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GENERALIZED SCALAR CURVATURE TYPE EQUATION ON COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. In this work, we investigate positive solutions for a quasilinear elliptic equation on complete manifold M. This equation extends to the p-Laplacian the equation of the prescribed scalar curvature. A minimizing sequence is constructed which converges to a non trivial solution belonging to $C^{1,\alpha}(K)$ for any compact set $K \subset M$ and some $\alpha \in (0,1)$.

1. Introduction

Let (M,g) be a complete Riemannian manifold of dimension $n \geq 3$, with bounded geometry, $R(x)$ its scalar curvature and $p \in (1,n)$. Let $H_1^p(M)$ be the standard Sobolev space endowed with the norm

$$\|u\|_{H_1^p(M)} = \|\nabla u\|_{L^p(M)} + \|u\|_{L^p(M)}.$$

In this paper, we seek for a positive solution $u \in H_{1,\mathrm{loc}}^p(M)$ to the equation

$$\Delta_p u + a(x)u^{p-1} = f(x)u^{p*-1}. \tag{1.1}$$

where $\Delta_p u = -div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u on M and $p^* = \frac{pn}{n-p}$.

Our results extend those of Druet [2] obtained in the case of compact manifolds. On complete Riemannian manifold conditions at infinity on f must be added.

In the case $p = 2$ and the function $a(x) = \frac{n-2}{n(n-1)}R(x)$, where $R(x)$ is the scalar curvature of the manifold M, the problem of the existence of a positive solution of the equation (1.1) is originated from the study of pointwise conformal deformation of Riemannian metric with prescribed scalar curvature. If in case $p = 2$, u is a positive solution of (1.1) on (M,g), then the scalar curvature of the pointwise conformal metric $g' = u^{\frac{4}{n-2}}g$ is $\frac{4(n-1)}{n-2}f$ (cf. [2]). The equation (1.1) is referred as the generalized scalar curvature type equation.

Our main result in this paper is as follows.

Theorem 1.1. *Let (M,g) be a complete non-compact Riemannian n-manifold with $n \geq 3$, $1 < p < n$ such that $p^2 < n$. Let $a,\ f \in C^\infty(M)$ be real valued function on M. Suppose that operator $L_p u = \Delta_p u + a(x)u^{p-1}$ is coercive. Under the following assumptions:*

1. *At a point x_o where f is maximal, we are in one of the following cases*

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(i) *$p < 2$, $n > 3p - 2$ and $a(x_o) < 0$*
(ii) *$p = 2$ and $\frac{8(n-1)}{(n-2)(n-4)} a(x_o) < \frac{-\Delta f(x_o)}{f(x_o)} + \frac{2R(x_o)}{n-4}$*
(iii) *$p > 2$ and $(\frac{n-3p+2}{p})\frac{\Delta f(x_o)}{f(x_o)} < R(x_o)$.*

(2) *(M, g) is of bounded geometry, that is: Ricci $> -c$, where $c \geq 0$ is a constant, and the injectivity radius is strictly positive.*
(3) *There exists a constant $C > 0$ such that*

$$|\nabla f| \leq Cf, |\nabla^2 f| \leq Cf, \int_M |a|^{\frac{n}{p}} dv_g \leq C \int_M f dv_g < \infty \text{ and } \int_M f^{p/p*} dv_g < \infty.$$

(4) *The functions a and f are bounded and f is strictly positive.*

Then, there exists a positive solution $u \in H^p_{1,loc}(M)$ of (1.1) such that $u \in C^{1,\alpha}(K)$ on any compact set K of M for some $\alpha \in (0, 1)$.

This article is organized as follows: in the second section we construct a sequence of minimizing weak solutions, in the third section we give sufficient geometric conditions to guarantee the strong convergence of the minimizing sequence. Using the Aubin's test functions, we show in the last section that these geometric conditions are satisfied.

2. Convergence of the minimizing sequence

In this section, we construct a sequence of weak solutions for (1.1). The following theorem has been proved in [2].

Theorem 2.1. *Let (M, g) be a Riemannian compact manifold $1 < p < n$ such that $p^2 < n$, and let a, $f \in C^\infty(M)$ be real functions on M. We assume that the operator $L_p u = \Delta_p u + a(x)u^{p-1}$ is coercive. If at a point x_o where f is maximal, we have one of the following cases*

(i) *$p < 2$, $n > 3p - 2$ and $a(x_o) < 0$*
(ii) *$p = 2$ and $\frac{8(n-1)}{(n-2)(n-4)} a(x_o) < \frac{-\Delta f(p)}{f(p)} + \frac{2R(x_o)}{n-4}$*
(iii) *$p > 2$ and $(\frac{n-3p+2}{p})\frac{\Delta f(x_o)}{f(x_o)} < R(x_o)$.*

Then, there exists a positive solution $u \in H^p_1(M)$ of (1.1) such that $u \in C^{1,\alpha}(M)$ for some $\alpha \in (0, 1)$.

Let Ω_j be an exhaustion of the complete manifold M by compact manifolds with smooth boundary such that $\Omega_j \subset \overset{o}{\Omega}_{j+1}$. Let u_j be the minimizer given by Theorem 2.1 for

$$\begin{gathered} \Delta_p u_j + a(x)u_j^{p-1} = \mu(\Omega_j) f u_j^{p*-1} \quad \text{in } \Omega_j \\ u_j > 0 \quad \text{in } \Omega_j \\ u_j = 0 \quad \text{on } \partial\Omega_j\,. \end{gathered} \tag{2.1}$$

By the monotone decreasness of $\mu(\Omega_j)$ and the coercivity of the operator $L_p u = \Delta_p u + a(x)u^{p-1}$, we have

$$\|u\|_{H^p_1(\Omega_j)} \leq \frac{1}{c}\mu(\Omega_1) \tag{2.2}$$

where $c > 0$ is a constant. Since (2.2) implies the boundedness of $\{u_i\}$ in $H^p_1(M)$, we can choose a subsequence of $\{u_i\}$ still denoted $\{u_i\}$ such that $u_i \to u$ weakly in $H^p_1(M)$

Proposition 2.2. *The sequence $\{u_i\}$ converges weakly on every compact set K of M to a solution $u \in C^{1,\alpha}(K)$ of*

$$\begin{gathered}\Delta_p u + a(x)u^{p-1} = fu^{p*-1} \quad \text{in } K \\ u > 0 \quad \text{in } K \\ u = 0 \quad \text{on } \partial K\end{gathered} \tag{2.3}$$

for some $\alpha \in (0,1)$.

To prove the boundedness of $\{u_i\}$in $C^{1,\alpha}(K)$, we use propositions from the paper of Druet [2] which have their origin in Tolksdorf [7] Guedda and Veron [3] and Vazquez [8].

Proposition 2.3. *Let (M,g) be a compact Riemannian n-manifold. Assume that $u \in H_1^p(M)$ is a solution of $\Delta_p u + a(x)u^{p-1} = f$, where $n \geq 2$, $1 < p < n$, $a(x) \in L^{\frac{n}{p}}(M)$ and $f \in L^{\frac{n}{p}}(M)$, then $u \in L^t(M)$ for $t \in [1,\infty)$.*

Proposition 2.4. *Let (M,g) be a compact Riemannian n-manifold. Assume $n \geq 2$, $1 < p < n$, $f \in L^s(M)$ for some $s > \frac{n}{p}$ and $u \in H_1^p(M)$ is a solution of $\Delta_p u = f$ on M. Then $u \in L^\infty(M)$.*

Proposition 2.5. *Let (M,g) be a compact Riemannian n-manifold and $h(x,r) \in C^o(M \times R)$. Assume $n \geq 2$, $1 < p < n$ and $\forall (x,r) \in M \times R$, $|h(x,r)| \leq C|r|^{p*-1} + D$. If $u \in H_1^p(M)$ is a solution of $\Delta_p u + h(x,u) = 0$, then $u \in C^{1,\alpha}(M)$. Moreover $\|u\|_{C^{1,\alpha}(M)} \leq \tilde{c}$, where $\tilde{c}$ is a constant depending only on $\|u\|_{L^\infty(M)}$ and $\|h(x,r)\|_{L^\infty(M)}$.*

Proof of Proposition 2.2. First we show that the sequence $\{u_j\}$ is bounded in $L^t(K)$ for any $t \in [1,+\infty)$. Involving Proposition 2.3, we have only to check that the sequence $\{a(x) - fu_j^{p*-p}\}$ is bounded in $L^{\frac{n}{p}}(K)$. We have

$$\begin{aligned}\int_K |a(x) - fu_j^{p*-p}|^{\frac{n}{p}} dv_g &\leq 2^{\frac{n}{p}-1} \int_K (|a(x)|^{\frac{n}{p}} + |f|^{\frac{n}{p}} u_j^{p*}) dv_g \\ &= 2^{\frac{n}{p}-1}\big[(\|a\|_{\frac{n}{p}}^K)^{\frac{n}{p}} + (\|f\|_\infty^K)^{\frac{n}{p}} (\|u_j\|_{p*}^K)^{p*}\big]\end{aligned}$$

where $\|u\|_p^K = \big(\int_K |u|^p dv_g\big)^{1/p}$. Since by the relation(3) the sequence$\{u_j\}$ is bounded in $L^p(K)$, so is in $L^{p*}(K)$, we have the desired conclusion.

