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ESTIMATES FOR THE MIXED DERIVATIVES OF THE GREEN FUNCTIONS ON HOMOGENEOUS MANIFOLDS OF NEGATIVE CURVATURE

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Abstract. We consider the Green functions for second-order left-invariant differential operators on homogeneous manifolds of negative curvature, being a semi-direct product of a nilpotent Lie group N and $A = \mathbb{R}^+$. We obtain estimates for mixed derivatives of the Green functions both in the coercive and non-coercive case. The current paper completes the previous results obtained by the author in a series of papers [14, 15, 16, 19].

1. Introduction

Let M be a connected and simply connected homogeneous manifold of negative curvature. Such a manifold is a solvable Lie group $S = NA$, a semi-direct product of a nilpotent Lie group N and an Abelian group $A = \mathbb{R}^+$. Moreover, for an H belonging to the Lie algebra $\mathfrak{a}$ of A, the real parts of the eigenvalues of $\mathrm{Ad}_{\exp H}|_{\mathfrak{n}}$, where $\mathfrak{n}$ is the Lie algebra of N, are all greater than 0. Conversely, every such a group equipped with a suitable left-invariant metric becomes a homogeneous Riemannian manifold with negative curvature (see [8]).

On S we consider a second-order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y.$$

We assume that $Y_0, Y_1, \dots, Y_m$ generate the Lie algebra $\mathfrak{s}$ of S and $Y \in \mathfrak{s}$. We can always make $Y_0, \dots, Y_m$ linearly independent and moreover, we can choose $Y_0, Y_1, \dots, Y_m$ so that $Y_1(e), \dots, Y_m(e)$ belong to $\mathfrak{n}$ (write $\mathcal{L}$ as $\sum_{i,j=0}^{\dim\mathfrak{n}} \alpha_{i,j} E_i E_j + \sum_{j=0}^{\dim\mathfrak{n}} \beta_j E_j$, $E_0 \in \mathfrak{a}$, $\{E_j\}$ is a basis of $\mathfrak{n}$, $\alpha_{i,j}, \beta_j \in \mathbb{R}$ and then rewrite $\mathcal{L}$ as a sum of squares). Let $\pi : S \to A = S/N$ be the canonical homomorphism. Then the image of $\mathcal{L}$ under π is a second-order left-invariant operator on $\mathbb{R}^+$,

$$\pi(\mathcal{L}) = (a\partial_a)^2 - \gamma a \partial_a,$$

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where $\gamma = \gamma_{\mathcal{L}} \in \mathbb{R}$. We say that a second order differential operator $\mathcal{L}$ on a Riemannian manifold is *non-coercive* (*coercive* resp.) if there is no $\varepsilon > 0$ such that $\mathcal{L}+\varepsilon\mathrm{Id}$ admits the Green function (if such an ε exists resp.). It is worth noting that our definition of coercivity is slightly different than that used e.g. in [1]. Namely, for us, $\mathcal{L}$ is coercive if it is weakly coercive in Ancona's terminology. There is a relation between the notion of coercivity property in the sense used in the theory of partial differential equations (i.e., that an appropriate bilinear form is coercive, [9]) and weak coercivity. For this the reader is referred to [1].

In this paper we shall study both coercive and non-coercive operators. In this case $\mathcal{L}$ can be written as

$$\mathcal{L} = \mathcal{L}_\gamma = \sum_j \Phi_a(X_j)^2 + \Phi_a(X) + a^2\partial_a^2 + (1-\gamma)\partial_a, \tag{1.1}$$

where $\gamma = \gamma_{\mathcal{L}} \in \mathbb{R}$, $X, X_1, \dots, X_m$ are left-invariant vector fields on N, moreover, $X_1, \dots, X_m$ are linearly independent and generate $\mathfrak{n}$,

$$\Phi_a = \mathrm{Ad}_{\exp(\log a)Y_0} = e^{(\log a)\,\mathrm{ad}_{Y_0}} = e^{(\log a)D},$$

where $D = \mathrm{ad}_{Y_0}$ is a derivation of the Lie algebra $\mathfrak{n}$ of the Lie group N such that the real parts d_j of the eigenvalues λ_j of D are positive. By multiplying $\mathcal{L}_\gamma$ by a constant, i.e., changing Y_0, we can make d_j arbitrarily large (see [5]).

