Electronic Journal of Differential Equations, Vol. 2004(2004), No. 141, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF SOLUTIONS FOR NONCONVEX FUNCTIONAL DIFFERENTIAL INCLUSIONS

VASILE LUPULESCU

Abstract. We prove the existence of solutions for the functional differential inclusion $x' \in F(T(t)x)$, where F is upper semi-continuous, compact-valued multifunction such that $F(T(t)x) \subset \partial V(x(t))$ on $[0,T]$, V is a proper convex and lower semicontinuous function, and $(T(t)x)(s) = x(t+s)$.

1. Introduction

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with the norm $\|\cdot\|$ and the scalar product $\langle\cdot,\cdot\rangle$. When I is a segment in $\mathbb{R}$, we denote by $\mathcal{C}(I,\mathbb{R}^m)$ the Banach space of continuous functions from I to $\mathbb{R}^m$ with the norm $\|x(.)\|_\infty := \sup\{\|x(t)\|; t \in I\}$. When σ is a positive number, we put $\mathcal{C}_\sigma := \mathcal{C}([-\sigma,0],\mathbb{R}^m)$ and for any $t \in [0,T]$, $T > 0$, we define the operator $T(t)$ from $\mathcal{C}([-\sigma,T],\mathbb{R}^m)$ to $\mathcal{C}_\sigma$ as $(T(t)x)(s) := x(t+s)$, $s \in [-\sigma,0]$.

Let Ω be a nonempty subset in $\mathcal{C}_\sigma$. For a given multifunction $F : \Omega \to 2^{\mathbb{R}^m}$ we consider the following functional differential inclusion:

$$x' \in F(T(t)x). \tag{1.1}$$

We recall that a continuous function $x(.) : [-\sigma,T] \to \mathbb{R}^m$ is said to be a solution of (1.1) if $x(.)$ is absolutely continuous on $[0,T]$, $T(t)x \in \Omega$ for all $t \in [0,T]$ and $x'(t) \in F(T(t)x)$ for almost all $t \in [0,T]$; see [8].

The functional differential equation (1.1) with F single-valued, has been studied by many authors; for results, references, and applications, see for example [9, 10].

The existence of solutions for the functional differential inclusion (1.1) was proved by Haddad [8] when F is upper semicontinuous with convex compact values. The nonconvex case in Banach space has been studied by Benchohra and Ntouyas [2]. The case when F is lower semicontinuous with compact value has been studied by Fryszkowski [7].

In this paper we prove the existence of solutions for functional differential inclusion (1.1) when F is upper semicontinuous, compact valued multifunction such that $F(\psi) \subset \partial V(\psi(0))$ for every $\psi \in \Omega$ and V is a proper convex and lower semicontinuous function. Our existence result contains Peano's existence theorem as a

2000 *Mathematics Subject Classification.* 34A60, 34K05, 34K25.
Key words and phrases. Functional differential inclusions; existence result.

Submitted September 24, 2004. Published November 29, 2004.

particular case. On the other hand, our result may be considered as an extension of the previous result of Bressan, Cellina and Colombo [3].

2. Preliminaries and statement of the main result

For $x \in \mathbb{R}^m$ and $r > 0$ let $B(x,r) := \{y \in \mathbb{R}^m; \|y - x\| < r\}$ be the open ball centered at x with radius r, and let $\overline{B}(x,r)$ be its closure. For $\varphi \in \mathcal{C}_\sigma$ let $B_\sigma(\varphi,r) := \{\psi \in \mathcal{C}_\sigma; \|\psi - \varphi\|_\infty < r\}$ and $\overline{B}_\sigma(\varphi,r) := \{\psi \in \mathcal{C}_\sigma; \|\psi - \varphi\|_\infty \le r\}$. For $x \in \mathbb{R}^m$ and for a closed subset $A \subset \mathbb{R}^m$ we denote by $d(x,A)$ the distance from x to A given by $d(x,A) := \inf\{\|y - x\|; y \in A\}$. Given a function $V : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ let

$$D(V) := \{x \in \mathbb{R}^m : V(x) < +\infty\}$$

be its effective domain. We say that V is proper function if $D(V)$ is nonempty.