Next, we show that $\{u_j\}$ is bounded in $L^\infty(K)$. According to Proposition 2.4, we have to show that the sequence $\{g_j\}$ given by $g_j(x) = -a(x)u_j(x)^{p-1} + f(x)u_j(x)^{p*-1}$, is bounded in $L^s(K)$ for some $s > \frac{n}{p}$. But this fact is a consequence of proposition 2.3.

Finally, we take $h(x,u_j) = a(x)u_j^{p-1} - f(x)u_j^{p*-1}$, and since by assumption the functions a and f are bounded on the manifold M, one has the boundness of the sequence $\{h(x,u_j(x)\}$ in the compact set K.

By proposition 2.5, $u_j \in C^{1,\alpha}(K)$ and $\|u_j\|_K^{1,\alpha} \leq c(p,n,K,\|g_j\|_{L^\infty(K)})$. The boundedness of $\{u_j\}$ in $L^\infty(K)$ implies that $\{g_j\}$ and $C(p,n,K,\|g_j\|_{L^\infty(K)})$ are bounded. Consequently, $\{u_j\}$ is bounded in $C^{1,\alpha}(K)$. So by Arzela-Ascoli theorem $\{u_j\}$ converges uniformly towards a weak solution u of (2.3) on each compact set. $\square$

3. Strong convergence

In this section, we have to show that the solution u is not trivial. To achieve this task, we give sufficient conditions that guarantee the strong convergence of minimizers constructed in the previous section. Let K be any compact set of the complete manifold M, $2K$ a compact set containing K and $\eta \in C^{\infty}(M)$ be the function

$$\eta(x) = \begin{cases} 0 & \text{on } K \\ 1 & \text{on } M - 2K\,. \end{cases}$$

Let $k > 1$ and $\{u_q\}$ be the sequence of minimizers given by Proposition 2.2 and $\|.\|_p$ be the $L^p(M)$-norm. we are going to estimate the ratio $\|\nabla(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})\|_p$.

Letting $\{\Omega_k\}$ be the exhaustion, of the complete manifold M, considered in the previous section. Denote by $\Lambda_k = \{u \in H_1^p(\Omega_k) : \int_{\Omega_k} f|u|^{p*} dv_g = 1\}$ and $I_k(u)$ the functional $I_k(u) = \int_{\Omega_k} (|\nabla u|^p + |u|^p) dv_g$.

Proposition 3.1. *Under the conditions (2), (3), (4), of Theorem 1.1 and*

$$(\sup_{M-K} f(x))^{p/p*} \inf_{u \in \Lambda_k} I_k(u) < K(n,p)^{-p},$$

the ratio $\|\nabla(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})\|_p$ *is bounded.*

Proof. For $p \geq 2$, using Simon's inequality [5], that is to say: for any vector fields X and Y on the manifold M,

$$|X+Y|^p \leq C_p \left\langle |X|^{p-2}X + |Y|^{p-2}Y, X+Y \right\rangle$$

where C_p is a constant depending on p and $\langle .,. \rangle$ denoting the metric. We get

$$\begin{aligned}
&\|\nabla(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})\|_p^p \\
&= \int_M |\nabla(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})|^p dv_g \\
&= \int_M |(u_q^{\frac{k+p-1}{p}} \nabla(\eta f^{\frac{1}{p*}}) + \frac{k+p-1}{p} (\eta f^{\frac{1}{p*}}) u_q^{\frac{k-1}{P}} \nabla u_q)|^p dv_g \\
&\leq C_p \int_M \Big[u_q^{(\frac{k+p-1}{p})(p-1)} |\nabla(\eta f^{\frac{1}{p*}})|^{p-2} \nabla(\eta f^{\frac{1}{p*}}) \\
&\quad + (\tfrac{k+p-1}{p})^{p-1} (\eta f^{\frac{1}{p*}})^{p-1} u_q^{(\frac{k-1}{p})(p-1)} |\nabla u_q|^{p-2} \nabla u_q \Big] \\
&\quad \times \big[u_q^{\frac{k+p-1}{p}} \nabla(\eta f^{\frac{1}{p*}}) + \tfrac{k+p-1}{p} (\eta f^{\frac{1}{p*}})^{p-1} u_q^{\frac{k-1}{p}} \nabla u_q \big] dv_g \\
&= C_p \Big[\int_M u_q^{k+p-1} |\nabla(\eta f^{\frac{1}{p*}})|^p dv_g + (\tfrac{k+p-1}{p})^p \int_M (\eta f^{\frac{1}{p*}})^p u_q^{k-1} |\nabla u_q|^p dv_g \\
&\quad \times \tfrac{k+p-1}{p} \int_M \eta f^{\frac{1}{p*}} u_q^{k+p-2} |\nabla(\eta f^{\frac{1}{p*}})|^{p-2} \langle \nabla(\eta f^{\frac{1}{p*}}), \nabla u_q \rangle dv_g \\
&\quad + (\tfrac{k+p-1}{p})^{p-1} \int_M (\eta f^{\frac{1}{p*}})^{p-1} u_q^k |\nabla u_q|^{p-2} \langle \nabla(\eta f^{\frac{1}{p*}}), \nabla u_q \rangle dv_g].
\end{aligned}$$

On the other hand,

$$\int_M \eta^p f^{p/p*} u_q^k \Delta_p u_q dv_g = k \int_M \eta^p f^{p/p*} u_q^{k-1} |\nabla u_q|^p dv_g$$

$$+ p \int_M (\eta f^{\frac{1}{p*}})^{p-1} u_q^k |\nabla u_q|^{p-2} \langle \nabla u_q, \nabla(\eta f^{\frac{1}{p*}})\rangle dv_g$$

and

$$\begin{aligned}
&\int_M \eta f^{\frac{1}{p*}} u_q^{k+p-1} \Delta_p(\eta f^{\frac{1}{p*}}) dv_g \\
&= \int_M u_q^{k+p-1} |\nabla(\eta f^{\frac{1}{p*}})|^p dv_g \\
&\quad + (k+p-1) \int_M (\eta f^{\frac{1}{p*}}) u_q^{k+p-2} |\nabla(\eta f^{\frac{1}{p*}})|^{p-2} \langle \nabla u_q, \nabla \eta f^{\frac{1}{p*}} \rangle dv_g
\end{aligned}$$

so

$$\begin{aligned}
&\|\nabla(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})\|_p^p \\
&\le C_p \Big[\int_M \eta f^{\frac{1}{p*}} u_q^{k+p-1} \Delta_p(\eta f^{\frac{1}{p*}}) dv_g + \frac{1}{k}(\frac{k+p-1}{p})^p \int_M \eta^p f^{p/p*} u_q^k \Delta_p u_q dv_g \\
&\quad - \tfrac{p-1}{p}(k+p-1) \int_M \eta f^{\frac{1}{p*}} u_q^{k+p-2} |\nabla(\eta f^{\frac{1}{p*}})|^{p-2} \langle \nabla u_q, \nabla(\eta f^{\frac{1}{p*}})\rangle dv_g \\
&\quad - (\tfrac{k+p-1}{p})^{p-1} \tfrac{p-1}{k} \int_M (\eta f^{\frac{1}{p*}})^{p-1} u_q^k |\nabla u_q|^{p-2} \langle \nabla u_q, \nabla(\eta f^{\frac{1}{p*}})\rangle dv_g \Big].
\end{aligned}$$

Multiplying (1.1) by $(\eta f^{\frac{1}{p*}})^p u_q^k$ and integrating over M, we get

$$\begin{aligned}
&\int_M (\eta f^{\frac{1}{p*}})^p u_q^k \Delta_p u_q dv_g \\
&= -\int_M a(x)(\eta f^{\frac{1}{p*}})^p u_q^{k+p-1} dv_g + \mu_{(\Omega_q)} \int_M (\eta f^{\frac{1}{p*}})^p f u_q^{k+p*-1} dv_g.
\end{aligned} \tag{3.1}$$

Using Hölder inequality, we obtain

$$\begin{aligned}
&\int_M (\eta f^{\frac{1}{p*}})^p f u_q^{k+p*-1} dv_g \\
&\le (\sup_{M-K} f)^{p/p*} \Big(\int_{M-K} f u_q^{p*} dv_g \Big)^{1-\frac{p}{p*}} \Big(\int_M (\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})^{p*} dv_g \Big)^{p/p*}.
\end{aligned} \tag{3.2}$$