Let $\mathcal{G}_\gamma(xa, yb)$ be the *Green function* for $\mathcal{L}_\gamma$. $\mathcal{G}_\gamma$ is (uniquely) defined by two conditions:

(i) $\mathcal{L}_\gamma\mathcal{G}_\gamma(\cdot, yb) = -\delta_{yb}$ as distributions (functions are identified with distributions via the right Haar measure),
(ii) for every $yb \in S$, $\mathcal{G}_\gamma(\cdot, yb)$ is a potential for $\mathcal{L}_\gamma$, i.e, is a positive superharmonic function such that its largest harmonic minorant is equal to zero (cf. [2]).

Let

$$\mathcal{G}_\gamma(x, a) := \mathcal{G}_\gamma(e, xa), \tag{1.2}$$

where e is the identity element of the group S. Since $\mathcal{L}_\gamma$ is left-invariant it is easily seen that

$$\mathcal{G}_\gamma(xa, yb) = \mathcal{G}_\gamma(e, yb(xa)^{-1}) = \mathcal{G}_\gamma(yb(xa)^{-1}).$$

In this article we call $\mathcal{G}_\gamma(x, a)$ defined in (1.2) the Green function for $\mathcal{L}_\gamma$.

The main goal of this paper is to give estimates for derivatives of the Green function (1.2) for $\mathcal{L}_\gamma$.

To illustrate the general set up, before we proceed further, we would like to give a simple and explicit example of the operator $\mathcal{L}$ in coordinates. Consider $S = \mathbb{R}^n \times \mathbb{R}^+$. Let $d_1, \dots, d_n$ be positive constants. For every $a > 0$, define $\Phi_a(\partial_{x_j}) = a^{d_j}\partial_{x_j}$. Then Φ_a on $\mathbb{R}^n$ becomes $\Phi_a(x) = \Phi_a(x_1, \dots, x_n) = (a^{d_1}x_1, \dots, a^{d_n}x_n)$ and we get on $\mathbb{R}^n$ a structure of the homogeneous group with the homogeneous dimension $Q = \sum d_j$ (see [7]). The multiplication law in S is given by the formula $(x, a)\cdot(y, b) = (x + \Phi_a(y), ab)$. In this example the operator (1.1) is $\mathcal{L} = \sum_j a^{2d_j}\partial_{x_j}^2 + a^2\partial_a^2 + (1-\gamma)a\partial_a$. The Green function for $\mathcal{L}$ is $\mathcal{G}((x, a), (y, b)) = \int_0^\infty p_t(x, a; y, b)dt$, where $p_t(x, a; y, b)$ is the heat diffusion kernel on S, such that $u(t, y, b) := p_t(x, a; y, b)$ is the minimal solution of $\mathcal{L}u = \partial_t u$, $u(0, y, b) = \delta_{(x,a)}(y, b)$ and $\delta_{(x,a)}(\cdot)$ stands for Dirac's delta.

Let us go back to the general setting. We are going to prove (or at least to sketch the proof of) the following estimates. Let $\gamma \geq 0$. For every neighborhood $\mathcal{U}$ of the identity e of NA there is a constant $C = C(\gamma)$ such that we have

$$|\partial_a^k \mathcal{X}^I \mathcal{G}_{-\gamma}(x,a)| \leq \begin{cases} C(|x|+a)^{-\|I\|-Q-\gamma}a^{-k} & \\ \times(1+|\log(|x|+a)^{-1}|)^{\|I\|_0} & \text{for } (x,a) \in (\mathcal{Q}\cup\mathcal{U})^c, \\ Ca^{-k} & \text{for } (x,a) \in \mathcal{Q}\setminus\mathcal{U} \end{cases} \tag{1.3}$$

