples:

First, the operator $P(\xi) = a^2 \xi_1^2 + \xi_2^2$ discussed in Example 4.2, with $a_\varepsilon = 1/|\log(\varepsilon)|$, is not WH-elliptic, so Theorem 5.5 does not apply, but Proposition 5.8 does, and so the operator is $\mathcal{G}^\infty$-hypoelliptic.

Second, the operator $P(\xi) = a^2 \xi^2 + 1$ was seen to be SH-elliptic in Remark 4.5, so it is $\mathcal{G}^\infty$-hypoelliptic by Theorem 5.5. However, if $a_\varepsilon = \varepsilon$ or any positive power thereof, it does not satisfy the log-type property required in Proposition 5.8. The corresponding homogeneous differential equation $(-\varepsilon^2 \partial_x^2 + 1) u_\varepsilon(x) = 0$ has the solution $u_\varepsilon(x) = C_1 \exp(x/\varepsilon) + C_2 \exp(-x/\varepsilon)$, which does not yield an element of $\mathcal{G}^\infty(\mathbb{R})$. This does not contradict the $\mathcal{G}^\infty$-hypoellipticity of the operator $P(D)$, because $(u_\varepsilon)_\varepsilon$ is not moderate and so does not represent a solution in $\mathcal{G}(\mathbb{R})$. We will take up this observation in Example 5.11 below.

If the coefficient γ entering the hypotheses of Proposition 5.8 is a zero divisor, the operator $P(\xi) = \xi^2 + \gamma$ may or may not be $\mathcal{G}^\infty$-hypoelliptic. It is so, if $\gamma \geq 0$ (similar to Remark 4.5). It is not, if $\gamma_\varepsilon \in \{0, -1/\varepsilon^2\}$ in an interlaced way as $\varepsilon \to 0$ (following Example 4.3).

Example 5.11. The regularity result of Theorem 5.5 can be used to prove nonexistence of solutions: For example, let $a \in \widetilde{\mathbb{R}}$ be the class of $a_\varepsilon = \varepsilon$. Then the homogeneous equation

$$\big(- a^2 \frac{d^2}{dx^2} + 1\big) u(x) = 0 \tag{5.5}$$

has no nontrivial solution in $\mathcal{G}(\Omega)$ on whatever open subset $\Omega \subset \mathbb{R}$. To see this, assume there is a solution, with representative $(u_\varepsilon)_\varepsilon$, say. Then

$$\big(- \varepsilon^2 \frac{d^2}{dx^2} + 1\big) u_\varepsilon(x) = n_\varepsilon(x)$$

for some $(n_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$. It follows that

$$u''_\varepsilon = \frac{1}{\varepsilon^2}\big(u_\varepsilon - n_\varepsilon\big)\,, \quad u^{(4)}_\varepsilon = \frac{1}{\varepsilon^2}\big(u''_\varepsilon - n''_\varepsilon\big) = \frac{1}{\varepsilon^4}\big(u_\varepsilon - n_\varepsilon\big) - \frac{1}{\varepsilon^2} n''_\varepsilon\,,$$

and so on, so that

$$\big(u^{(2\alpha)}_\varepsilon - \frac{1}{\varepsilon^{2\alpha}} u_\varepsilon\big)_\varepsilon \in \mathcal{N}(\Omega)$$

for every $\alpha \in \mathbb{N}$. On the other hand, the operator in (5.5) is SH-elliptic, hence $\mathcal{G}^\infty$-hypoelliptic. Therefore, there is $N \geq 0$ such that $u^{(2\alpha)}_\varepsilon = O(\varepsilon^{-N})$ for every $\alpha \in \mathbb{N}$. It follows that $u_\varepsilon = O(\varepsilon^{-N+2\alpha})$ for every $\alpha \in \mathbb{N}$, whence $(u_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$ and $u = 0$ in $\mathcal{G}(\Omega)$. Observe that the nonexistence result depends crucially on the asymptotic properties of the generalized coefficient a^2. If we take $a \in \widetilde{\mathbb{R}}$ invertible such that $1/a$ is of logarithmic type then equation 5.5 is nontrivially solvable (see Example 5.9).

6. Micro-elliptic regularity for first order operators with variable coefficients of slow scale

For partial differential operators with smooth coefficients elliptic regularity can be considered a special instance of microlocal non-characteristic regularity. This is expressed in terms of a general relation combining the wave front sets of the distributional solution and of the right-hand side in the PDE with the characteristic set of the operator. It states that for any partial differential operator $P(x, D)$ with C^∞ coefficients and distribution u we have the following inclusion relation:

$$\mathrm{WF}(u) \subseteq \mathrm{WF}(Pu) \cup \mathrm{Char}(P) \tag{6.1}$$

(for subsets of $T^*(\Omega)\setminus 0$, the cotangent space over Ω with the zero section removed.) Recall that $\mathrm{Char}(P) = P_m^{-1}(0) \cap T^*(\Omega) \setminus 0$ and thus depends only on the principal symbol of the operator.

The concept of microlocal regularity of a Colombeau function follows the classical idea of adding information about directions of rapid decrease in the frequency domain upon localization in space (cf. [8, 10, 14]). It refines $\mathcal{G}^\infty$ regularity in the same way as the distributional wave front set does with C^∞ regularity, i.e., the projection of the (generalized) wave front set into the base space equals the (generalized) singular support.

We briefly recall the definition of the generalized wave front set of a Colombeau function: $u \in \mathcal{G}(\Omega)$ is said to be *microlocally regular* at $(x_0, \xi_0) \in T^*(\Omega) \setminus 0$ if (for a representative $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{\mathrm{M}}(\Omega)$) there is an open neighborhood U of x_0 and a conic neighborhood Γ of ξ_0 such that for all $\varphi \in \mathcal{D}(U)$ it holds that $\mathcal{F}(\varphi u)$ is rapidly decreasing in Γ, i.e., $\exists N \in \mathbb{R}\ \forall l \in \mathbb{N}_0\ \exists C > 0\ \exists \varepsilon_0 > 0$:

$$|(\varphi u_\varepsilon)\widehat{\ }(\xi)| \le C\varepsilon^{-N}(1+|\xi|)^{-l} \quad \forall \xi \in \Gamma, \forall \varepsilon \in (0,\varepsilon_0). \tag{6.2}$$

(Here, $\widehat{\ }$ denotes classical Fourier transform and $\mathcal{F}(\varphi u)$ the corresponding generalized Fourier transform of the compactly supported Colombeau function φu.) The *generalized wave front set* of u, denoted $\mathrm{WF}_g(u)$, is defined as the complement (in $T^*(\Omega) \setminus 0$) of the set of pairs (x_0, ξ_0) where u is microlocally regular.