Let $V : \mathbb{R}^m \to \mathbb{R}$ be a proper convex and lower semicontinuous function. The multifunction $\partial V : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, defined by

$$\partial V(x) := \{\xi \in \mathbb{R}^m; V(y) - V(x) \ge \langle \xi, y - x\rangle, \quad \forall y \in \mathbb{R}^m\}, \tag{2.1}$$

is called subdifferential (in the sense of convex analysis) of the function V.

We say that a multifunction $F : \Omega \subset \mathcal{C}_\sigma \to 2^{\mathbb{R}^m}$ is upper semicontinuous if for every $\varphi \in \Omega$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(\psi) \subset F(\varphi) + B(0,\varepsilon), \quad \forall \psi \in \Omega \cap B_\sigma(\varphi,\delta).$$

The definition of the upper semicontinuous multifunctions is the same as [6, Definition 1.2].

For a multifunction $F : \Omega \to 2^{\mathbb{R}^m}$ we consider the functional differential inclusion (1.1) under the following assumptions:

- (H1) $\Omega \subset \mathcal{C}_\sigma$ is an open set and F is upper semicontinuous with compact values;
- (H2) There exists a a proper convex and lower semicontinuous function $V : \mathbb{R}^m \to \mathbb{R}$ such that

$$F(\psi) \subset \partial V(\psi(0)) \text{ for every } \psi \in \Omega. \tag{2.2}$$

Remark. A convex function $V : \mathbb{R}^m \to \mathbb{R}$ is continuous in the whole space $\mathbb{R}^m$ [11, Corollary 10.1.1] and almost everywhere differentiable [11, Theorem 25.5]. Therefore, (H2) restricts strongly the multivaluedness of F.

Our main result is the following:

Theorem 2.1. *If $F : \Omega \to 2^{\mathbb{R}^m}$ and $V : \mathbb{R}^m \to \mathbb{R}$ satisfy assumptions (H1) and (H2) then for every $\varphi \in \Omega$ there exists $T > 0$ and $x(.) : [-\sigma, T] \to \mathbb{R}^m$ a solution of the functional differential inclusion* (1.1) *such that $T(0)x = \varphi$ on $[-\sigma, 0]$.*

3. Proof of the main result

Let $\varphi \in \Omega$ be arbitrarily fixed. Since the multifunction $x \to \partial V(x)$ is locally bounded [4, Proposition 2.9], there exists $r > 0$ and $M > 0$ such that V is Lipschitz continuous with constant M on $B(\varphi(0), r)$. Since Ω is an open set we can choose r such that $\overline{B}_\sigma(\varphi, r) \subset \Omega$. Moreover, by [1, Proposition 1.1.3], F is locally bounded; therefore, we can assume that

$$\sup\{\|y\| : y \in F(\psi),\ \psi \in B(\varphi, r)\} \le M\,. \tag{3.1}$$

Since φ is continuous on $[-\sigma, 0]$ we can choose $\eta > 0$ such that

$$\|\varphi(t) - \varphi(s)\| < r/4 \text{ for all } t, s \in [-\sigma, 0] \text{ with } |t - s| < \eta. \tag{3.2}$$

Let $0 < T \le \min\{\eta, r/4M\}$. We shall prove the existence of a solution of (1.1) defined on the interval $[-\sigma, T]$. For this, we define a family of approximate solutions and we prove that a subsequence converges to a solution of (1.1).