and

$$|\partial_a^k \mathcal{X}^I \mathcal{G}_{\gamma}(x,a)| \leq \begin{cases} C(|x|+a)^{-\|I\|-Q-\gamma}a^{\gamma-k} & \\ \times(1+|\log(|x|+a)^{-1}|)^{\|I\|_0} & \text{for } (x,a) \in (\mathcal{Q}\cup\mathcal{U})^c, \\ Ca^{\gamma-k} & \text{for } (x,a) \in \mathcal{Q}\setminus\mathcal{U}, \end{cases} \tag{1.4}$$

where $|\cdot|$ stands for a "homogeneous norm" on N, $\mathcal{Q} = \{|x| \leq 1, a \leq 1\}$, $\|I\|$ is a suitably defined length of the multi-index I and $\|I\|_0$ is a certain number depending on I and the nilpotent part of the derivation D. In particular, $\|I\|_0$ is equal to 0 if the action of $A = \mathbb{R}^+$ on N, given by Φ_a, is diagonal or, if $I = 0$. $\mathcal{X}_1, \dots, \mathcal{X}_n$ is an appropriately chosen basis of $\mathfrak{n}$. For the precise definitions of all the notions that have appeared here see Sect. 2.

It should be said that the estimate for the Green function itself (i.e., I=0) with $\gamma > 0$, also from below, was proved by E. Damek in [4] and then by the author for $\gamma = 0$ in [19] but at that time it was impossible to prove analogous estimate for derivatives. The reason was that we did not have sufficient estimates for the derivatives of the transition probabilities of the evolution on N generated by an appropriate operator which appears as the "horizontal" component of the diffusion on $N \times \mathbb{R}^+$ generated by $a^{-2}\mathcal{L}_{-\gamma}$ (cf. [5]). These estimates have been obtained by the author in [20] and eventually led up to the estimates for derivatives of the Green functions in the non-coercive case, i.e., $\gamma = \gamma_{\mathcal{L}} = 0$ (see [15] for derivatives with respect to N and A-variables separately and [16] for the mixed derivatives which required a slightly different approach). Next, in [14] the results from [15] have been used to get estimates for derivatives in the coercive case. This note completes the previous works of the author in that we provide a proof of the estimates which is valid for both the coercive and non-coercive cases.

The proofs of (1.3) and (1.4) require both analytic and probabilistic techniques. Some of them have been introduced in [6, 5] and [19].

The structure of the paper is as follows. In Sect. 2 we set the notation and give all necessary definitions. In particular, we recall a definition of the Bessel process which appears as the"vertical" component of the diffusion generated by $a^{-2}\mathcal{L}_{-\gamma}$ on $N \times \mathbb{R}^+$ as well as the notion of the evolution on N generated by an appropriate operator which appears as the"horizontal" component of the diffusion on $N \times \mathbb{R}^+$ mentioned in the above (cf. [5, 15]). Moreover, we state Theorem 2.1 which is the main tool in the proof of Theorem 3.1.

Finally, in Sect. 3 we state the estimates (1.3), (1.4) precisely (see Theorem 3.1) and we give a sketch of its proofs.

2. Preliminaries.

NA **groups.** Good reference for this topic are [6, 5] and [7]. Let N be a connected and simply connected nilpotent Lie group. Let D be a derivation of the Lie algebra $\mathfrak{n}$ of N. For every $a \in \mathbb{R}^+$ we define an automorphism Φ_a of $\mathfrak{n}$ by the formula

$$\Phi_a = e^{(\log a)D}.$$

Writing $x = \exp X$ we put

$$\Phi_a(x) := \exp \Phi_a(X).$$

Let $\mathfrak{n}^{\mathbb{C}}$ be the complexification of $\mathfrak{n}$. Define

$$\mathfrak{n}^{\mathbb{C}}_\lambda = \{X \in \mathfrak{n}^{\mathbb{C}} : \exists k > 0 \text{ such that } (D - \lambda I)^k = 0\}.$$