Remark 6.1. (i) Since $\mathcal{F}(\varphi u)$ is temperate it suffices to require an estimate of the form (6.2) for all $\xi \in \Gamma$ with $|\xi| \ge r_\varepsilon$ where $(r_\varepsilon)_\varepsilon$ is of slow scale. Indeed, there are $M \in \mathbb{R}$ and $\varepsilon_1 > 0$ such that

$$|(\varphi u_\varepsilon)\widehat{\ }(\xi)| \le \varepsilon^{-M}(1+|\xi|)^{M} \quad \forall \xi \in \mathbb{R}^n, \forall \varepsilon \in (0,\varepsilon_1).$$

When $|\xi| \le r_\varepsilon$ the right-hand side is bounded as follows:

$$\varepsilon^{-M}(1+|\xi|)^M \le \varepsilon^{-M}(1+r_\varepsilon)^{M+l}(1+|\xi|)^{-l} \le \varepsilon^{-M-1}(1+|\xi|)^{-l}.$$

Hence taking $\max(M+1,N)$, $\min(\varepsilon_0,\varepsilon_1)$ as new N, ε_0 one arrives at (6.2) valid for all $\xi \in \Gamma$.
(ii) If $v \in \mathcal{G}_c(\Omega)$ and $(r_\varepsilon)_\varepsilon$ is not of slow scale then the rapid decrease property (6.2) does not follow from the corresponding estimates in the regions $|\xi| \ge r_\varepsilon$. Consider, for example, a model delta net $v_\varepsilon = \rho(./\varepsilon)/\varepsilon$ where $\rho \in \mathcal{D}(\mathbb{R})$. Then $\widehat{v_\varepsilon}$ is not rapidly decreasing in $\xi > 0$ and $\xi < 0$ but

$$\begin{aligned}|\widehat{v_\varepsilon}(\xi)| = |\widehat{\rho}(\varepsilon\xi)| = |\int e^{-i\varepsilon x\xi}\rho(x)\,dx| = (\varepsilon|\xi|)^{-l}|\int e^{-i\varepsilon x\xi}\rho^{(l)}(x)\,dx| \\ \le C_l(\varepsilon|\xi|)^{-l} = C_l(\varepsilon\sqrt{|\xi|})^{-l}|\xi|^{-l/2} \le C_l|\xi|^{-l/2} \quad \text{if } |\xi| \ge \frac{1}{\varepsilon^2} =: r_\varepsilon.\end{aligned}$$

In view of the examples in Section 4 one cannot expect to obtain an extension of (6.1) to arbitrary operators with $\mathcal{G}^\infty$-coefficients by designing a notion of generalized characteristic set based solely on the principal part. In general, the lower order terms of the symbol do have an effect on the regularity properties, even for smooth principal part: consider the symbol $P_\varepsilon(\xi) = \xi - 1/\varepsilon$ whose corresponding operator admits the non-regular solution $u_\varepsilon(x) = \exp(ix/\varepsilon)$ to the homogeneous equation. However, in case of first order operators with variable coefficients a direct approach shows sufficiency of two further assumptions to restore a microlocal regularity relation of the type (6.1). Both are requirements of slow scale: One about regularity of the coefficients and the other in terms of lower bounds on the principal symbol over conic regions.

An identification of adequate conditions in the general case, in particular, finding an appropriate notion of characteristic set of the operator, remains open. As suggested by the results and examples of Sections 3 and 4 the latter might have to include the influence of lower order terms in an essential way.

We first introduce an auxiliary notion to replace $\mathrm{Char}(P)$ in our variant of relation (6.1) for first order operators.

Definition 6.2. Let $P(x,D)$ be a partial differential operator of order m with coefficients in $\mathcal{G}(\Omega)$ and let $(x_0,\xi_0) \in T^*(\Omega) \setminus 0$. P is said to be *W-elliptic with slow scales* at (x_0,ξ_0), *W_{sc}-elliptic* in short, if for some representative $(P_\varepsilon(x,\xi))_\varepsilon$

the following holds: there is an open neighborhood U of x_0, a conic neighborhood Γ of ξ_0, slow scale nets $(s_\varepsilon)_\varepsilon$, $(r_\varepsilon)_\varepsilon$ with $s_\varepsilon > 0$, $r_\varepsilon > 0$, and $\varepsilon_0 > 0$ such that

$$|P_\varepsilon(x,\xi)| \geq s_\varepsilon^{-1}(1+|\xi|)^m \tag{6.3}$$

for all $x \in U$, $\xi \in \Gamma$, $|\xi| \geq r_\varepsilon$, $\varepsilon \in (0,\varepsilon_0)$. We denote by W_{sc}-Ell(P) the subset of pairs in $T^*(\Omega) \setminus 0$ where P is W_{sc}-elliptic.

Remark 6.3. (i) In the theory of propagation of singularities for operators with smooth coefficients it is the complement of the ellipticity set, namely the characteristic set Char(P), which plays a dominant role. In view of the remarks at the beginning of this section and the variety of ellipticity notions used in the Colombeau context so far we refrain from introducing (yet another) notion of *generalized characteristic set*. Finding an appropriate definition for a theory of propagation of singularities for operators with nonsmooth coefficients is the subject of ongoing and future research. For the purpose of the present paper we prefer the notation W_{sc}-Ell$(P)^{\mathrm{c}}$, the complement of W_{sc}-Ell(P) in $T^*(\Omega) \setminus 0$.
(ii) Clearly, W_{sc}-ellipticity implies W-ellipticity. The symbol $P_\varepsilon(\xi) = \varepsilon\xi$, which is SH-elliptic, shows that the converse does not hold.

Example 6.4. (i) $P_\varepsilon(\xi) = \xi - 1/\varepsilon$ gives W_{sc}-Ell$(P)^{\mathrm{c}} = \mathbb{R}\times\mathbb{R}\setminus 0$ but W_{sc}-Ell$(P_1)^{\mathrm{c}} = \emptyset$, whereas $Q_\varepsilon(\xi) = \xi - \log(1/\varepsilon)$ yields W_{sc}-Ell$(Q)^{\mathrm{c}} = \emptyset$. Slightly more general, let $(r_\varepsilon)_\varepsilon$, $(s_\varepsilon)_\varepsilon$ be slow scale nets, $|r_\varepsilon| > 0$, then $P_\varepsilon(\xi) = r_\varepsilon^{-1}\xi - s_\varepsilon$ is W_{sc}-elliptic at every $(x_0,\xi_0) \in \mathbb{R}\times\mathbb{R}\setminus 0$.
(ii) For the operator $P_\varepsilon(\xi) = \varepsilon\xi_1 + i\xi_2$ (from Example 4.1) we obtain W_{sc}-Ell$(P)^{\mathrm{c}} = \mathbb{R}^2 \times (\mathbb{R}\times\{0\}) \setminus 0$ since (6.3) is valid with constant s_ε and r_ε in any cone $|\xi_2| \geq c|\xi_1| > 0$. We observe that the wave front set of the solution u to $Pu = 0$ is a subset of W_{sc}-Ell$(P)^{\mathrm{c}}$. Indeed, we have $u_\varepsilon(x_1,x_2) = \exp(ix_1/\varepsilon)\cdot\exp(-x_2)$ and application of [10, Lemma 5.1] proves the inclusion.