The first term of the right-hand side of (3.1) is estimated as

$$\begin{aligned}
&\int_M a(x)(\eta f^{\frac{1}{p*}})^p u_q^{k+p-1} dv_g \\
&\le \Big(\int_{M-K} |a(x)|^{\frac{n}{p}} dv_g \Big)^{\frac{p}{n}} \Big(\int_{M-K} (\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})^{p*} dv_g \Big)^{p/p*}.
\end{aligned}$$

Since by assumption

$$\Big(\int_M |a(x)|^{\frac{n}{p}} dv_g \Big)^{p/n} < C \int_M f dv_g < \infty,$$

we choose the compact set K so that

$$\int_{M-K} f dv_g < \frac{\varepsilon}{C}.$$

Then

$$\begin{aligned}
&\int_M (\eta f^{\frac{1}{p*}})^p u_q^k \Delta_p u_q dv_g \\
&\leq \Big(((\sup_{M-K} f)^{p/p*} \int_M f u_q^{p*})^{1-\frac{p}{p*}} + \varepsilon \Big) \Big(\int_M \big(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}}\big)^{p*} dv_g \Big)^{p/p*}.
\end{aligned} \tag{3.3}$$

On the other hand we have

$$\nabla(\eta f^{\frac{1}{p*}}) = f^{\frac{1}{p*}} \nabla \eta + \frac{1}{p*} \eta f^{\frac{1}{p*}-1} \nabla f$$

and since by assumption $|\nabla f| \leq C$, we obtain

$$|\nabla \eta f^{\frac{1}{p*}}| \leq f^{\frac{1}{p*}} |\nabla \eta| + \frac{\eta c}{p*} f^{\frac{1}{p*}} \leq C f^{\frac{1}{p*}}$$

where C is a universal constant. So

$$\begin{aligned}
&\int_M \eta f^{\frac{1}{p*}} u_q^{k+p-2} |\nabla(\eta f^{\frac{1}{p*}})|^{p-2} \langle \nabla(\eta f^{\frac{1}{p*}}), \nabla u_q \rangle dv_g \\
&\leq C \int_{M-K} f^{\frac{p}{p*}} u_q^{k+p-2} |\nabla u_q| dv_g.
\end{aligned} \tag{3.4}$$

Using Hölder inequality we obtain that the right-hand side of this inequality is bounded above by

$$\begin{aligned}
&\int_{M-K} f^{p/p*} u_q^{k+p-2} |\nabla u_q| dv_g \\
&\leq \Big(\int_{M-K} |\nabla u_q|^p dv_g \Big)^{\frac{1}{p}} \Big(\int_{M-K} (f^{p/p*} u_q^{k+p-2} \Big)^{\frac{p}{p-1}} dv_g)^{1-\frac{1}{p}}.
\end{aligned} \tag{3.5}$$

Applying Hölder's inequality again, we get

$$\begin{aligned}
&\int_{M-K} (f^{p/p*} u_q^{k+p-2})^{\frac{p}{p-1}} dv_g \\
&\leq \Big(\int_{M-K} u_q^{p*} dv_g \Big)^{\frac{p(k+p-2)}{p*(p-1)}} \Big(\int_{M-K} f^{\frac{p(p-1)}{p*(p-1)-p(k+p-2)}} dv_g \Big)^{1-\frac{p(k+p-2)}{p*(p-1)}} \\
&\leq (\sup_{M-K} f)^{\frac{p(k+p-2)}{p*(p-1)}} \Big(\int_{M-K} u_q^{p*} dv_g \Big)^{\frac{p(k+p-2)}{p*(p-1)}} \Big(\int_{M-K} f^{p/p*} dv_g \Big)^{1-\frac{p(k+p-2)}{p*(p-1)}}.
\end{aligned} \tag{3.6}$$

As above, we get

$$\begin{aligned}
&\int_M (\eta f^{\frac{1}{p*}})^{p-1} u_q^k |\nabla u_q|^{p-2} \langle \nabla u_q, \nabla(\eta f^{\frac{1}{p*}}) \rangle dv_g \\
&\leq C \int_{M-K} f^{p/p*} u_q^k |\nabla u_q|^{p-1} dv_g \\
&\leq C \Big(\int_{M-K} f^{p/p*} u_q^{p*} \Big)^{\frac{k}{p*}} \Big(\int_{M-K} f^{p/p*} |\nabla u_q|^{\frac{p*(p-1)}{p*-k}} \Big)^{1-\frac{k}{p*}}.
\end{aligned}$$

Since $\alpha = (p-1)\frac{p*}{p*-k}$, we have $p-\alpha = \frac{p(p*-k)-p*(p-1)}{p*-k} = \frac{p*-pk}{p*-k} > 0$ and

$$\begin{aligned} &\int_{M-K} f^{p/p*}|\nabla u_q|^\alpha dv_g \\ &\le \Big(\int_{M-K}|\nabla u_q|^p dv_g\Big)^{\frac{\alpha}{p}}\Big(\int_{M-K} f^{\frac{p^2}{p*(p-\alpha)}}dv_g\Big)^{1-\frac{\alpha}{p}} \\ &\le (\sup_{M-K} f)^{\frac{p(p-1)}{p*-pk}}\Big(\int_{M-K}|\nabla u_q|^p dv_g\Big)^{\frac{\alpha}{p}}\Big(\int_{M-K} f^{p/p*}dv_g\Big)^{1-\frac{\alpha}{p}}. \end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned} \Delta_p(\eta f^{\frac{1}{p*}}) &= -\operatorname{div}(|\nabla(\eta f^{\frac{1}{p*}})|^{p-2}\nabla(\eta f^{\frac{1}{p*}})) \\ &= |\nabla \eta f^{\frac{1}{p*}}|^{p-2}\Delta(\eta f^{\frac{1}{p*}}) - \operatorname{trace}\big(\nabla|\nabla(\eta f^{\frac{1}{p*}})|^{p-2}\otimes\nabla(\eta f^{\frac{1}{p*}})\big) \end{aligned}$$

and

$$\begin{aligned} \Delta(\eta f^{\frac{1}{p*}}) &= f^{\frac{1}{p*}}\Delta\eta + \eta\Delta f^{\frac{1}{p*}} - trace(\nabla\eta\otimes\nabla f^{\frac{1}{p*}}) \\ &\le f^{\frac{1}{p*}}\Delta\eta + \frac{1}{p*}(1-\frac{1}{p*})\eta f^{\frac{1}{p*}-2}|\nabla f|^2 + \frac{1}{p*}\eta f^{\frac{1}{p*}-1}\Delta f + \frac{1}{p*}f^{\frac{1}{p*}-1}|\nabla f| \end{aligned}$$

then

$$|\Delta(\eta f^{\frac{1}{p*}})| \le C f^{\frac{1}{p*}}$$

and

$$|\nabla(\eta f^{\frac{1}{p*}})|^{p-2}|\Delta(\eta f^{\frac{1}{p*}})| \le C f^{\frac{p-1}{p*}}.$$

From

$$|\nabla|\nabla(\eta f^{\frac{1}{p*}})|^{p-2}| = (p-2)|\nabla(\eta f^{\frac{1}{p*}})|^{p-3}|\nabla|\nabla(\eta f^{\frac{1}{p*}})||$$

and Kato's inequality, we deduce that

$$\big|\nabla|\nabla(\eta f^{\frac{1}{p*}})|^{p-2}\big| \le (p-2)|\nabla(\eta f^{\frac{1}{p*}})|^{p-3}|\nabla^2(\eta f^{\frac{1}{p*}})|.$$

Now, since

$$\begin{aligned} &\nabla^2(\eta f^{\frac{1}{p*}}) \\ &= f^{\frac{1}{p*}}\nabla^2\eta + \frac{2}{p*}f^{\frac{1}{p*}-1}\nabla\eta\otimes\nabla f + \frac{1}{p*}(1-\frac{1}{p*})\eta f^{\frac{1}{p*}-2}\nabla f\otimes\nabla f + \frac{1}{p*}\nabla^2 f, \end{aligned}$$

we obtain

$$\big|\nabla|\nabla(\eta f^{\frac{1}{p*}})|^{p-2}\big| \le C f^{\frac{p-1}{p*}}.$$

Finally, we get

$$|\Delta_p(\eta f^{\frac{1}{p*}})| \le C f^{\frac{p-1}{p*}}$$

and

$$\begin{aligned} &\int_M \eta f^{\frac{1}{p*}}u_q^{k+p-1}\Delta_p(\eta f^{\frac{1}{p*}})dv_g \\ &\le C\int_{M-K} f^{p/p*}u_q^{k+p-1} \\ &\le C\Big(\int_{M-K} u_q^{p*}dv_g\Big)^{\frac{k+p-1}{p*}}\Big(\int_{M-K} f^{\frac{p*}{p*k-p+1}}dv_g\Big)^{1-\frac{k+p-1}{p*}}. \end{aligned} \tag{3.8}$$