Then

$$\mathfrak{n} = \bigoplus_{\mathrm{Im}\lambda \geq 0} V_\lambda, \tag{2.1}$$

where

$$V_\lambda = \begin{cases} V_{\bar\lambda} = (\mathfrak{n}^{\mathbb{C}}_\lambda \oplus \mathfrak{n}^{\mathbb{C}}_{\bar\lambda}) \cap \mathfrak{n} & \text{if } \mathrm{Im}\lambda \neq 0, \\ \mathfrak{n}^{\mathbb{C}}_\lambda \cap \mathfrak{n} & \text{if } \mathrm{Im}\lambda = 0. \end{cases}$$

We assume that the real parts d_j of the eigenvalues λ_j of the matrix D are strictly greater than 0. We define the number

$$Q = \sum_j \mathrm{Re}\ \lambda_j = \sum_j d_j \tag{2.2}$$

and we refer to this as a *"homogeneous dimension"* of N. In this paper $D = \mathrm{ad}_{Y_0}$ (see Introduction). Under the assumption on positivity of d_j, (2.1) is a gradation of $\mathfrak{n}$.

We consider a group S which is a *semi-direct* product of N and the multiplicative group $A = \mathbb{R}^+ = \{\exp tY_0 : t \in \mathbb{R}\}$:

$$S = NA = \{xa : x \in N, a \in A\}$$

with multiplication given by the formula

$$(xa)(yb) = (x\Phi_a(y)\ ab).$$

On N we define a *"homogeneous norm"*, $|\cdot|$ (cf. [6, 5]) as follows. Let $(\cdot,\cdot)$ be a fixed inner product in $\mathfrak{n}$. We define a new inner product

$$\langle X, Y\rangle = \int_0^1 \Big(\Phi_a(X), \Phi_a(Y)\Big)\frac{da}{a} \tag{2.3}$$

and the corresponding norm

$$\|X\| = \langle X, X\rangle^{1/2}.$$

We put

$$|X| = \left(\inf\{a > 0 : \|\Phi_a(X)\| \geq 1\}\right)^{-1}.$$

One can easily show that for every $Y \neq 0$ there exists precisely one $a > 0$ such that $Y = \Phi_a(X)$ with $|X| = 1$. Then we have $|Y| = a$.

Finally, we define the homogeneous norm on N. For $x = \exp X$ we put

$$|x| := |X|.$$

Notice that if the action of $A = \mathbb{R}^+$ on N (given by Φ_a) is diagonal the norm we have just defined is the usual homogeneous norm on N and the number Q in (2.2) is simply the homogeneous dimension of N (see [7]).

Having all that in mind we define appropriate derivatives (see also [6]). We fix an inner product (2.3) in $\mathfrak{n}$ so that V_{λ_j}, $j = 1, \dots, k$ are mutually orthogonal and an orthonormal basis $\mathcal{X}_1, \dots, \mathcal{X}_n$ of $\mathfrak{n}$. The enveloping algebra $\mathfrak{U}(\mathfrak{n})$ of $\mathfrak{n}$ is identified with the polynomials in $\mathcal{X}_1, \dots, \mathcal{X}_n$. In $\mathfrak{U}(\mathfrak{n})$ we define $\langle \mathcal{X}_1 \otimes \dots \otimes \mathcal{X}_r, \mathcal{Y}_1 \otimes \dots \otimes \mathcal{Y}_r\rangle =$

$\prod_{j=1}^r \langle \mathcal{X}_j, \mathcal{Y}_j \rangle$. Let V_j^r be the symmetric tensor product of r copies of V_{λ_j}. For $I = (i_1, \dots, i_k) \in (\mathbb{N} \cup \{0\})^k$ let

$$\mathcal{X}^I = \mathcal{X}_1^{(i_1)} \dots \mathcal{X}_k^{(i_k)}, \text{ where } \mathcal{X}_j^{(i_j)} \in V_j^{i_j}.$$

Then for $\mathcal{X} \in V_{\lambda_j}$,

$$\|\Phi_a(\mathcal{X})\| \le c \exp(d_j \log a + D_j \log(1 + |\log a|)),$$

where $d_j = \mathrm{Re}\lambda_j$ and $D_j = \dim V_{\lambda_j} - 1$, and so

$$\|\Phi_a(\mathcal{X}^I)\| \le \exp\Big(\sum_{j=1}^k i_j(d_j \log a + D_j \log(1 + |\log a|))\Big) \prod_{j=1}^k \|\mathcal{X}_j^{(i_j)}\| \tag{2.4}$$