Note that $\operatorname{sing\,supp}_g(u) = \mathbb{R}^2$ (by direct inspection of derivatives) but the inclusion $\mathrm{WF}_g(u) \subset \mathrm{W}_{\mathrm{sc}}$-Ell$(P)^{\mathrm{c}}$ is nevertheless strict since $f_\varepsilon(x_1) = \exp(ix_1/\varepsilon)$ has the 'half-sided' wave front set $\mathrm{WF}_g(f) = \mathbb{R}\times\mathbb{R}_+$. (To see this one easily checks that $(\varphi\cdot\exp(i./\varepsilon))\widehat{\ }(\xi) = \widehat{\varphi}(\xi - 1/\varepsilon)$ is rapidly decreasing if and only if $\xi < 0$.)
(iii) Let $P_\varepsilon(x,y,\partial_x,\partial_y) = \partial_y - a_\varepsilon(x,y)\partial_x$ where $a_\varepsilon \in \mathcal{E}_{\mathrm{M}}(\mathbb{R}^2)$ is real-valued and bounded uniformly with respect to ε. Put $c_1 = \inf_{x,y,\varepsilon} a_\varepsilon$, $c_2 = \sup_{x,y,\varepsilon} a_\varepsilon$ then any pair $((x_0,y_0),(\xi_0,\eta_0)) \in \mathbb{R}^2\times\mathbb{R}^2\setminus 0$ with $\eta_0 \notin [c_1,c_2]\cdot\{\xi_0\}$ is a point of W_{sc}-ellipticity of P.

The second slow scale condition used in the theorem to follow is introduced as a strong regularity property of the prospective coefficients in the operators.

Definition 6.5. $v \in \mathcal{G}(\Omega)$ is said to be of *slow scale* if it has a representative $(v_\varepsilon)_\varepsilon \in \mathcal{E}_{\mathrm{M}}(\Omega)$ with the following property: $\forall K \Subset \Omega\ \forall\alpha \in \mathbb{N}_0^n\ \exists$ slow scale net $(r_\varepsilon)_\varepsilon$, $r_\varepsilon > 0\ \exists\varepsilon_0 > 0$ such that

$$|\partial^\alpha v_\varepsilon(x)| \leq r_\varepsilon \quad \forall x \in K, \forall\varepsilon \in (0,\varepsilon_0). \tag{6.4}$$

Remark 6.6. (i) Any Colombeau function of slow scale is in $\mathcal{G}^\infty$ but clearly the converse does not hold.
(ii) Examples of functions of slow scale are obtained by logarithmically scaled embeddings: If $v \in \mathcal{S}'$, $\rho \in \mathcal{S}$ is a mollifier and we put $\rho^\varepsilon(x) := (\log(1/\varepsilon))^n \cdot \rho(\log(1/\varepsilon)x)$ then $v_\varepsilon := v * \rho^\varepsilon$ defines the Colombeau function $v = [(v_\varepsilon)_\varepsilon]$ which is of slow scale. This occurs in applications, e.g., when considering hyperbolic PDEs with discontinuous or nonsmooth coefficients ([9, 12, 15]).

Theorem 6.7. *Let $P(x,D)$ be a first order partial differential operator with coefficients of slow scale. Then we have for any $u \in \mathcal{G}(\Omega)$*

$$\mathrm{WF}_g(u) \subseteq \mathrm{WF}_g(Pu) \cup W_{sc}\text{-}Ell(P_1)^c. \tag{6.5}$$

Remark 6.8. (i) We point out that the zero order terms of the symbol do not appear in the determination of $\mathrm{W_{sc}\text{-}Ell}(P_1)^{\mathrm{c}}$. In this respect (6.5) is closer to the classical relation (6.1) than can be expected in more general situations. If $\mathrm{W_{sc}\text{-}Ell}(P_1)^{\mathrm{c}} = \emptyset$ this can be considered a special version of an elliptic regularity result not covered by Theorem 5.5 above.
(ii) In Example 6.4, (ii) above the inclusion relation (6.5) is strict. On the other hand, as trivial examples like $P = 1$ and $P = \frac{d}{dx}$ show we may also have equality in (6.5).

The proof of Theorem 6.7 will be based on an integration by parts technique as in the classical regularization of oscillatory integrals. In case of Colombeau functions regularity is coupled to the asymptotic behavior with respect to ε. Thus we have to carefully observe the interplay of this parameter with the spatial variables when estimating Fourier integrals. The following auxiliary result will be useful in this context.

Lemma 6.9. *Let $v \in \mathcal{G}(\Omega)$ be microlocally regular at $(x_0,\xi_0) \in T^*(\Omega)\setminus 0$ with conic neighborhood $U \times \Gamma$ such that (6.2) holds. Let M be a set and assume that $(g^\nu)_{\nu\in M}$ is a family of Colombeau functions $g^\nu \in \mathcal{G}(\Omega)$ with the following properties:*

1. *$\exists S \Subset U$ $\forall \nu \in M$: $\mathrm{supp}_g(g^\nu) \subseteq S$.*
2. *$(g^\nu)_{\nu\in M}$ satisfies a uniform $\mathcal{G}^\infty$-property, i.e., $\forall K \Subset \Omega$ $\forall \alpha \in \mathbb{N}_0^n$ $\exists q \geq 0$ $\exists \varepsilon_0 > 0$ such that*

$$|\partial^\alpha g^\nu_\varepsilon(x)| \leq \varepsilon^{-q} \quad \forall x \in K, \forall \nu \in M, \forall \varepsilon \in (0,\varepsilon_0).$$

Then $\mathcal{F}(g^\nu v)$ is rapidly decreasing in Γ uniformly with respect to $\nu \in M$. To be more precise, for any choice of $\psi \in \mathcal{D}(U)$ with $\psi = 1$ on S there is a real number N', depending only on ψ and v, such that $\mathcal{F}(g^\nu v)$ satisfies (6.2) with $C = 1$ and uniform ε-power $-N'$ for all $\nu \in M$.

Remark 6.10. *Note that condition (2) is satisfied in particular if each derivative of g^ν has slow scale bounds uniformly with respect to $\nu \in M$.*

Proof. The idea of the following proof is to view this as a special case of Theorem 3.1 in [10] when ν is fixed. However, to determine the precise ε-growth we have to refine the estimates along the way appropriately.

First, we note that if Γ_0 is a closed conic neighborhood of ξ_0 such that $\Gamma_0 \subseteq \Gamma \cup \{0\}$ then one can find $c > 0$ with the following property: $\xi \in \Gamma_0$, $\eta \in \Gamma^c \Rightarrow |\xi - \eta| \geq c|\eta|$. (See [7, proof of Lemma 8.1.1] or [10, Lemma 3.1 (i)] for details.)