Sobolev's inequality leads to

$$\Big(\int_{M-K} u_q^{p*}dv_g\Big)^{p/p*}$$

$$
\begin{aligned}
&\le (K(n,p)^p+\varepsilon)\int_{M-K}|\nabla u_q|^p dv_g + A\int_{M-K} u_q^p dv_g\\
&\le (K(n,p)^p+\varepsilon)\Big(\int_{M-K}|\nabla u_q|^p dv_g + \frac{A}{K(n,p)^p+\varepsilon}\int_{M-K} u_q^p dv_g\Big).
\end{aligned}
$$

From the coercivity of the operator $L_p u = -\Delta_p u - a(x)|u|^{p-2}u$, we get

$$
\begin{aligned}
&\Big(\int_{M-K} u_q^{p*} dv_g\Big)^{p/p*}\\
&\le \frac{1}{c}(K(n,p)^p+\varepsilon)\max\big(1, \frac{A}{K(n,p)^p+\varepsilon}\big)\int_{M-K}(|\nabla u_q|^p + u_q^p)dv_g\\
&\le \tilde{C}\int_M (|\nabla u_q|^p + u_q^p)dv_g,
\end{aligned}
$$

where $\tilde{C} = \frac{1}{c}(K(n,p)^p+\varepsilon)\max\big(1, \frac{A}{K(n,p)^p+\varepsilon}\big)$ and by construction of the sequence $\{u_q\}$, which has a compact support in Ω_q, we have

$$
\int_M (|\nabla u_q|^p + a(x)u_q^p)dv_g = \lambda_q
$$

hence

$$
\Big(\int_{M-K} u_q^{p*} dv_g\Big)^{p/p*} \le \tilde{C}\lambda_q.
$$

Since by assumption the Lagrange multipliers satisfy

$$
\lambda_q < \frac{1}{K(n,p)^p(\sup_{M-K} f)^{p/p*}},
$$

we have

$$
\Big(\int_{M-K} u_q^{p*} dv_g\Big)^{p/p*} \le C. \tag{3.9}
$$

Combining inequalities (5) to (13) we obtain

$$
\begin{aligned}
\|\nabla(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})\|_p^p &\le \lambda_q\Big(\big(\sup_{M-K} f\big)^{p/p*}\big(\int_{M-K} f u_q^{p*} dv_g\big)^{1-\frac{p}{p*}} + \varepsilon\Big)\\
&\quad\times\Big(\int_{M-K}\big(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}}\big)^{p*} dv_g\Big)^{p/p*} + C\,.
\end{aligned}
$$

Using Sobolev's inequality, this expression is bounded by

$$
\begin{aligned}
&\lambda_q\Big(\big(\sup_{M-K} f\big)^{p/p*}\big(\int_{M-K} f u_q^{p*} dv_g\big)^{1-\frac{p}{p*}} + \varepsilon\Big)\\
&\times\Big((K(n,p)^p+\varepsilon)\|\nabla(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})\|_p^p + A\|\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}}\|_p^p\Big) + C\,,
\end{aligned} \tag{3.10}
$$

where $K(n,p)$ is the best constant in the Sobolev's inequality. For the last term in (3.10), we write

$$
\begin{aligned}
&\int_{M-K} f^{p/p*} u_q^{k+p-1} dv_g\\
&\le \Big(\int_{M-K} u_q^{p*} dv_g\Big)^{\frac{\kappa+p-1}{p*}}\Big(\int_{M-K} f^{\frac{p}{p*-k-p+1}} dv_g\Big)^{1-\frac{\kappa+p-1}{p*}}\\
&\le \big(\sup_{M-K} f\big)^{\frac{p(1-p-k)}{p*^2}}\Big(\int u_q^{p*} dv_g\Big)^{\frac{\kappa+p-1}{p*}}\Big(\int f^{p/p*} dv_g\Big)^{1-\frac{\kappa+p-1}{p*}} < \infty.
\end{aligned}
$$

From the assumption on the Lagrange multipliers, $\|\nabla(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})\|_p^p$ is bounded.

In the case $1 < p < 2$, the Simon's inequality writes

$$|X+Y|^p \leq C_p \left\langle |X|^{p-2}X + |Y|^{p-2}Y, X+Y \right\rangle^{\frac{p}{2}} (|X|^p + |Y|^p)^{1-\frac{p}{2}}$$

where X, Y are any vector fields on the manifold M.

Putting

$$\begin{gathered} X = u_q^{\frac{k+p-1}{p}} \nabla(\eta f^{\frac{1}{p*}}), \\ Y = \frac{k+p-1}{p} (\eta f^{\frac{1}{p*}}) u_q^{\frac{k-1}{P}} \frac{k+p-1}{p} (\eta f^{\frac{1}{p*}}) u_q^{\frac{k-1}{P}} \nabla u_q \end{gathered}$$

we get

$$\begin{aligned} &\Big| u_q^{\frac{k+p-1}{p}} \nabla(\eta f^{\frac{1}{p*}}) + \frac{k+p-1}{p} (\eta f^{\frac{1}{p*}}) u_q^{\frac{k-1}{P}} \nabla u_q \Big|^p dv_g \\ &\leq C_p \Big[\Big(u_q^{(\frac{k+p-1}{p})(p-1)} |\nabla(\eta f^{\frac{1}{p*}})|^{p-2} \nabla(\eta f^{\frac{1}{p*}}) \\ &\quad + (\tfrac{k+p-1}{p})^{p-1} (\eta f^{\frac{1}{p*}})^{p-1} u_q^{(\frac{k-1}{p})(p-1)} |\nabla u_q|^{p-2} \nabla u_q \Big) \\ &\quad \times \Big(u_q^{\frac{k+p-1}{p}} \nabla(\eta f^{\frac{1}{p*}}) + \frac{k+p-1}{p} (\eta f^{\frac{1}{p*}}) u_q^{\frac{k-1}{P}} \nabla u_q \Big) \Big]^{p/2} \\ &\quad \times \Big[u_q^{k+p-1} |\nabla(\eta f^{\frac{1}{p*}})|^p + (\frac{k+p-1}{p})^p (\eta f^{\frac{1}{p*}})^p u_q^{k-1} |\nabla u_q|^p \Big]^{1-\frac{p}{2}}. \end{aligned}$$

Then

$$\begin{aligned} &\|\nabla(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})\|_p^p \\ &\leq C_p \int_M \Big[u_q^{k+p-1} |\nabla(\eta f^{\frac{1}{p*}})|^p + (\frac{k+p-1}{p})^p (\eta f^{\frac{1}{p*}})^p u_q^{k-1} |\nabla u_q|^p \\ &\quad + \frac{k+p-1}{p} (\eta f^{\frac{1}{p*}}) u_q^{k+p-2} |\nabla(\eta f^{\frac{1}{p*}})|^{p-2} \langle \nabla u_q, \nabla(\eta f^{\frac{1}{p*}}) \rangle \\ &\quad + (\frac{k+p-1}{p})^{p-1} (\eta f^{\frac{1}{p*}})^{p-1} u_q^k |\nabla u_q|^{p-2} \langle \nabla u_q, \nabla(\eta f^{\frac{1}{p*}}) \rangle \Big]^{p/2} \\ &\quad \times \Big[u_q^{k+p-1} |\nabla(\eta f^{\frac{1}{p*}})|^p + (\frac{k+p-1}{p})^p (\eta f^{\frac{1}{p*}})^p u_q^{k-1} |\nabla u_q|^p \Big]^{1-\frac{p}{2}} dv_g. \end{aligned}$$

And by Hölder's inequality, the above expression is less than or equal to

$$\begin{aligned} &C_p \Big(\int_M \Big[u_q^{k+p-1} |\nabla(\eta f^{\frac{1}{p*}})|^p + (\frac{k+p-1}{p})^p (\eta f^{\frac{1}{p*}})^p u_q^{k-1} |\nabla u_q|^p \\ &+ \frac{k+p-1}{p} (\eta f^{\frac{1}{p*}}) u_q^{k+p-2} |\nabla(\eta f^{\frac{1}{p*}})|^{p-2} \langle \nabla u_q, \nabla(\eta f^{\frac{1}{p*}}) \rangle \\ &+ (\frac{k+p-1}{p})^{p-1} (\eta f^{\frac{1}{p*}})^{p-1} u_q^k |\nabla u_q|^{p-2} \langle \nabla u_q, \nabla(\eta f^{\frac{1}{p*}}) \rangle \Big] dv_g \Big)^{2/p} \\ &\times \Big(\int_M (u_q^{k+p-1} |\nabla(\eta f^{\frac{1}{p*}})|^p + (\frac{k+p-1}{p})^p (\eta f^{\frac{1}{p*}})^p u_q^{k-1} |\nabla u_q|^p) dv_g \Big)^{1-\frac{p}{2}}. \end{aligned}$$