First, for a fixed $n \in \mathbb{N}^*$, we set

$$x_n(t) = \varphi(t),\ t \in [-\sigma, 0]. \tag{3.3}$$

Furthermore, we partition $[0, T]$ by points $t_n^j := \frac{jT}{n}$, $j = 0, 1, \dots, n$, and, for every $t \in [t_n^j, t_n^{j+1}]$, we define

$$x_n(t) := x_n^j + (t - t_n^j)y_n^j, \tag{3.4}$$

where $x_n^0 = x_n(0) := \varphi(0)$ and

$$x_n^j = x_n^{j-1} + \frac{T}{n} y_n^{j-1}, \tag{3.5}$$

$$y_n^j \in F(T(t_n^j)x_n) \tag{3.6}$$

for every $j \in \{1, 2, \dots, n\}$. It is easy to see that for every $j \in \{1, 2, \dots, n\}$ we have

$$x_n^j = \varphi(0) + \frac{T}{n}(y_n^0 + y_n^1 + \dots + y_n^{j-1}). \tag{3.7}$$

By (3.1) and (3.7) we infer $\|x_n^j - \varphi(0)\| \le \frac{jT}{n} M < r/4$, proving that

$$x_n(t_n^j) = x_n^j \in B(\varphi(0), r/4) \tag{3.8}$$

for every $j \in \{1, 2, \dots, n\}$.

By (3.1) and (3.4) we have that

$$\|x_n(t) - x_n(t_n^j)\| = \|x_n(t) - x_n^j\| \le \frac{jT}{n} M < \frac{r}{4}, \tag{3.9}$$

for every $j \in \{0, 1, \dots, n\}$. Hence, from (3.8) and (3.9) we deduce that

$$\|x_n(t) - \varphi(0)\| \le \|x_n(t) - x_n(t_n^j)\| + \|x_n(t_n^j) - \varphi(0)\| < \frac{r}{2}$$

and so

$$x_n(t) \in B(\varphi(0), \frac{r}{2}), \text{ for every } t \in [0, T]. \tag{3.10}$$

Moreover, by (3.1), (3.4) and (3.6), we have $\|x_n'(t)\| \le M$ for every $t \in [0, T]$ and so the sequence (x_n') is bounded in $L^2([0, T], \mathbb{R}^m)$.

For $t, s \in [0, T]$, we have

$$\|x_n(t) - x_n(s)\| \le \Big| \int_s^t \|x_n'(\tau)\| d\tau \Big| \le M|t - s|$$

so that the sequence (x_n) is equiuniformly continuous. Hence, by Theorem 0.3.4 in [1], there exists a subsequence, still denoted by (x_n), and an absolute continuous function $x : [0, T] \to \mathbb{R}^m$ such that:

(i) (x_n) converges uniformly on $[0, T]$ to x;
(ii) (x_n') converges weakly in $L^2([0, T], \mathbb{R}^m)$ to x'.

Moreover, since by (3.3) all functions x_n agree with φ on $[-\sigma, 0]$, we can obviously say that $x_n \to x$ on $[-\sigma, T]$, if we extend x in such a way that $x \equiv \varphi$ on $[-\sigma, 0]$. Also, it is clearly that $T(0)x = \varphi$ on $[-\sigma, 0]$.

Further on, if we define $\theta_n(t) = t_n^j$ for all $t \in [t_n^j, t_n^{j+1}]$ then, by (3.4) and (3.6), we have

$$x_n'(t) \in F(T(\theta_n(t))x_n), \text{ a.e. on } [0, T]. \tag{3.11}$$

and, by (3.8),

$$x_n(\theta_n(t)) \in B(\varphi(0), \frac{r}{4}), \text{ for every } t \in [0,T]. \tag{3.12}$$

Also, since $|\theta_n(t) - t| \le \frac{T}{n}$ for every $t \in [0,T]$, then $\theta_n(t) \to t$ uniformly on $[0,T]$. Moreover, by the uniformly converges of (x_n) and (θ_n), we deduce that $x_n(\theta_n(t)) \to x(t)$ uniformly on $[0,T]$.