Let $\xi \in \Gamma_0$ and consider

$$|(g^\nu_\varepsilon v_\varepsilon)\widehat{\ }(\xi)| = |(g^\nu_\varepsilon \psi v_\varepsilon)\widehat{\ }(\xi)| = |\widehat{g^\nu_\varepsilon} * (\psi v_\varepsilon)\widehat{\ }(\xi)| \leq \int |\widehat{g^\nu_\varepsilon}(\xi-\eta)||(\psi v_\varepsilon)\widehat{\ }(\eta)|\, d\eta.$$

We split the integration into two parts according to the cases $\eta \in \Gamma$ and $\eta \in \Gamma^c$.

From the exchange formula $\eta^\alpha \widehat{g^\nu_\varepsilon}(\eta) = (\widehat{D^\alpha g^\nu_\varepsilon})(\eta)$ we deduce that the (global) rapid decrease estimates for $\widehat{g^\nu_\varepsilon}$ are uniform with respect to ν. This and condition (2) yields that $\forall p \in \mathbb{N}_0$ $\exists \varepsilon_0 > 0$:

$$|\widehat{g^\nu_\varepsilon}(\zeta)| \leq \varepsilon^{-q}(1+|\zeta|)^{-p} \quad \forall \zeta \in \mathbb{R}^n, \forall \varepsilon \in (0,\varepsilon_0), \forall \nu \in M. \tag{6.6}$$

Integrating over $\eta \in \Gamma$ we have rapid decrease of $|(\psi v_\varepsilon)\hat{\ }(\eta)|$ and hence the integrand is bounded as follows for some $N \in \mathbb{N}_0$ and for all $l, k \in \mathbb{N}_0$ and some $\varepsilon_0 > 0$:

$$|\widehat{g_\varepsilon^\nu}(\xi - \eta)||(\psi v_\varepsilon)\hat{\ }(\eta)| \le \varepsilon^{-q-N}(1 + |\xi - \eta|^2)^{-k/2}(1 + |\eta|^2)^{-l/2} \quad \forall \eta \in \Gamma, \forall \varepsilon \in (0, \varepsilon_0).$$

Peetre's inequality gives $(1 + |\xi - \eta|^2)^{-k/2} \le 2^{k/2}(1 + |\xi|^2)^{-k/2}(1 + |\eta|^2)^{k/2}$ and we obtain for all $k \in \mathbb{N}_0$

$$\begin{aligned}\int_\Gamma &|\widehat{g_\varepsilon^\nu}(\xi - \eta)||(\psi v_\varepsilon)\hat{\ }(\eta)|\, d\eta \\ &\le \varepsilon^{-N-q}\, 2^{k/2}(1 + |\xi|^2)^{-k/2} \int (1 + |\eta|^2)^{(k-l)/2}\, d\eta \le \varepsilon^{-N-q-1}(1 + |\xi|)^{-k} \quad (6.7)\end{aligned}$$

when $\varepsilon \in (0, \varepsilon_1)$, ε_1 small enough, and for all $\nu \in M$, if $l > k + n$.

In the integral over $\eta \in \Gamma^c$ we use the facts that $(\psi v_\varepsilon)\hat{\ }$ is temperate (in the Colombeau sense) and the cone inequality $|\xi - \eta| \ge c|\eta|$. There is $M \in \mathbb{N}_0$ and $\varepsilon_2 > 0$ such that

$$|(\psi v_\varepsilon)\hat{\ }(\eta)| \le \varepsilon^{-M}(1 + |\eta|^2)^{M/2} \quad \forall \eta \in \mathbb{R}^n, \forall \varepsilon \in (0, \varepsilon_2), \tag{6.8}$$

which in combination with (6.6) gives the following upper bound of the integrand for $k, l \in \mathbb{N}_0$ arbitrary, $\varepsilon \in (0, \varepsilon_3)$, ε_3 small enough, and some $C' > 0$:

$$\begin{aligned}|\widehat{g_\varepsilon^\nu}(\xi - \eta)||(\psi v_\varepsilon)\hat{\ }(\eta)| &\le \varepsilon^{-q-M}(1 + |\xi - \eta|^2)^{-k/2-l/2}(1 + |\eta|^2)^{M/2} \\ &\le \varepsilon^{-q-M} C'(1 + |\xi|)^{-k}(1 + |\eta|)^{M+k-l}\end{aligned}$$

where we have applied Peetre's inequality to the factor $(1 + |\xi - \eta|^2)^{-k/2}$ and the cone inequality in estimating the factor $(1 + |\xi - \eta|^2)^{-l/2}$. Thus we obtain

$$\begin{aligned}\int_{\Gamma^c} &|\widehat{g_\varepsilon^\nu}(\xi - \eta)||(\psi v_\varepsilon)\hat{\ }(\eta)|\, d\eta \\ &\le C'\varepsilon^{-M-q}(1 + |\xi|)^{-k} \int (1 + |\eta|)^{M+k-l}\, d\eta \le \varepsilon^{-M-q-1}(1 + |\xi|)^{-k} \quad (6.9)\end{aligned}$$

if $l > M + k + n$, $\varepsilon \in (0, \varepsilon_4)$, ε_4 small enough, and for all $\nu \in M$. Combining (6.7) and (6.9) we have shown that for all $k \in \mathbb{N}_0$, $\varepsilon_5 := \min(\varepsilon_1, \varepsilon_4)$

$$|(g_\varepsilon^\nu v_\varepsilon)\hat{\ }(\xi)| \le (1 + |\xi|)^{-k}(\varepsilon^{-N-q-1} + \varepsilon^{-M-q-1}) \quad \forall \xi \in \Gamma_0, \forall \nu \in M, \forall \varepsilon \in (0, \varepsilon_5).$$

Since Γ_0 was an arbitrary closed conic neighborhood of ξ_0 in $\Gamma \cup \{0\}$ we may put $N' = \max(N, M) + q + 1$ and the Lemma is proved. □

Proof of Theorem 6.7. Let $(x_0, \xi_0) \in T^*(\Omega) \setminus 0$ be in the complement of the right-hand side of (6.5) and choose $U \ni x_0$ open, $\Gamma \ni \xi_0$ a conic and closed neighborhood such that both defining properties of $\mathrm{WF}_g(Pu)^c$ as well as of $\mathrm{W_{sc}\text{-}Ell}(P_1)$ are satisfied when $(x, \xi) \in U \times \Gamma$. Let $\varphi \in \mathcal{D}(U)$ and denote by $\bar{\xi}$ the projection of $\xi \neq 0$ onto the sphere S^{n-1}. We will show that the function $t \mapsto (\varphi u_\varepsilon)\hat{\ }(t\bar{\xi})$ is rapidly decreasing when $t \ge \max(1, r_\varepsilon)$ uniformly with respect to $\bar{\xi} \in \Gamma \cap S^{n-1}$. This suffices to prove the theorem by Remark 6.1(i).