Bessel process. Let $\sigma(t)$ denote the Bessel process with a parameter $\alpha \ge 0$ (cf. [11]), i.e., a continuous Markov process with the state space $[0, +\infty)$ generated by $\partial_a^2 + \frac{2\alpha+1}{a}\partial_a$. The transition function with respect to the measure $y^{2\alpha+1}dy$ is given (cf. [3, 11]) by:

$$p_t(x,y) = \begin{cases} \frac{1}{2t}\exp\big(\frac{-x^2-y^2}{4t}\big) I_\alpha\big(\frac{xy}{2t}\big)\frac{1}{(xy)^\alpha} & \text{for } x, y > 0, \\ \frac{1}{2^\alpha (2t)^{\alpha+1}\Gamma(\alpha+1)}\exp\big(\frac{-y^2}{4t}\big) & \text{for } x = 0,\ y > 0, \end{cases}$$

where

$$I_\alpha(x) = \sum_{k=0}^\infty \frac{(x/2)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}$$

is *the Bessel function* (see [10]). Therefore for $x \ge 0$ and a measurable set $B \subset (0, \infty)$:

$$\mathbf{P}_x(\sigma(t) \in B) = \int_B p_t(x,y) y^{2\alpha+1} dy.$$

If $\sigma(t)$ is the Bessel process with a parameter α starting from x, i.e. $\sigma(0) = x$, then we will write that $\sigma(t) \in \mathrm{BESS}_x(\alpha)$ or simply $\sigma(t) \in \mathrm{BESS}(\alpha)$ if the starting point is not important or is clear from the context.

Properties of the Bessel process are very well known and their proofs are rather standard. They can be found e.g. in [11, 5, 18, 17]. However, in our paper we will not explicitly make use of any particular property of the Bessel process.

Evolutions. Let $X, X_1, \dots, X_m$ be as in (1.1). Let $\sigma : [0, \infty) \longrightarrow [0, \infty)$ be a continuous function such that $\sigma(t) > 0$ for every $t > 0$. We consider the family of evolutions operators $L_{\sigma(t)} - \partial_t$, where

$$L_{\sigma(t)} = \sigma(t)^{-2}\Big(\sum_j \Phi_{\sigma(t)}(X_j)^2 + \Phi_{\sigma(t)}(X)\Big). \tag{2.5}$$

For the main result of the paper we are mainly interested in the operator (2.5) with $\sigma(t)$ being a trajectory of an appropriate Bessel process.

Since we assume $X_1, \dots, X_m$ being linearly independent, we select $X_{m+1}, \dots, X_n$ so that $X_1, \dots, X_n$ form a basis of $\mathfrak{n}$. For a multi-index $I = (i_1, \dots, i_n)$, $i_j \in \mathbb{Z}^+$ and the basis $X_1, \dots, X_n$ of the Lie algebra $\mathfrak{n}$ of N we write: $X^I = X_1^{i_1} \dots X_n^{i_n}$ and $|I| = i_1 + \dots + i_n$. For $k = 0, 1, \dots, \infty$ we define:

$$C^k = \{f : X^I f \in C(N), \text{ for } |I| < k+1\}$$

and

$$C^k_\infty = \{f \in C^k : \lim_{x \to \infty} X^I f(x) \text{ exists for } |I| < k+1\}.$$

For $k < \infty$ the space C^k_∞ is a Banach space with the norm

$$\|f\|_{C^k_\infty} = \sum_{|I| \le k} \|X^I f\|_{C(N)}.$$

Let $\{U^\sigma(s,t) : 0 \le s \le t\}$ be the unique family of bounded operators on $C_\infty = C^0_\infty$ which satisfy

(i) $U^\sigma(s,s) = I$,
(ii) $U^\sigma(s,r)U^\sigma(r,t) = U^\sigma(s,t)$, $s < r < t$,
(iii) $\partial_s U^\sigma(s,t)f = -L_{\sigma(s)}U^\sigma(s,t)f$ for every $f \in C_\infty$,
(iv) $\partial_t U^\sigma(s,t)f = U^\sigma(s,t)L_{\sigma(t)}f$ for every $f \in C_\infty$,
(v) $U^\sigma(s,t) : C^2_\infty \longrightarrow C^2_\infty$.