Arguing as in the case $p \geq 2$, we obtain that $\|\nabla(\eta f^{\frac{1}{p*}} u_q^{\frac{k+p-1}{p}})\|_p^p$ is bounded. $\square$

4. Generic Theorem

Letting K be any compact set of the complete manifold M, we formulate in this section a generic theorem. First, we establish

Lemma 4.1. *Assume that every subsequence of $\{u_q\}$ which converges in $L^p(M)$ with $p>1$, converges to 0. Also assume there exists a constant $C>0$, independent of q such that $\|\nabla(\eta f^{\frac{1}{p*}}u_q^{\frac{k+p-1}{p}})\|_p^p \le C$, $k>1$. Then*
$$\lim_{q\to\infty}\sup\int_M(\eta f^{\frac{1}{p*}}u_q)^{p*}dv_g=0.$$

Proof. Suppose that $\lim_{q\to\infty}\sup\int_M(\eta f^{\frac{1}{p*}}u_q)^{p*}>0$. Using Hölder's inequality we obtain
$$\begin{aligned}&\int_M(\eta f^{\frac{1}{p*}}u_q)^{p*}dv_g\\&\le\sup_{M-K}f\Big(\int_M\big(\eta f^{\frac{1}{p*}}u_q^{\frac{k+p-1}{p}}\big)^{p*}\Big)^{\frac{n(p-1)+p}{n(k+p-1)}}\Big(\int_M u_q^{\frac{n(k+p-1)}{nk-p}}dv_g\Big)^{\frac{nk-p}{n(k+p-1)}}\end{aligned}$$
then
$$\lim_{q\to\infty}\sup\int_M u_q^{\frac{n(k+p-1)}{nk-p}}dv_g>0.$$
A contradiction with the fact that every subsequence of u_q converging in $L^p(M)$, $p>1$, converges to 0. □

As a consequence of the above lemma, we obtain the following generic theorem. Denote by $\Lambda=\{u\in H_1^p(M):\ \int_M f|u|^{p*}dv_g=1\}$ and $I(u)$ is the functional given by $I(u)=\int_M(|\nabla u|^p+|u|^p)dv_g$ where M is a complete Riemannian manifold.

Theorem 4.2. *Let (M,g) be a complete Riemannian manifold of bounded geometry, $1<p<n$, and let $a,\ f\in C^\infty(M)$ be real functions on M with $f>0$. We assume that:*

- (i) *The operator $L_pu=\Delta_pu+a(x)u^{p-1}$ is coercive*
- (ii) *Conditions (3) and (4) of Theorem 1.1 at infinity on f are satisfied*
- (iii) $(\sup_M f)^{\frac{p}{p*}}\inf_{u\in\Lambda}I(u)<K(n,p)^{-p}$.

Then (1.1) possesses a positive solution $u\in C^{1,\alpha}(K)$ for any compact set $K\subset M$ and some $\alpha\in(0,1)$.

Proof. Suppose that
$$\mu f(x))^{p/p*}K(n,p)^p\lim_{q\to\infty}\sup\int_{B(x_o,\delta)}fu_q^{p*}dvg<1$$
then by Lemma8, we get that
$$\lim_{q\to\infty}\sup\int_{B(x_o,\delta)}fu_q^{p*}dv_{tg}=0$$
which contradicts the fact that
$$\int_M fu_q^{p*}dv_g=1. \tag{4.1}$$
In fact
$$\int_M fu_q^{p*}dv_g=\int_{\cup_{i=1}^\infty B(x_i,\delta)}fu_q^{p*}dv_g\le\sum_{i=1}^\infty\int_{B(x_i,\delta)}fu_q^{p*}dv_g \tag{4.2}$$

where $M = \cup_{i=1}^{\infty} B(x_i, \delta)$.

So for sufficient large q the last term in (4.2) is strictly smaller that 1. Consequently

$$\mu f(x))^{p/p*} K(n,p)^p \lim_{q\to\infty} \sup \int_{B(x_o,\delta)} f u_q^{p*} dv_g \geq 1$$

and since by assumption $\mu f(x))^{p/p*} K(n,p)^p < 1$, we obtain

$$\lim_{q\to\infty} \sup \int_{B(x_o,\delta)} f u_q^{p*} dv_g > 1.$$

which is a contradiction with (4.1).

Then the condition that every subsequence of the sequence of minimizers $\{u_q\}$ which converges has 0 as a limit is false and the theorem is proved. $\square$

Examples of functions satisfying the conditions of Theorem 4.2. The conditions at infinity in Theorem 4.2 are satisfied, for example by functions decreasing like power functions: $f \sim r^{-q}$, $\nabla f \sim \rho^{-q-1}$ and $\nabla^2 f \sim r^{-q-2}$ with $q > n\frac{p*}{p}$. Since $\int_M f^{p/p*} dv_g < +\infty$ implies that $\frac{1}{r^{(1-\frac{1}{n})q+1-n}}$ is integrable.

If the function a decays at infinity as r^{-q} , then the condition that $\int_M f^{\frac{n}{p}} dv_g \leq C \int_M f dv_g < +\infty$ implies that the decay rate q satisfies $q > p$.

5. Test functions

In this section we give the proof of our main result (Theorem 1.1). For this task we check that the condition (iii) of the generic theorem proved in section 4 is satisfied.

Let K be any compact set of the manifold M and $x_o \in M - K$ be the maximum on of the function f as given in Theorem 1.1. Let $r = d(x_o, x)$ the distance function from x_o to any point x in the manifold $M - K$.

Let $\delta > 0$ be smaller than the injectivity radius; for $\epsilon > 0$, we consider the test function

$$u_\varepsilon(x) = \begin{cases} (\varepsilon + r^{\frac{p}{p-1}})^{1-\frac{n}{p}} - (\varepsilon + \delta^{\frac{p}{p-1}})^{1-\frac{n}{p}} & \text{if } r < \delta \\ 0 & \text{if } r \geq \delta\,. \end{cases}$$

Note that the function u_ε was introduced by Aubin in [1]. We have

$$|\nabla u_\varepsilon(x)|^p = \begin{cases} (\frac{n-p}{p-1})^p (\varepsilon + r^{\frac{p}{p-1}})^{-\frac{n}{p}} r^{(\frac{p}{p-1})} & \text{if } r < \delta \\ 0 & \text{if } r \geq \delta \end{cases}$$

so

$$\int_{B(x_o,\delta)} |\nabla u_\varepsilon(x)|^p dv_g = (\frac{n-p}{p-1})^p \int_0^\delta (\varepsilon + r^{\frac{p}{p-1}})^{-n} r^{n+\frac{1}{p-1}} dr \int_{S^{n-1}(r)} d\Omega. \tag{5.1}$$

where $d\Omega$ denotes the element volume on the sphere $S^{n-1}(r)$.

Let $S(r) = \int_{S^{n-1}(r)} d\Omega$. Taking into account the expansion of the determinant in a system of geodesic coordinates at a point x_o, we get

$$\sqrt{g} = 1 - R_{ij} x^i x^j + o(r^2)\,.$$

A computation in [1] gives us

$$S(r) = \omega_{n-1}(1 - \frac{R}{6n} r^2 + o(r^2))$$

where w_{n-1} is the volume of the standard unit sphere S^{n-1} in R^n. The integral (5.1) becomes

$$\begin{aligned}&\int_{B(x_o,\delta)}|\nabla u_\varepsilon(x)|^p dv_g\\ &=(\frac{n-p}{p-1})^p\omega_{n-1}\int_0^\delta(\varepsilon+r^{\frac{p}{1-p}})^{-n}r^{n+\frac{1}{p-1}}\big(1-\frac{R}{6n}r^2+o(r^2)\big)dr.\end{aligned}$$

Letting $s=r\varepsilon^{\frac{1-p}{p}}$, we get

$$\begin{aligned}\int_{B(x_o,\delta)}|\nabla u_\varepsilon(x)|^p dv_g=(\frac{n-p}{p-1})^p\omega_{n-1}\varepsilon^{1-\frac{n}{p}}\int_0^{\delta\varepsilon^{\frac{1-p}{p}}}&\big(1+s^{\frac{p}{1-p}}\big)^{-n}s^{n+\frac{1}{p-1}}\\ \times\big(1-\frac{R}{6n}s^2\varepsilon^{\frac{2(p-1)}{p}}&+o(s^2\varepsilon^{\frac{2(p-1)}{p}})\big)ds,\end{aligned} \tag{5.2}$$

set

$$\begin{gathered}I_p^q=\int_0^\infty t^{q-1}(1+t)^{-p}dt\quad\text{with } p-q-1>0,\\ B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt\quad\text{with } p>0,\ q>0\,.\end{gathered}$$