Let $P_\varepsilon(x, D) = \sum_{j=1}^n a_j^\varepsilon(x) D_j + a_0^\varepsilon(x)$ and observe that $P_{1\varepsilon}(x, D)\exp(-ix\xi) = -\exp(-ix\xi)P_{1,\varepsilon}(x, \xi)$. This suggests to define the first order differential operator

$L_{\varepsilon,\xi}(x,D)$ with parameters ε, ξ by

$$\begin{aligned} L_{\varepsilon,\xi}(x,D) &:= \Big(\frac{-1}{P_{1,\varepsilon}(x,\xi)} P_{1,\varepsilon}(x,D)\Big)^t \\ &= \frac{1}{P_{1,\varepsilon}(x,\xi)} P_{1,\varepsilon}(x,D) + \sum_{j=1}^n D_j(\frac{a_j^\varepsilon(x)}{P_{1,\varepsilon}(x,\xi)}) \\ &= \frac{1}{P_{1,\varepsilon}(x,\xi)} P_{1,\varepsilon}(x,D) + q_0^\varepsilon(x,\xi). \end{aligned} \tag{6.10}$$

To avoid heavy notation in the calculations below we will henceforth denote the operator $L_{\varepsilon,\xi}(x,D)$ simply by L.

Let $\xi = t\bar\xi$ with $t \geq \max(1, r_\varepsilon)$. Note that $P_{1,\varepsilon}(x,\xi) = P_{1,\varepsilon}(x, r_\varepsilon\bar\xi) \cdot t/r_\varepsilon$, $q_0^\varepsilon(x,\xi) = q_0^\varepsilon(x, r_\varepsilon\bar\xi) \cdot r_\varepsilon/t$ and $1/P_{1,\varepsilon}(x, r_\varepsilon\bar\xi)$ as well as $q_0^\varepsilon(x, r_\varepsilon\bar\xi)$ satisfy slow scale estimates in x uniformly with respect to $\bar\xi \in \Gamma \cap S^{n-1}$ (in the sense of (2) in Lemma 6.9).

Integrating by parts and applying the Leibniz rule for $P_{1,\varepsilon}$ we have

$$\begin{aligned} &(\varphi u_\varepsilon)\widehat{\ }(\xi) = \int e^{-ix\xi} L(\varphi u_\varepsilon)(x)\,dx \\ &= \int e^{-ix\xi}\Big(\frac{1}{P_{1,\varepsilon}(x,\xi)}\big(P_{1,\varepsilon}\varphi(x)\cdot u_\varepsilon(x) + \varphi(x)\cdot P_{1,\varepsilon}u_\varepsilon(x)\big) + q_0^\varepsilon(x,\xi)\varphi(x)u_\varepsilon(x)\Big)\,dx. \end{aligned}$$

We rewrite the middle term using $P_{1,\varepsilon}u_\varepsilon = P_\varepsilon u_\varepsilon - a_0^\varepsilon u_\varepsilon$ and obtain

$$\begin{aligned} (\varphi u_\varepsilon)\widehat{\ }(\xi) &= \int e^{-ix\xi}\Big(\frac{P_{1,\varepsilon}\varphi(x) - a_0^\varepsilon(x)\varphi(x)}{P_{1,\varepsilon}(x,\xi)} + q_0^\varepsilon(x,\xi)\varphi(x)\Big)\cdot u_\varepsilon(x)\,dx \\ &\quad + \frac{r_\varepsilon}{t}\int e^{-ix\xi}\frac{\varphi(x)}{P_{1,\varepsilon}(x,\bar\xi)} P_\varepsilon u_\varepsilon(x)\,dx \\ &=: I_1^\varepsilon(\xi) + \frac{r_\varepsilon}{t} h_1^\varepsilon(\bar\xi,\xi). \end{aligned}$$

The factor within parentheses in the first integral is $L\varphi(x) - a_0^\varepsilon(x)\varphi(x)/P_{1,\varepsilon}(x,\xi)$ and will be abbreviated as $\varphi_{1,\varepsilon(x,\xi)}$.

Choose $\psi \in \mathcal{D}(U)$ with $\psi = 1$ on $S := \operatorname{supp}(\varphi)$. We put $g_\varepsilon^{\bar\xi}(x) := \varphi(x)/P_{1,\varepsilon}(x,\bar\xi)$ and observe that this defines a family of functions satisfying properties (1) and (2) of Lemma 6.9 (with $M = \Gamma \cap S^{n-1}$). Since $h_1^\varepsilon(\bar\xi,\xi) = (g_\varepsilon^{\bar\xi} P_\varepsilon u_\varepsilon)\widehat{\ }(\xi)$ we deduce from the same lemma (with $v = Pu$) that $\eta \mapsto h_1^\varepsilon(\bar\xi,\eta)$ is rapidly decreasing when $\eta \in \Gamma$, with uniform ε-power, say $-K$, when $\bar\xi$ varies in $\Gamma \cap S^{n-1}$. In particular, $t\bar\xi \in \Gamma$ and hence $t \mapsto h_1^\varepsilon(\bar\xi, t\bar\xi)$ enjoys the same decrease estimate.

In $I_1^\varepsilon(\xi) = \int \exp(-ix\xi)\varphi_{1,\varepsilon}(x,\xi)u_\varepsilon(x)\,dx$ we have $\varphi_{1,\varepsilon}(x,\xi) = \varphi_{1,\varepsilon}(x, r_\varepsilon\bar\xi)\cdot r_\varepsilon/t$ and the $\bar\xi$-parameterized $\mathcal{E}_{\mathrm{M}}^\infty$-net $(\varphi_{1,\varepsilon}(.,r_\varepsilon\bar\xi))_\varepsilon$ also satisfies conditions (1), (2) in Lemma 6.9, as does $\varphi_{2,\varepsilon}(.,\xi)\cdot t^2/r_\varepsilon^2 := L_{\varepsilon,r_\varepsilon\bar\xi}\varphi_{1,\varepsilon}(.,r_\varepsilon\bar\xi) - a_0^\varepsilon\varphi_{1,\varepsilon}(.,r_\varepsilon\bar\xi)/P_{1,\varepsilon}(.,r_\varepsilon\bar\xi)$. Another integration by parts in $I_1^\varepsilon(\xi)$ gives

$$\begin{aligned} I_1^\varepsilon(\xi) &= \int e^{-ix\xi}\varphi_{2,\varepsilon}(x,\xi)\cdot u_\varepsilon(x)\,dx + (\frac{r_\varepsilon}{t})^2\int e^{-ix\xi}\frac{\varphi_{1,\varepsilon}(x,r_\varepsilon\bar\xi)}{P_{1,\varepsilon}(x,\bar\xi)} P_\varepsilon u_\varepsilon(x)\,dx \\ &=: I_2^\varepsilon(\xi) + \frac{r_\varepsilon^2}{t^2} h_2^\varepsilon(\bar\xi,\xi). \end{aligned}$$