Note that $U^\sigma(s,t)$ is a convolution operator. Namely, $U^\sigma(s,t)f = f * p^\sigma(t,s)$, where $p^\sigma(t,s)$ is a smooth density of a probability measure. By ii) we have $p^\sigma(t,r) * p^\sigma(r,s) = p^\sigma(t,s)$ for $t > r > s$. The existence of the family $U^\sigma(s,t)$ follows from [12].

For $\alpha \ge 0$, on a direct product $G = N \times \mathbb{R}^+$ we consider the following operator

$$\mathbf{L}_\alpha = a^{-2}\sum_j \Phi_a(X_j)^2 + a^{-2}\Phi_a(X) + \partial_a^2 + \frac{2\alpha+1}{a}\partial_a.$$

For $f \in C^\infty_c(G)$, we define on G the following function

$$u(t,x,a) := \mathbf{E}_a U^\sigma(0,t) f(x,\sigma(t)), \tag{2.6}$$

where the expectation is taken with respect to the distribution of the Bessel process $\sigma(t)$ starting from a. The following theorem, which gives the formula for the solution of the heat equation for L_α in terms of the evolution on N driven by the Bessel process, is one of the main tool in the proof of our results.

Theorem 2.1. *Let $u = u(t,x,a)$ be a function on G defined by* (2.6)*. Then*

$$\mathbf{L}_\alpha u = \partial_t u, \quad u(0,x,a) = f(x,a). \tag{2.7}$$

Moreover, there exists the only one bounded from below solution u of (2.7)*.*

Proof. For the proof of the first part of Theorem 2.1 see [5]. The uniqueness of the bounded from below solution u follows by some kind of the maximum principle which is a modification of Theorem. 3.1.1 in [13]. The proof for a diagonal action given in [17] can be easily generalized. □

3. The main result and its proof.

In this section we obtain pointwise estimates for derivatives of the Green function (1.2).

For a positive $\delta < 1/2$ define

$$\begin{aligned} T_\delta =& \{(x,a) \in N \times \mathbb{R}^+ : 1-\delta < a < 1+\delta, |x| < \delta\}, \\ \mathcal{Q} =& \{(x,a) \in N \times \mathbb{R}^+ : |x| \le 1, a \le 1\}. \end{aligned}$$

Theorem 3.1. *For a multi-index $I = (i_1, \dots, i_k)$, $\gamma \ge 0$, $k \in \mathbb{Z}^+$ and all operators $\mathcal{X}^I = \mathcal{X}_1^{(i_1)} \dots \mathcal{X}_k^{(i_k)}$, where $\mathcal{X}_j^{(i_j)} \in V_j^{i_j}$, with $\|\mathcal{X}^I\| \le 1$, there are constants C such* *that*

$$|\partial_a^k \mathcal{X}^I \mathcal{G}_{-\gamma}(x,a)| \le \begin{cases} C(|x|+a)^{-\|I\|-Q-\gamma} a^{-k} & \\ \times(1+|\log(|x|+a)^{-1}|)^{\|I\|_0} & \textit{for } (x,a) \in (\mathcal{Q}\cup T_\delta)^c, \\ Ca^{-k} & \textit{for } (x,a) \in \mathcal{Q}\setminus T_\delta \end{cases}$$

and

$$|\partial_a^k \mathcal{X}^I \mathcal{G}_{\gamma}(x,a)| \le \begin{cases} C(|x|+a)^{-\|I\|-Q-\gamma} a^{\gamma-k} & \\ \times(1+|\log(|x|+a)^{-1}|)^{\|I\|_0} & \textit{for } (x,a) \in (\mathcal{Q}\cup T_\delta)^c, \\ Ca^{\gamma-k} & \textit{for } (x,a) \in \mathcal{Q}\setminus T_\delta \end{cases}$$

where $\|I\| = \sum_{j=1}^k i_j d_j$, $d_j = Re\lambda_j$, *and* $\|I\|_0 = \sum_{j=1}^k i_j D_j$, $D_j = \dim V_{\lambda_j} - 1$.