Put $t=s^{\frac{p}{p-1}}$, then the integral (5.2) becomes

$$\begin{aligned}&\int_{B(x_o,\delta)}|\nabla u_\varepsilon(x)|^p dv_g\\ &=\frac{p-1}{p}(\frac{n-p}{p-1})^p\omega_{n-1}\varepsilon^{1-\frac{n}{p}}\Big[\int_0^{\delta^{\frac{p}{p-1}}\varepsilon^{-1}}(1+t)^{-n}t^{n(1-\frac{1}{p})}dt\\ &\quad-\varepsilon^{2(1-\frac{1}{p})}\frac{R(x_o)}{6n}\int_0^{\delta^{\frac{p}{p-1}}\varepsilon^{-1}}(1+t)^{-n}t^{(n+2)(1-\frac{1}{p})}dt+o(\varepsilon^{\frac{2(p-1)}{p}})\Big]\\ &=\frac{p-1}{p}(\frac{n-p}{p-1})^p\omega_{n-1}\varepsilon^{1-\frac{n}{p}}\Big[\int_0^\infty(1+t)^{-n}t^{n(1-\frac{1}{p})}dt\\ &\quad-\varepsilon^{2(1-\frac{1}{p})}\frac{R(x_o)}{6n}\int_0^\infty(1+t)^{-n}t^{(n+2)(1-\frac{1}{p})}dt-\int_{\delta^{\frac{p}{p-1}}\varepsilon^{-1}}^\infty(1+t)^{-n}t^{n(1-\frac{1}{p})}dt\\ &\quad+\varepsilon^{2(1-\frac{1}{p})}\frac{R(x_o)}{6n}\int_{\delta^{\frac{p}{p-1}}\varepsilon^{-1}}^\infty(1+t)^{-n}t^{(n+2)(1-\frac{1}{p})}dt+o(\varepsilon^{\frac{2(p-1)}{p}})\Big].\end{aligned}$$

We have

$$\lim_{\varepsilon\to0}\int_{\delta^{\frac{p}{p-1}}\varepsilon^{-1}}^\infty(1+t)^{-n}t^{n(1-\frac{1}{p})}dt=0$$

and if $n+2>3p$, then

$$\lim_{\varepsilon\to0}\int_{\delta^{\frac{p}{p-1}}\varepsilon^{-1}}^\infty(1+t)^{-n}t^{(n+2)(1-\frac{1}{p})}dt=0\,.$$

So

$$\begin{aligned}&\int_{B(x_o,\delta)}|\nabla u_\varepsilon(x)|^p dv_g\\ &=\frac{p-1}{p}(\frac{n-p}{p-1})^p\omega_{n-1}\varepsilon^{1-\frac{n}{p}}\Big[I_n^{n(1-\frac{1}{p})}-\varepsilon^{2(1-\frac{1}{p})}\frac{R(x_o)}{6n}I_n^{(n+2)(1-\frac{1}{p})}+o\big(\varepsilon^{2(1-\frac{1}{p})}\big)\Big].\end{aligned}$$

On the other hand, a simple computation, for ($p > q+1$), gives the following formula

$$I_p^q = B(q+1, p-q-1) = \frac{\Gamma(q+1)\Gamma(p-q-1)}{\Gamma(p)},$$

where Γ denotes the Euler function from which we obtain the following relation

$$I_n^{(n+2)(1-\frac{1}{p})} = \frac{\Gamma((n+2)(1-\frac{1}{p})+1)\Gamma(\frac{n+2}{p}-3)}{\Gamma(n(1-\frac{1}{p})+1)\Gamma(\frac{n}{p}-1)} I_n^{n(1-\frac{1}{p})} = a(n,p) I_n^{n(1-\frac{1}{p})}.$$

Finally the equality (5.2) becomes

$$\begin{aligned}
&\int_{B(x_o,\delta)} |\nabla u_\varepsilon(x)|^p dv_g \\
&= \frac{p-1}{p}(\frac{n-p}{p-1})^p \omega_{n-1}\varepsilon^{1-\frac{n}{p}} I_n^{n(1-\frac{1}{p})}[1-\varepsilon^{2(1-\frac{1}{p})}a(n,p)\frac{R(x_o)}{6n}+o(\varepsilon^{2(1-\frac{1}{p})})].
\end{aligned} \tag{5.3}$$

The expansion of $\int_{B(x_o,\delta)} a u_\varepsilon^p dv_g$ is computed in the same way as above,

$$\begin{aligned}
&\int_{B(x_o,\delta)} a u_\varepsilon^p dv_g \\
&= \int_0^\delta u_\varepsilon^p r^{n-1} dr \int_{S^{n-1}(r)} a\sqrt{g}\, d\Omega \\
&= \int_0^\delta \big((\varepsilon + r^{\frac{p}{p-1}})^{1-\frac{n}{p}} - \nu\big)^p r^{n-1} dr \\
&\quad \times \int_{S^{n-1}(r)} \big(a(x_o) + \frac{1}{2}\nabla_{ij}a(x_o)x^i x^j + o(r^2)\big)\big(1 - \frac{1}{6}R_{ij}(x_o)x^i x^j + o(r^2)\big) d\Omega \\
&= \omega_{n-1} a(x_o) \int_0^\delta r^{n-1}(\varepsilon + r^{\frac{p}{p-1}})^{p-n}\Big[1 - p\nu(\varepsilon + r^{\frac{p}{p-1}})^{\frac{n}{p}-1} + o\big((\varepsilon + r^{\frac{p}{p-1}})^{\frac{n}{p}-1}\big)\Big] \\
&\quad \times \Big[1 - (\frac{R(x_o)}{6n} + \frac{\Delta a(x_o)}{2nR(x_o)})r^2 + o(r^2)\Big] dr\,.
\end{aligned}$$

Putting $s = r^{\frac{1-p}{p}}$, we get

$$\begin{aligned}
&\int_{B(x_o,\delta)} a u_\varepsilon^p dv_g \\
&= \omega_{n-1} a(x_o) \varepsilon^{p-\frac{n}{p}} \int_0^{\delta\varepsilon^{\frac{1-p}{p}}} s^{n-1}(1 + s^{\frac{p}{p-1}})^{p-n}\Big[1 - p\nu\varepsilon^{\frac{n}{p}-1}(1 + s^{\frac{p}{p-1}}) \\
&\quad + o(1 + s^{\frac{p}{p-1}})^{\frac{n}{p}-1}\varepsilon^{\frac{n}{p}-1}\Big]\Big[1 - (\frac{R(x_o)}{6n} + \frac{\Delta a(x_o)}{2na(x_o)})s^2\varepsilon^{2(1-\frac{1}{p})} + \varepsilon^{2(1-\frac{1}{p})}o(s^2)\Big] ds.
\end{aligned}$$

Letting $t = s^{\frac{p}{p-1}}$, we get

$$\begin{aligned}
&\int_{B(x_o,\delta)} a u_\varepsilon^p dv_g \\
&= \frac{p-1}{p}\omega_{n-1} a(x_o)\varepsilon^{p-\frac{n}{p}} \int_0^{\delta^{\frac{p}{p-1}}\varepsilon^{-1}} t^{n(1-\frac{1}{p})-1}(1+t)^{p-n} \\
&\quad \times \Big[1 - p\nu\varepsilon^{\frac{n}{p}-1}(1+t) + o(1+t)\varepsilon^{\frac{n}{p}-1}\Big]
\end{aligned}$$

$$\times \Big[1-\big(\frac{R(x_o)}{6n}+\frac{\Delta a(x_o)}{2na(x_o)}\big)t^{2(1-\frac{1}{p})}\varepsilon^{2(1-\frac{1}{p})}+\varepsilon^{2(1-\frac{1}{p})}o\big(t^{2(1-\frac{1}{p})}\big)\Big]dt$$

$$=\frac{p-1}{p}\omega_{n-1}a(x_o)\varepsilon^{p-\frac{n}{p}}\int_0^{\delta^{\frac{p}{p-1}}\varepsilon^{-1}}t^{n(1-\frac{1}{p})-1}(1+t)^{p-n}dt+o(\varepsilon^{p-\frac{n}{p}})\,.$$

Since $p<n^2$, we have

$$\lim_{\varepsilon\to 0}\int_{\delta^{\frac{p}{p-1}}\varepsilon^{-1}}^{\infty}t^{n(1-\frac{1}{p})-1}(1+t)^{p-n}dt=0\,;$$

therefore,

$$\int au_\varepsilon^p dv_g=\frac{p-1}{p}\omega_{n-1}a(x_o)\varepsilon^{p-\frac{n}{p}}I_{n-p}^{n(1-\frac{1}{p})-1}+o(\varepsilon^{p-\frac{n}{p}}).$$