Now, we have to estimate $\|(T(\theta_n(t))x_n)(s) - \varphi(s)\|$ for each $s \in [-\sigma, 0]$. If $-\theta_n(t) \le s \le 0$, then $\theta_n(t) + s \ge 0$ and so there exists $j \in \{0, 1, \dots, n-1\}$ such that $\theta_n(t) + s \in [t_n^j, t_n^{j+1}]$. Thus, by (3.2), (3.10) and by the fact that $|\theta_n(t) - t| \le T$ and $|s| \le T$, we have

$$\begin{aligned}
\|(T(\theta_n(t))x_n)(s) - \varphi(s)\| &= \|x_n(\theta_n(t)+s) - \varphi(s)\| \\
&\le \|x_n(\theta_n(t)+s) - \varphi(0)\| + \|\varphi(s) - \varphi(0)\| \\
&< \frac{3r}{4} < r\,.
\end{aligned}$$

If $-\sigma \le s \le -\theta_n(t)$ then $s + \theta_n(t) \le 0$ and by (3.2) we have

$$\|(T(\theta_n(t))x_n)(s) - \varphi(s)\| = \|\varphi(\theta_n(t)+s) - \varphi(s)\| \le \frac{r}{4} < r.$$

Therefore,

$$T(\theta_n(t))x_n \in B(\varphi, r), \quad \text{for every } t \in [0,T]. \tag{3.13}$$

Let us denote the modulus continuity of a function ψ defined on interval I of $\mathbb{R}$ by

$$\omega(\psi, I, \varepsilon) := \sup\{\|\psi(t) - \psi(s)\|; s, t \in I, |s-t| < \varepsilon\},\ \varepsilon > 0.$$

Then we have:

$$\begin{aligned}
\|T(\theta_n(t))x_n - T(t)x_n\|_\infty &= \sup_{-\sigma \le s \le 0} \|x_n(\theta_n(t)+s) - x_n(t+s)\| \\
&\le \omega(x_n, [-\sigma, T], \frac{T}{n}) \\
&\le \omega(\varphi, [-\sigma, 0], \frac{T}{n}) + \omega(x_n, [0,T], \frac{T}{n}) \\
&\le \omega(\varphi, [-\sigma, 0], \frac{T}{n}) + \frac{T}{n}M;
\end{aligned}$$

hence

$$\|T(\theta_n(t))x_n - T(t)x_n\|_\infty \le \delta_n \quad \text{for every } t \in [0,T], \tag{3.14}$$

where $\delta_n := \omega(\varphi, [-\sigma, 0], \frac{T}{n}) + \frac{T}{n}M$. Thus, by continuity of φ, we have $\delta_n \to 0$ as $n \to \infty$ and hence

$$\|T(\theta_n(t))x_n - T(t)x_n\|_\infty \to 0 \text{ as } n \to \infty.$$

Therefore, since the uniform convergence of x_n to x on $[-\sigma, T]$ implies

$$T(t)x_n \to T(t)x \quad \text{uniformly on } [-\sigma, 0], \tag{3.15}$$

we deduce that

$$T(\theta_n(t))x_n \to T(t)x \quad \text{in } \mathcal{C}_\sigma\,. \tag{3.16}$$

Moreover, by (3.13) and (3.16), we have that $T(t)x \in \overline{B}_\sigma(\varphi, r) \subset \Omega$. Also, by (3.11) and (3.14), we have

$$d((T(t)x_n, x_n'(t)), \operatorname{graph}(F)) \le \delta_n \quad \text{for every } t \in [0.T]. \tag{3.17}$$

By (H2), (ii), (3.16) and [1, Theorem 1.4.1], we obtain

$$x'(t) \in coF(T(t)x) \subset \partial V(x(t)) \quad \text{a.e. on } [0,T], \tag{3.18}$$

where co stands for the closed convex hull.