Let $\alpha \ge 0$ and $\gamma \ge 0$. Along with the operator $\mathcal{L}_{-\gamma}$ defined in (1.1) we consider the corresponding operator $\mathbf{L}_\alpha$,

$$\mathbf{L}_\alpha = a^{-2}\sum_j \Phi_a(X_j)^2 + a^{-2}\Phi_a(X) + \partial_a^2 + \frac{2\alpha+1}{a}\partial_a = a^{-2}\mathcal{L}_{-\gamma},$$

where $\alpha = \gamma/2$. The Green function G_α for $\mathbf{L}_\alpha$ is given by

$$G_\alpha(x,a;y,b) = \int_0^\infty p_t(x,a;y,b)dt,$$

where

$$T_t f(x,a) = \int f(y,b)p_t(x,a;y,b)dyb^{2\alpha+1}db$$

is the heat semigroup on $L^2(N\times\mathbb{R}^+, dyb^{2\alpha+1}db)$ given by Theorem 2.1 with the infinitesimal generator $\mathbf{L}_\alpha$.

On $N\times\mathbb{R}^+$ we define *dilations*:

$$D_t(x,a) = (\Phi_t(x), ta), \qquad t>0.$$

It is not difficult to check that although the operator $\mathbf{L}_\alpha$ is not left-invariant it has some homogeneity with respect to the family of dilations introduced above:

$$\mathbf{L}_\alpha(f\circ D_t) = t^2\mathbf{L}_\alpha f\circ D_t.$$

This implies that

$$G_\alpha(x,a;y,b) = t^{-Q-2\alpha}G_\alpha(D_{t^{-1}}(x,a);D_{t^{-1}}(y,b)). \tag{3.1}$$

It turns out (see (1.17) in [5]) that

$$\mathcal{G}_{-\gamma}(x,a) = G_{\gamma/2}(e,1;x,a) = G^*_{\gamma/2}(x,a;e,1), \tag{3.2}$$

where G^*_α is the Green function for the operator

$$\mathbf{L}^*_\alpha = a^{-2}\sum_j \Phi_a(X_j)^2 - a^{-2}\Phi_a(X) + \partial_a^2 + \frac{2\alpha+1}{a}\partial_a$$

formally conjugate to $\mathbf{L}_\alpha$ with respect to the measure $a^{2\alpha+1}dadx$. Moreover,

$$G^*_\alpha(x,a;e,1) = \lim_{\eta\to 0}\int_0^\infty \mathbf{E}_1 p^\sigma(t,0)(x)m_\alpha(I_{a,\eta})^{-1}1_{I_{a,\eta}}(\sigma_t)dt,$$

where

$$m_\alpha(I) = \int_I a^{2\alpha+1}da$$

and the expectation is taken with respect to the distribution of the Bessel process with the parameter α starting from 1, i.e., $\mathrm{BESS}_1(\alpha)$ on the space $C((0,\infty),(0,\infty))$, $p^\sigma(t,0)$ is the transition function of the evolution generated by the operator (2.5) and $I_{a,\eta} = [a-\eta, a+\eta]$.

Since $\mathcal{L}_{-\gamma}(\cdot) = a^{-\gamma}\mathcal{L}_\gamma(a^\gamma \cdot)$ it follows that

$$\mathcal{G}_\gamma(xa, yb) = a^\gamma \mathcal{G}_{-\gamma}(xa, yb) b^{-\gamma} \tag{3.3}$$

and therefore, by (3.2) and (3.3),

$$\mathcal{G}_\gamma(x, a) = G^*_{\gamma/2}(x, a; e, 1) a^\gamma.$$

Before we go to the proof of our main result we note the following important proposition which gives estimates on the set $\mathcal{Q} \setminus T_\delta$ of some functional of the evolution p^σ which plays the crucial role in the proof of Theorem 3.1.