Again, $t \mapsto h_2^\varepsilon(\bar\xi, t\bar\xi)$ is seen to be rapidly decreasing uniformly in $\bar\xi$ with ε-power $-K$ (the same K as above). Successively, after k steps we arrive at

$$(\varphi u_\varepsilon)\widehat{\ }(\xi) = I_k^\varepsilon(\xi) + \sum_{j=1}^{k} (\frac{r_\varepsilon}{t})^j h_j^\varepsilon(\bar\xi, t\bar\xi)$$

where $t \geq r_\varepsilon$ and $h_j^\varepsilon(\bar\xi, t\bar\xi) = O(t^{-k}\varepsilon^{-K})$ when $0 < \varepsilon < \varepsilon_0$ uniformly in $\bar\xi$ ($1 \leq j \leq k$), and

$$I_k^\varepsilon(\xi) = (\frac{r_\varepsilon}{t})^k \cdot \int e^{-ix\xi} \varphi_{k,\varepsilon}(x, r_\varepsilon\bar\xi) u_\varepsilon(x)\, dx$$

with $\|\varphi_{k,\varepsilon}(., r_\varepsilon\bar\xi)\|_{\mathrm{L}^\infty} = O(\varepsilon^{-1})$ uniformly in $\bar\xi$. Since $r_\varepsilon^k = O(\varepsilon^{-1})$ and $\sup |u_\varepsilon(x)| = O(\varepsilon^{-M})$ (sup over $x \in \mathrm{supp}(\varphi)$) for some M we finally find that

$$|(\varphi u_\varepsilon)\widehat{\ }(\xi)| = O(t^{-k}\varepsilon^{-M-2}) + O(t^{-k}\varepsilon^{-K}) = O(t^{-k}\varepsilon^{-N})$$

uniformly in $\bar\xi$ when $t \geq \max(1, r_\varepsilon)$ and $N := \max(M+2, K)$ with arbitrary k. □

Remark 6.11. (i) The simple examples $P_\varepsilon = \frac{d}{dx} - i/\varepsilon$ and $Q_\varepsilon = \varepsilon\frac{d}{dx} - i$, both admitting $u_\varepsilon(x) = \exp(ix/\varepsilon)$ as solution to the homogeneous equation, show that neither the slow scale condition on the coefficients nor the W_{sc}-ellipticity can be dropped in Theorem 6.7.
(ii) We may use this opportunity to point out that Theorem 6.7 establishes corresponding claims made earlier in [10, Examples 2.1 and 4.1] and in [9, Theorem 23 (i)] independently of the references cited therein (cf. Example 6.4(iii)).

7. Solvability of PDEs with coefficients in $\widetilde{\mathbb{C}}$

In this section, we present a necessary and sufficient condition on the symbol $P(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$ ($a_\alpha \in \widetilde{\mathbb{C}}$) for the corresponding PDE to be solvable in Ω for arbitrary compactly supported right-hand sides. More precisely, we investigate the property

$$\forall f \in \mathcal{G}_c(\Omega) : \exists u \in \mathcal{G}(\Omega) : \quad P(D)u = f. \tag{7.1}$$

A sufficient condition for general solvability on $\Omega = \mathbb{R}^n$ has been given in [14, Theorem 2.4], in the special case of the existence of fundamental solutions already in [13]. We restate it here in a slightly simplified form. (The simplification is an immediate consequence of the characterization of invertible generalized numbers.)

Theorem 7.1 ([14, Theorem 2.4])**.** *Assume that there is $\xi_0 \in \mathbb{R}^n$ such that*

$$P_m(\xi_0) \textit{ is invertible in } \widetilde{\mathbb{C}}. \tag{7.2}$$

Then $P(D)$ is solvable in $\mathbb{R}^n$, i.e.,

$$\forall f \in \mathcal{G}(\mathbb{R}^n) : \exists u \in \mathcal{G}(\mathbb{R}^n) : \quad P(D)u = f. \tag{7.3}$$

The proof in [14] uses a complex Fourier integral representation (within a partition of unity) of a solution candidate. We will take a different approach while restricting to the case of compactly supported right-hand side; on the other hand, we thereby gain a relaxation of condition (7.2), which will, in addition, turn out to be a characterization of solvable operators.

Before stating the new solvability condition we briefly discuss the relations among various properties of the symbol. First, we note the simple fact that condition (7.2) is implied by any of the ellipticity conditions introduced above.

Proposition 7.2. *Every W-elliptic symbol $P(\xi)$ satisfies (7.2).*

Proof. Recall that from Proposition 3.3 that the W-ellipticity of P is equivalent to the S-ellipticity of the principal part P_m. Hence there is N, R, and ε_0 such that $|P_{\varepsilon,m}(\xi)| \geq \varepsilon^N(1+|\xi|)^m$ when $0<\varepsilon<\varepsilon_0$ and $|\xi| \geq R$. Picking ξ_0 with $|\xi_0| \geq R$ arbitrary we obtain (7.2), after readjusting N and ε_0 accordingly. □

Clearly, condition (7.2) is strictly weaker than W-ellipticity, as can be seen from the example $P(\xi_1,\xi_2)=\xi_1-\xi_2$. The following example shows that (7.2) is not necessary for solvability.

Example 7.3. Let $P(\xi)=a\xi+i$, where $0 \neq a \in \widetilde{\mathbb{R}}$ is the zero divisor with the following representative

$$a_\varepsilon = \begin{cases} 0 & \text{if } 1/\varepsilon \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $P_1(\xi)=a\xi$ cannot be invertible (for any ξ), but $P(D)u=f\in\mathcal{G}$ is always solvable. A solution is given by the class of the representative

$$u_\varepsilon(x) = \begin{cases} -if_\varepsilon(x) & \text{if } 1/\varepsilon \in \mathbb{N}, \\ i\int_0^x e^{(x-y)} f_\varepsilon(y)\,dy & \text{otherwise.} \end{cases}$$

Note, however, that here $|P(\xi)|^2 = a^2\xi^2+1$ is invertible (for arbitrary ξ) thanks to the lower order term.