From the formulae

$$I_{n-p}^{n(1-\frac{1}{p})-1}=\frac{\Gamma\big(n(1-\frac{1}{p})\big)\Gamma(\frac{n}{p}-p)}{\Gamma(n-p)}\,,$$

$$I_n^{n(1-\frac{1}{p})}=\frac{\Gamma\big(n(1-\frac{1}{p})+1\big)\Gamma(\frac{n}{p}-1)}{\Gamma(n)}=\frac{n(1-\frac{1}{p})\Gamma\big(n(1-\frac{1}{p})\big)\Gamma(\frac{n}{p}-1)}{\Gamma(n)}$$

we deduce that

$$I_{n-p}^{n(1-\frac{1}{p})-1}=\frac{\Gamma(n)\Gamma(\frac{n}{p}-p)}{n(1-\frac{1}{p})\Gamma(n-p)\Gamma(\frac{n}{p}-1)}I_n^{n(1-\frac{1}{p})}=b(p,n)I_n^{n(1-\frac{1}{p})}.$$

Finally, we get

$$\int_{B(x_o,\delta)}au_\varepsilon^p dv_g=\varepsilon^{p-\frac{n}{p}}\frac{p-1}{p}\omega_{n-1}a(x_o)b(p,n)I_n^{n(1-\frac{1}{p})}+o(\varepsilon^{p-\frac{n}{p}}). \tag{5.4}$$

Now, we compute the term

$$\begin{aligned}
&\int_{B(x_o,\delta)}fu_\varepsilon^{p*}dv_g\\
&=\int_0^\delta r^{n-1}\big((\varepsilon+r^{\frac{p}{p-1}})^{1-\frac{n}{p}}-\nu\big)^{p*}dr\int_{S^{n-1}(r)}f\sqrt{g}d\Omega\\
&=\int_0^\delta r^{n-1}\big((\varepsilon+r^{\frac{p}{p-1}})^{1-\frac{n}{p}}-\nu\big)^{p*}dr\\
&\quad\times\int_{S^{n-1}(r)}\big(f(x_o)+\frac{1}{2}\nabla_{ij}f(x_o)x^ix^j+o(r^2)\big)\big(1-\frac{1}{6}R_{ij}(x_o)x^ix^j+o(r^2)\big)d\Omega\\
&=\int_0^\delta r^{n-1}\big((\varepsilon+r^{\frac{p}{p-1}})^{1-\frac{n}{p}}-\nu\big)^{p*}dr\\
&\quad\times\int_{S^{n-1}(r)}\Big[f(x_o)+\big(\frac{1}{2}\nabla_{ij}f(x_o)-f(x_o)\frac{R_{ij}(x_o)}{6}\big)x^ix^j+o(r^2)\Big]d\Omega\\
&=\omega_{n-1}f(x_o)\int_0^\delta r^{n-1}\big((\varepsilon+r^{\frac{p}{p-1}})^{1-\frac{n}{p}}-\nu\big)^{p*}\big(1-\big(\frac{R(x_o)}{6n}+\frac{\Delta f(x_o)}{2nf(x_o)}\big)r^2+o(r^2)\big)dr\\
&=\omega_{n-1}f(x_o)\int_0^\delta r^{n-1}(\varepsilon+r^{\frac{p}{p-1}})^{-n}\big(1-\big(\frac{R(x_o)}{6n}+\frac{\Delta f(x_o)}{2nf(x_o)}\big)r^2+o(r^2)\big)dr\,.
\end{aligned}$$

Letting $s = r\varepsilon^{\frac{1-p}{p}}$, we get

$$\int_{B(x_o,\delta)} fu_\varepsilon^{p*}dv_g = \frac{p-1}{p}\omega_{n-1}f(x_o)\varepsilon^{\frac{-n}{p}}\int_0^{\delta\varepsilon^{\frac{1-p}{p}}} s^{n-1}(1+s^{\frac{p}{p-1}})^{-n} \times\big[1-\big(\frac{R(x_o)}{6n}+\frac{\Delta f(x_o)}{2nf(x_op)}\big)s^2\varepsilon^{2(1-\frac{1}{p})}+o(s^2)\varepsilon^{2(1-\frac{1}{p})}\big]ds\,.$$

Setting $t = s^{\frac{p}{p-1}}$, we obtain

$$\begin{aligned}&\int_{B(x_o,\delta)} fu_\varepsilon^{p*}dv_g\\ &= \frac{p-1}{p}\omega_{n-1}f(x_o)\varepsilon^{\frac{-n}{p}}\int_0^{\delta^{\frac{p}{p-1}}\varepsilon^{-1}} t^{n(1-\frac{1}{p})-1}(1+t)^{-n}\\ &\quad\times\big(1-\big(\frac{R(x_o)}{6n}+\frac{\Delta f(x_o)}{2nf(x_o)}\big)t^{2(1-\frac{1}{p})}\varepsilon^{2(1-\frac{1}{p})}+o(t^{2(1-\frac{1}{p})})\varepsilon^{2(1-\frac{1}{p})}\big)dt.\end{aligned}$$

Since

$$\lim_{\varepsilon\to 0}\int_{\delta^{\frac{p}{p-1}}\varepsilon^{-1}}^{\infty} t^{n(1-\frac{1}{p})-1}(1+t)^{-n}dt = 0$$

and

$$\lim_{\varepsilon\to 0}\int_{\delta^{\frac{p}{p-1}}\varepsilon^{-1}}^{\infty} t^{(n+2)(1-\frac{1}{p})-1}(1+t)^{-n}dt = 0$$

provided that $p < 1+\frac{n}{2}$, which is the case if $p^2 < n$, we deduce that

$$\int_{B(x_o,\delta)} fu_\varepsilon^{p*}dv_g = \frac{p-1}{p}\omega_{n-1}f(x_o)\varepsilon^{\frac{-n}{p}}\Big[I_n^{n(1-\frac{1}{p})-1} - \big(\frac{R(x_o)}{6n}+\frac{\Delta f(x_o)}{2nf(x_o)}\big)I_n^{(n+2)(1-\frac{1}{p})-1}\varepsilon^{2(1-\frac{1}{p})}+o\big(\varepsilon^{2(1-\frac{1}{p})}\big)\Big].$$

Hence, by putting

$$c(n,p) = \frac{\Gamma\big((n+2)(1-\frac{1}{p})\big)\Gamma\big(\frac{n-2p+1}{p}\big)}{\Gamma\big(n(1-\frac{1}{p})\big)\Gamma\big(\frac{n}{p}\big)}$$

one has

$$\begin{aligned}&\int_{B(x_o,\delta)} fu_\varepsilon^{p*}dv_g\\ &= \varepsilon^{\frac{-n}{p}}\frac{p-1}{p}\omega_{n-1}f(x_o)I_n^{n(1-\frac{1}{p})-1}\big[1-\big(\frac{R(x_o)}{6n}+\frac{\Delta f(x_o)}{2nf(x_o)}\big)c(p,n)I_n^{n(1-\frac{1}{p})}\\ &\quad+\varepsilon^{2(1-\frac{1}{p})}+o\big(\varepsilon^{2(1-\frac{1}{p})}\big)\big]\end{aligned}$$

and since

$$I_n^{n(1-\frac{1}{p})-1} = \frac{\Gamma(\frac{n}{p})}{n(n-\frac{1}{p})\Gamma(\frac{n}{p}-1)}I_n^{n(1-\frac{1}{p})} = d(n,p)I_n^{n(1-\frac{1}{p})}$$

it follows that

$$\begin{aligned}&\int_{B(x_o,\delta)} fu_\varepsilon^{p*}dv_g\\ &= \varepsilon^{\frac{-n}{p}}\frac{p-1}{p}\omega_{n-1}f(x_o)d(n,p)I_n^{n(1-\frac{1}{p})}\end{aligned}$$

$$\times \big[1-\big(\frac{R(x_o)}{6n}+\frac{\Delta f(x_o)}{2nf(x_o)}\big)c(n,p)\varepsilon^{2(1-\frac{1}{p})}+o\big(\varepsilon^{2(1-\frac{1}{p})}\big)\big].$$

From equalities (5.3) and (5.4), we get

$$\begin{aligned}
&\int_{B(x_o,\delta)}(|\nabla u_\varepsilon^p|+au_\varepsilon^p)dv_g\\
&=\frac{p-1}{p}\omega_{n-1}\big(\frac{n-p}{p-1}\big)^p\varepsilon^{1-\frac{n}{p}}I_n^{n(1-\frac{1}{p})}\big[1-\frac{a(n,p)}{6n}R(x_o)\varepsilon^{2(1-\frac{1}{p})}\\
&\quad+b(n,p)\big(\frac{n-p}{p-1}\big)^pa(x_o)\varepsilon^{p-1}+o\big(\varepsilon^{2(1-\frac{1}{p})}\big)+o\big(\varepsilon^{p-1}\big)\big]\,.
\end{aligned}\tag{5.5}$$