Proposition 3.2. *i) For every $1 > \delta > 0$ and for every multi-index I such that $|I| > 0$ there exists a constant $C > 0$ such that for every $(x,a) \in (\mathcal{Q} \setminus T_\delta) \cap \{(x,a) \in N \times \mathbb{R}^+ : a \le 1-\delta\}$ and for every $0 \le l \le k-1$,*

$$\sup_{0<\eta<\delta/2} \Big| \int_0^\infty \mathbf{E}_1 X^I p^\sigma(t,0)(x) \partial_a^l m_\alpha(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma_t) dt \Big| \le C a^{-k}.$$

ii) For every $0 < \chi_0 \le 1$, $0 < r_0 \le 1$ and for every multi-index I such that $|I| > 0$ there exists a constant $C > 0$ such that for every $\chi \le \chi_0$, for every $(x,a) \in \{0 < a \le 1, r_0 \le |x| \le 1\}$, and for every $0 \le l \le k-1$,

$$\sup_{0<\eta<1} \Big| \int_0^\infty \mathbf{E}_\chi X^I p^\sigma(t,0)(x) \partial_a^l m_\alpha(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma_t) dt \Big| \le C a^{-k}.$$

iii) For every $1 > \delta > 1/2$ and for every multi-index I such that $|I| > 0$ there exists a constant $C > 0$ such that for every $\chi \le 1/2 - \delta$, for every $(x,a) \in \{(1-\delta)/2 \le a \le 1/2\}$ and for every $0 \le l \le k-1$,

$$\sup_{0<\eta<\delta/4} \Big| \int_0^\infty \mathbf{E}_\chi X^I p^\sigma(t,0)(x) \partial_a^l m_\alpha(I_{a,\eta})^{-1} \partial_a^{k-l} 1_{I_{a,\eta}}(\sigma_t) dt \Big| \le C a^{-k}.$$

Sketch of the proof. The case $\alpha = 0$ has been proved in [16]. (See the proof of Proposition 3.1 in [16]. We take the opportunity to say that the formulation of Proposition 3.1 in [16] is wrong and that it should have been stated exactly as above with $\alpha = 0$.) The generalization from $\alpha = 0$ to an arbitrary $\alpha > 0$ is (almost) straightforward. One only needs to imitate the proof of Proposition 3.1 in [16]. $\square$

After this preparatory facts we are ready to give

Sketch of the proof of Theorem 3.1. Let $0 < \delta < 1/2$, $k \in \mathbb{Z}^+$ and a multi-index I be fixed.

Case 1. We consider the set

$$S_1 = \mathcal{Q} \setminus T_\delta.$$

By Proposition 3.2 and Theorem 2.1 it follows that there exists a positive constant C such that

$$|\partial_a^k \mathcal{X}^I G^*_{\gamma/2}(x, a; e, 1)| \le C a^{-k} \tag{3.4}$$

for every $(x,a) \in \tilde{S}_1 := S_1 \cap \{(x,a) \in N \times \mathbb{R}^+ : a \leq 1-\delta\}$. But $S_1 \setminus \mathrm{Int}\tilde{S}_1$ is a compact set and $G^*_{\gamma/2}$ is a continuous function so we get (3.4) on S_1. Therefore, on S_1 we have that

$$|\partial_a^k \mathcal{X}^I \mathcal{G}_{-\gamma}(x,a)| = |\partial_a^k \mathcal{X}^I G^*_{\gamma/2}(x,a;e,1)| \leq Ca^{-k}$$

and

$$|\partial_a^k \mathcal{X}^I \mathcal{G}_{\gamma}(x,a)| = |\partial_a^k \mathcal{X}^I G^*_{\gamma/2}(x,a;e,1)a^{\gamma}| \leq Ca^{\gamma-k}.$$

Case 2. Outside the set S_1 we imitate the proof of Theorem 3.1 in [14], where derivative with respect to $\mathcal{X}$ have been considered. The homogeneity (3.1) of the Green function plus (2.4) play the crucial role. □

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