The key property of the (generalized) symbol $P(\xi)$, which will turn out to be equivalent to (7.1), is specified in terms of its associated (generalized) temperate weight function, which we define in analogy with [5, 2.1, Example 2] (see also [6, Example 10.1.3])

Definition 7.4. If P is the symbol of a PDO of order m with coefficients in $\widetilde{\mathbb{C}}$ we define $\widetilde{P}^2 := \sum_{\alpha\leq m} \partial^\alpha P \cdot \overline{\partial^\alpha P} \in \mathcal{G}(\mathbb{R}^n)$ or, alternatively, in terms of representatives

$$\widetilde{P}^2_\varepsilon(\xi) := \sum_{\alpha\leq m} |\partial^\alpha_\xi P_\varepsilon(\xi)|^2. \tag{7.4}$$

Note that, contrary to the classical case, we avoid taking the square root, but prefer to stick closely to the classical notation.

Lemma 7.5. *If there is $\xi_0 \in \mathbb{R}^n$ such that $\widetilde{P}^2(\xi_0)$ is invertible (in $\widetilde{\mathbb{C}}$) then $\widetilde{P}^2$ is invertible in $\mathcal{G}(\mathbb{R}^n)$ and $\widetilde{P} := \sqrt{\widetilde{P}^2}$ is a well-defined Colombeau function. More precisely, there exist $d>0$, $N\geq 0$, $\varepsilon_0\in(0,1)$ such that*

$$\widetilde{P}^2_\varepsilon(\xi) \geq \varepsilon^N(1+d|\xi_0-\xi|)^{-2m} \quad \forall \xi\in\mathbb{R}^n, \forall\varepsilon\in(0,\varepsilon_0). \tag{7.5}$$

Proof. By [5, Equation (2.1.10)] there is a constant $d>0$ (independent of $\varepsilon\in(0,1]$) such that

$$\widetilde{P}^2_\varepsilon(\xi+\eta) \leq (1+d|\eta|)^{2m}\widetilde{P}^2_\varepsilon(\xi) \quad \forall \xi,\eta\in\mathbb{R}^n, \forall\varepsilon\in(0,1].$$

Since $\widetilde{P}^2(\xi_0)$ is invertible we have for some $N>0$ and $\varepsilon_0\in(0,1]$ that $\varepsilon^N \leq \widetilde{P}^2_\varepsilon(\xi_0)$ when $\varepsilon\in(0,\varepsilon_0)$. Therefore, substituting $\eta=\xi_0-\xi$ in the inequality above we obtain

$$\varepsilon^N \leq \widetilde{P}^2_\varepsilon(\xi_0) \leq (1+d|\xi_0-\xi|)^{2m}\widetilde{P}^2_\varepsilon(\xi)$$

for all $\xi \in \mathbb{R}^n$ and $\varepsilon \in (0, \varepsilon_0)$. This proves (7.5) and shows that the square root of $\widetilde{P}_\varepsilon^2$ is smooth (and moderate). Moreover, (7.5) yields the invertibility of $\widetilde{P}^2$ as a generalized function on $\mathbb{R}^n$. □

The symbol in Example 7.3 defines an invertible weight function (e.g. $\widetilde{P}^2(0) = a^2 + 1$), whereas condition (7.2) is not satisfied. The following proposition shows that, in general, the invertibility of $\widetilde{P}^2$ is a (strictly) weaker condition.

Proposition 7.6. *If $P(\xi)$ is a generalized symbol satisfying (7.2) then its associated weight function $\widetilde{P}^2$ is invertible.*

Proof. Observe that

$$\frac{1}{n}\Big(\sum_{|\alpha|=m} |a_\alpha^\varepsilon|\Big)^2 \le \sum_{|\alpha|=m} (\alpha!)^2 |a_\alpha^\varepsilon|^2 = \sum_{|\alpha|=m} |\partial_\xi^\alpha P_\varepsilon(\xi)|^2 \le \widetilde{P}_\varepsilon^2(\xi).$$

Let $\xi_0 \in \mathbb{R}^n$ be arbitrary. There is $C > 0$ (dependent only on n and m) such that

$$|P_{\varepsilon,m}(\xi_0)| = |\sum_{|\alpha|=m} a_\alpha^\varepsilon \xi_0^\alpha| \le C|\xi_0|^m \sum_{|\alpha|=m} |a_\alpha^\varepsilon|.$$

Thus the invertibility of $P_m(\xi_0)$ implies the invertibility of $\widetilde{P}_\varepsilon^2(\xi_0)$. □

Theorem 7.7. *Assume that $\widetilde{P}^2$ is invertible at some $\xi_0 \in \mathbb{R}^n$. Then for every $f \in \mathcal{G}_c(\Omega)$ there is a solution $u \in \mathcal{G}(\Omega)$ to the equation $P(D)u = f$.*

Proof. Recall the definition of the Banach spaces $\mathcal{B}_{p,k}$ (cf. [5, Ch.II] or [6, Ch.10]), where $1 \le p \le \infty$ and k is a temperate weight function (cf. [5, Ch.II]). These are the spaces of temperate distributions on $\mathbb{R}^n$ given by

$$\mathcal{B}_{p,k} = \{v \in \mathcal{S}'(\mathbb{R}^n) \mid k \cdot \widehat{v} \in \mathrm{L}^p(\mathbb{R}^n)\}$$

and equipped with the norm $\|v\|_{p,k} = \|k\widehat{v}\|_{L^p}/(2\pi)^{n/p}$. Furthermore, the corresponding local space $\mathcal{B}_{p,k}^{\mathrm{loc}}$ consists of the distributions $w \in \mathcal{D}'(\mathbb{R}^n)$ such that $\varphi w \in \mathcal{B}_{p,k}$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

We apply [5, Theorem 3.1.1] to obtain a regular fundamental solution for the operator P_ε, for each $\varepsilon \in (0, 1]$. More precisely, we have the following. Let $c > 0$ be arbitrary; there is a constant $C > 0$, depending only on n, m, and c, such that $\forall \varepsilon \in (0, 1]$ $\exists E_\varepsilon \in \mathcal{B}^{\mathrm{loc}}_{\infty,\widetilde{P}_\varepsilon}$ such that $P_\varepsilon(D)E_\varepsilon = \delta_0$ and $E_\varepsilon/\cosh(c|.|) \in \mathcal{B}_{\infty,\widetilde{P}_\varepsilon}$ with norm estimate

$$\|E_\varepsilon/\cosh(c|.|)\|_{\infty,\widetilde{P}_\varepsilon} \le C. \tag{7.6}$$

Choose a representative $(f_\varepsilon)_\varepsilon \in f$ and $K_0 \Subset \Omega$ such that $\mathrm{supp}(f_\varepsilon) \subseteq K_0$ for all ε. Define

$$u_\varepsilon := (E_\varepsilon * f_\varepsilon)|_\Omega \in \mathrm{C}^\infty(\Omega), \varepsilon \in (0, 1].$$

By construction, $P_\varepsilon(D)u_\varepsilon = f_\varepsilon$, so it remains to be shown that $(u_\varepsilon)_\varepsilon$ is moderate.