Since the functions $t \to x(t)$ and $t \to V(x(t))$ are absolutely continuous, we obtain from [4, Lemma 3.3] and (3.18) that

$$\frac{d}{dt}V(x(t)) = \|x'(t)\|^2 \quad \text{a.e. on } [0,T];$$

therefore,

$$V(x(T)) - V(x(0)) = \int_0^T \|x'(t)\|^2 dt. \tag{3.19}$$

On the other hand, since

$$x_n'(t) = y_n^j \in F(T(t_n^j)x_n) \subset \partial V(x_n(t_n^j))$$

for every $t \in [t_n^j, t_n^{j+1}]$ and for every $j \in \{0, 1, \dots, n-1\}$, it follows that

$$\begin{aligned} V(x_n(t_n^{j+1})) - V(x_n(t_n^j)) &\geq \langle x_n'(t), x_n(t_n^{j+1}) - x_n(t_n^j)\rangle \\ &= \langle x_n'(t), \int_{t_n^j}^{t_n^{j+1}} x_n'(t)dt\rangle = \int_{t_n^j}^{t_n^{j+1}} \|x'(t)\|^2 dt. \end{aligned}$$

By adding the n inequalities above, we obtain

$$V(x_n(T)) - V(x(0)) \geq \int_0^T \|x_n'(t)\|^2 dt$$

and passing to the limit as $n \to \infty$, we obtain

$$V(x(T)) - V(x(0)) \geq \limsup_{n\to\infty} \int_0^T \|x_n'(t)\|^2 dt. \tag{3.20}$$

Therefore, by b(3.19) and (3.20),

$$\int_0^T \|x'(t)\|^2 dt \geq \limsup_{n\to\infty} \int_0^T \|x_n'(t)\|^2 dt$$

and, since (x_n') converges weakly in $L^2([0,T], \mathbb{R}^m)$ to x', by applying [5, Proposition III.30], we obtain that (x_n') converges strongly in $L^2([0,T], \mathbb{R}^m)$. Hence there exists a subsequence, still denote by (x_n'), which converges pointwiese a.e. to x'.

Since, by (H1), the graph of F is closed [1, Proposition 1.1.2], by (3.17),

$$\lim_{n\to\infty} d((T(t)x_n, x_n'(t)), \operatorname{graph}(F)) = 0,$$

we obtain

$$x'(t) \in F(T(t)x) \quad \text{a.e. on } [0,T]\,.$$

Therefore, the functional differential inclusion (1.1) has solutions.

References

[1] J. P. Aubin and A. Cellina; *Differential inclusions*, Berlin, Springer-Verlag, 1984.

[2] M. Benchohra and S. K. Ntouyas; *Existence results for functional differential inclusions*, Electron. J. Diff. Eqns., Vol. 2001(2001), No. 41, pp. 1-8.

[3] A. Bressan, A. Cellina and G. Colombo; *Upper semicontinuous differential inclusions without convexity*, Proc. Amer. Math. Soc., 106(1984), 771-775.

[4] H. Brezis; *Operateurs Maximaux Monotones et Semigroups de Contractions Dans Les Espaces de Hilbert*, Amsterdam, North-Holland, 1973.

[5] H. Brezis, Analyse Functionelle, Théorie et Applications, Paris, Masson, 1983.

[6] K. Deimling; *Multivalued Differential Equations*, Walter de Gruyter, Berlin, New York, 1992.

[7] A. Fryszokwski; *Existence of solutions of functional differential inclusions in nonconvex case*, Ann. Pol. Math., 45(1985), 121-124.

[8] G. Haddad; *Topological Properties of Sets Solutions for Functional Differential Inclusions*, Nonlinear Analysis, T.M.A, 5(12)(1981), 1349-1366.

[9] J. K. Hale and S.M.V. Lumel; *Introduction to functional differential equations*, Springer-Verlag, 1993.

[10] V. Lakshmikantham and S. Leela; *Nonlinear differential equations in abstract spaces*, Pergamon Press, Oxford, New York, 1981.

[11] R. T. Rockafeller; *Convex Analysis*, Princeton University Press, 1970.

Vasile Lupulescu

"Constantin Brâncuşi" University of Târgu-Jiu, Bulevardul Republicii, nr. 1, 1400 Târgu-Jiu, Romania

E-mail address: vasile@utgjiu.ro