Since

$$\begin{aligned}
&\Big(\int_{B(x_o,\delta)}fu_\varepsilon^{p*}dv_g\Big)^{\frac{p}{p*}}\\
&=\varepsilon^{1-\frac{n}{p}}\big(\frac{p-1}{p}\omega_{n-1}f(x_o)d(n,p)I_n^{n(1-\frac{1}{p})}\big)^{p/p*}\\
&\quad\times\big[1-\big(\frac{R(x_o)}{6n}+\frac{\Delta f(x_o)}{2nf(x_o)}\big)c(n,p)\varepsilon^{2(1-\frac{1}{p})}+o\big(\varepsilon^{2(1-\frac{1}{p})}\big)\big]^{p/p*}
\end{aligned}$$

we get

$$\begin{aligned}
&\frac{\int_{B(x_o,\delta)}(|\nabla u_\varepsilon^p|+au_\varepsilon^p)dv_g}{\big(\int_{B(x_o,\delta)}fu_\varepsilon^{p*}dv_g\big)^{p/p*}}\\
&=\big(\frac{p-1}{p}\omega_{n-1}f(x_o)I_n^{n(1-\frac{1}{p})}\big)^{\frac{p}{n}}\big(\frac{n-p}{p-1}\big)^p\big(d(n,p)f(x_o)\big)^{-\frac{p}{p*}}\\
&\quad\times\big[1-\big(\frac{a(n,p)}{6n}a(x_o)\varepsilon^{2(1-\frac{1}{p})}-b(n,p)a(x_o)\varepsilon^{p-1}\big)+o\big(\varepsilon^{2(1-\frac{1}{p})}\big)+o\big(\varepsilon^{p-1}\big)\big]\\
&\quad\times\big[1+\frac{p}{p*}\big(\frac{R(x_o)}{6n}+\frac{\Delta f(x_o)}{2nf(p)}\big)c(n,p)\varepsilon^{2(1-\frac{1}{p})}+o\big(\varepsilon^{2(1-\frac{1}{p})}\big)\big].
\end{aligned}\tag{5.6}$$

Now, since the function

$$\phi=\big(1+r^{\frac{p}{p-1}}\big)^{1-\frac{n}{p}}$$

realizes the best constant in the Sobolev's imbedding $H_1^p(R^n)\subset L^{p*}(R^n)$, that is

$$\Big(\int_{R^n}\phi^{p*}dx\Big)^{p/p*}=K(n,p)^p\int_{R^n}|\phi|^pdx\ ,$$

we get

$$\begin{aligned}
&\big(\frac{n-p}{p-1}\big)^p\omega_{n-1}\int_0^\infty(1+r^{\frac{p}{p-1}})^{-n}r^{n+\frac{1}{p-1}}dr\\
&=K(n,p)^{-p}\Big(\omega_{n-1}\int_0^\infty(1+r^{\frac{p}{p-1}})^{-n}r^{n-1}dr\Big)^{p/p*}.
\end{aligned}$$

By letting $t=r^{\frac{p}{p-1}}$, we have

$$\begin{aligned}
&\frac{p-1}{p}\big(\frac{n-p}{p-1}\big)^p\omega_{n-1}\int_0^\infty(1+t)^{-n}t^{n(1-\frac{1}{p})}dt\\
&=K(n,p)^{-p}\Big(\omega_{n-1}\frac{p-1}{p}\int_0^\infty(1+t)^{-n}t^{n(1-\frac{1}{p})-1}dr\Big)^{p/p*}.
\end{aligned}$$

Therefore,

$$\frac{p-1}{p}(\frac{n-p}{p-1})^p\omega_{n-1}I_n^{n(1-\frac{1}{p})} = K(n,p)^{-p}\big(\frac{p-1}{p}\omega_{n-1}d(n,p)I_n^{n(1-\frac{1}{p})}\big)^{p/p*}$$

which implies

$$K(n,p)^{-p} = (\frac{n-p}{p-1})^p\big(\frac{p-1}{p}\omega_{n-1}I_n^{n(1-\frac{1}{p})}\big)^{\frac{p}{n}}d(n,p)^{-\frac{p}{p*}}.$$

Then the equality (5.6) becomes

$$\begin{aligned}
&\frac{\int_{B(x_o,\delta)}(|\nabla u_\varepsilon^p| + au_\varepsilon^p)dv_g}{\big(\int_{B(x_o,\delta)} fu_\varepsilon^{p*}dv_g\big)^{\frac{p}{p*}}} \\
&= K(n,p)^{-p}f(x_o)^{-\frac{p}{p*}}\Big[1 - \big(\frac{a(n,p)}{6n}R(x_o)\varepsilon^{2(1-\frac{1}{p})} \\
&\quad - (\frac{p-1}{n-p})^p b(n,p)R(x_o)\varepsilon^{p-1}\big) + o\big(\varepsilon^{2(1-\frac{1}{p})}\big) + o(\varepsilon^{p-1})\Big] \\
&\quad \times\big[1 + \frac{p}{p*}(\frac{R(x_o)}{6n} + \frac{\Delta f(x_o)}{2nf(x_o)})c(n,p)\varepsilon^{2(1-\frac{1}{p})} + o(\varepsilon^{2(1-\frac{1}{p})})\big] \\
&= K(n,p)^{-p}f(x_o)^{-\frac{p}{p*}}\Big[1 - \Big\{\big(\frac{a(n,p)}{6n} - \frac{p}{p*}\frac{c(n,p)}{6n}\varepsilon^{2(1-\frac{1}{p})}\big)R(x_o) \\
&\quad - (\frac{p-1}{n-p})^p b(n,p)a(x_o)\varepsilon^{p-1} - \frac{p}{p*}\frac{\Delta f(x_o)}{2nf(x_o)}c(n,p)\varepsilon^{2(1-\frac{1}{p})}\Big\} \\
&\quad + o\big(\varepsilon^{p-1}\big) + o\big(\varepsilon^{2(1-\frac{1}{p})}\big)\Big].
\end{aligned} \tag{5.7}$$

If $1 < p < 2$ and $n + 2 > 3p$, the bracket in the equality (5.7) is equivalent to

$$1 + (\frac{p-1}{n-p})^p b(n,p)a(x_o)\varepsilon^{p-1}.$$

Then, if $a(x_o) < 0$, we get

$$1 + (\frac{p-1}{n-p})^p b(n,p)a(x_o)\varepsilon^{p-1} < 1. \tag{5.8}$$

If $p = 2$, the bracket reads

$$(\frac{a(n,2)}{6n} - \frac{n-2}{n}\frac{c(n,2)}{6n})R(x_o) - (\frac{1}{n-2})^2 b(n,2)a(x_o) - \frac{n-2}{n}\frac{\Delta f(x_o)}{2nf(x_o)}c(n,2)$$

where the quantities $a(n,2)$, $b(n,2)$ and $c(n,2)$ are replaced by their respective expressions. The condition

$$(\frac{a(n,2)}{6nc(n,2)} - \frac{n-2}{6n^2})R(x_o) - (\frac{1}{n-2})^2\frac{b(n,2)}{c(n,2)}a(x_o) - \frac{n-2}{n}\frac{\Delta f(x_o)}{2nf(x_o)} > 0$$

implies

$$\begin{aligned}
&(\frac{(n+2)(n-2)}{6n^2(n-4)} - \frac{n-2}{6n^2})R(x_o) \\
&- (\frac{1}{n-2})^2\frac{4(n-1)(n-2)^2}{n^2(n-4)}a(x_o) - \frac{n-2}{n}\frac{\Delta f(x_o)}{2nf(x_o)} > 0;
\end{aligned}$$

that is,

$$\frac{\Delta f(x_o)}{f(x_o)} - \frac{2}{n-4}a(x_o) + \frac{8(n-1)}{(n-2)(n-4)}R(x_o) < 0. \tag{5.9}$$

Now for, $p > 2$, the bracket in question is equivalent to

$$\Big(\frac{a(n,p)}{6nc(n,p)} - \frac{n-p}{6n^2}\Big)R(x_o) - \frac{n-p}{n}\frac{\Delta f(x_o)}{2nf(x_o)}.$$

The condition

$$\Big(\frac{(n+2)(n-p)}{6n^2(n-3p+2)} - \frac{n-p}{6n^2}\Big)R(x_o) - \frac{n-p}{2n^2}\frac{\Delta f(x_o)}{f(x_o)} > 0$$

becomes

$$\frac{\Delta f(x_o)}{f(x_o)} < \frac{p}{n-3p+2}R(x_o). \tag{5.10}$$

Each of the conditions (5.8), (5.9) and (5.10) assures that

$$\frac{\int_{B(x_o,\delta)}(|\nabla u_\varepsilon^p| + au_\varepsilon^p)dv_g}{\big(\int_{B(x_o,\delta)} fu_\varepsilon^{p*}dv_g\big)^{p/p*}} < K(n,p)^{-p}f(x_o)^{-p/p*}$$

and a fortiori the condition (iii) of the generic theorem is satisfied. Therefore, our main theorem (Theorem 1.1) is proved.

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