Let $K \Subset \Omega$, $\alpha \in \mathbb{N}_0^n$, $x \in K$ arbitrary. From the definition of u_ε we obtain

$$\begin{aligned} |\partial^\alpha u_\varepsilon(x)| &= |E_\varepsilon * \partial^\alpha f_\varepsilon(x)| = |\langle E_\varepsilon, \partial^\alpha f_\varepsilon(x - .)\rangle| \\ &= |\langle \frac{E_\varepsilon}{\cosh(c|.|)}, \big((\partial^\alpha f_\varepsilon) \cdot \cosh(c|x - .|)\big)(x - .)\rangle| = |\big(e_\varepsilon * g_{\varepsilon,x}^\alpha\big)(x)| \end{aligned}$$

where we have put $e_\varepsilon := E_\varepsilon / \cosh(c|.|)$ and $g^\alpha_{\varepsilon,x}(y) := \partial^\alpha f_\varepsilon(y) \cosh(c|x-y|)$. Note that we have $e_\varepsilon \in \mathcal{B}_{\infty,\widetilde{P}_\varepsilon}$ and $g^\alpha_{\varepsilon,x} \in \mathcal{D}(K_0) \subset (\mathcal{B}_{1,1/\widetilde{P}_\varepsilon} \cap \mathcal{E}'(K_0))$; now [5, Theorem 2.2.6] yields that $e_\varepsilon * g^\alpha_{\varepsilon,x} \in \mathcal{B}_{1,1} = \mathcal{F}^{-1}(\mathrm{L}^1) \subset \mathrm{L}^\infty$ and hence we have

$$\begin{aligned}|\big(e_\varepsilon * g^\alpha_{\varepsilon,x}\big)(x)| &\le \|e_\varepsilon * g^\alpha_{\varepsilon,x}\|_{\mathrm{L}^\infty} \le \|e_\varepsilon * g^\alpha_{\varepsilon,x}\|_{1,1} \\ &\le \|e_\varepsilon\|_{\infty,\widetilde{P}_\varepsilon} \|g^\alpha_{\varepsilon,x}\|_{1,1/\widetilde{P}_\varepsilon} \le C \|g^\alpha_{\varepsilon,x}\|_{1,1/\widetilde{P}_\varepsilon}.\end{aligned}$$

We have to establish a moderate upper bound for the last factor, that is

$$\begin{aligned}\|g^\alpha_{\varepsilon,x}\|_{1,1/\widetilde{P}_\varepsilon} &= \big\|\widehat{g^\alpha_{\varepsilon,x}}/\widetilde{P}_\varepsilon\big\|_{\mathrm{L}^1} \\ &= \int \frac{|\widehat{g^\alpha_{\varepsilon,x}}(\xi)|}{\widetilde{P}_\varepsilon(\xi)}\, d\xi \le \varepsilon^{-N/2} \int |\widehat{g^\alpha_{\varepsilon,x}}(\xi)| \cdot (1 + d|\xi_0 - \xi|)^m \, d\xi, \quad \varepsilon \in (0, \varepsilon_0),\end{aligned}$$

where we have made use of Lemma 7.5 (and the notation there).

A direct calculation, using Leibniz' rule, the support properties of f_ε, and noting that the factor $\cosh(c|x - .|)$ is ε-independent, shows that the family $(g^\alpha_{\varepsilon,x})_{x\in K}$ has moderate upper bounds (with respect to ε) in every semi-norm of $\mathcal{S}(\mathbb{R}^n)$. By the continuity of the Fourier transform we conclude that for every $l \in \mathbb{N}$ there is $M \ge 0$ and $C_l > 0$ such that

$$|\widehat{g^\alpha_{\varepsilon,x}}(\xi)| \le C_l (1 + |\xi|)^{-l} \varepsilon^{-M},$$

when ε is sufficiently small (and uniformly in $x \in K$). Choosing $l > m + n$ we may use this bound in the integrand above and arrive at

$$\|g^\alpha_{\varepsilon,x}\|_{1,1/\widetilde{P}_\varepsilon} \le C' \varepsilon^{-(2M+N)/2},$$

where C' is some constant depending only on n, m, l, and ξ_0, and ε is small. Combining these estimates, we have shown that

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| \le C C' \varepsilon^{-(2M+N)/2}$$

if ε is sufficiently small. □

The above solvability proof was based on a careful analysis and extension of classical (distributional) constructions. Curiously enough, we can show a converse implication by simple reasoning with properties of the ring of generalized numbers.

Theorem 7.8. *If $f \in \mathcal{G}(\Omega)$ such that $f(x_0)$ is invertible (in $\widetilde{\mathbb{C}}$) for some $x_0 \in \Omega$ and $P(D)u = f$ is solvable in $\mathcal{G}(\Omega)$ then $\widetilde{P}^2$ is invertible (in $\mathcal{G}(\Omega)$).*

Proof. Assume that $\widetilde{P}^2$ is not invertible. Then due to Lemma 7.5, in particular, $\widetilde{P}^2(0)$ cannot be invertible. Since $\partial^\alpha P_\varepsilon(0) = \alpha! a^\varepsilon_\alpha$ we have

$$\widetilde{P}^2_\varepsilon(0) = \sum_{|\alpha|\le m} (\alpha!)^2 |a^\varepsilon_\alpha|^2.$$

Since non-invertible generalized numbers are zero divisors we may choose a representative of $\widetilde{P}^2(0)$, say $(b_\varepsilon)_\varepsilon$, which vanishes on a zero sequence of ε-values, i.e., $b_{\nu_k} = 0$ ($k \in \mathbb{N}$) for some sequence $(\nu_k)_k \in (0,1]^{\mathbb{N}}$ with $\nu_k \to 0$ as $k \to \infty$. We have for all q that $|b_\varepsilon - \widetilde{P}^2_\varepsilon(0)| = O(\varepsilon^q)$ ($\varepsilon \to 0$). Since all terms in the above sum representation of $\widetilde{P}^2_\varepsilon(0)$ are nonnegative, we deduce that $|a^{\nu_k}_\alpha| = O(\nu_k^q)$ for all q ($k \to \infty$). (In fact, we may choose representatives of a_α, $|\alpha| \le m$, which vanish along the same zero sequence.)

Define the generalized number $c \in \widetilde{\mathbb{R}}$ by the representative

$$c_\varepsilon = \begin{cases} 1 & \text{if } \varepsilon = \nu_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $c \neq 0$ but $c \cdot a_\alpha = 0$ ($|\alpha| \leq m$) by construction. If $f(x_0)$ is invertible and $P(D)u = f$ we arrive at the following contradiction

$$0 \neq c \cdot f(x_0) = c \cdot P(D)u(x_0) = \sum_\alpha c a_\alpha D^\alpha u(x_0) = 0.$$

□

Corollary 7.9. *The solvability property (7.1) for $P(D)$ with generalized constant coefficients holds if and only if $\widetilde{P}^2$ is invertible (at some point in $\mathbb{R}^n$).*

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