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THE EIGENVALUE PROBLEM FOR A SINGULAR QUASILINEAR ELLIPTIC EQUATION

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Abstract. We show that many results about the eigenvalues and eigenfunctions of a quasilinear elliptic equation in the non-singular case can be extended to the singular case. Among these results, we have the first eigenvalue is associated to a $C^{1,\alpha}(\Omega)$ eigenfunction which is positive and unique (up to a multiplicative constant), that is, the first eigenvalue is simple. Moreover the first eigenvalue is isolated and is the unique positive eigenvalue associated to a non-negative eigenfunction. We also prove some variational properties of the second eigenvalue.

1. Introduction

In this paper, we shall study the eigenvalue problem of the singular quasilinear elliptic equation

$$\begin{gathered} -\operatorname{div}(|x|^{-ap}|Du|^{p-2}Du) = \lambda|x|^{-(a+1)p+c}|u|^{p-2}u, \quad \text{in } \Omega \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with C^1 boundary, $0 \in \Omega$, $1 < p < n$, $0 \le a < (n-p)/p$, and $c > 0$.

For $a = 0, c = p$, there are many results about the eigenvalues and eigenfunctions of problem (1.1), such as λ_1 is associated to a $C^{1,\alpha}(\Omega)$ eigenfunction which is positive in Ω and unique (up to a multiplicative constant), that is, λ_1 is simple. Moreover λ_1 is isolated, and is the unique positive eigenvalue associated to a non-negative eigenfunction (cf. [11, 1, 6] and references therein).

In this paper, we will show that many results about the eigenvalues and eigenfunctions in the case where $a = 0, c = p$ can be extended to the more general case where $0 \le a < (n-p)/p$, $c > 0$. The starting point of the variational approach to these problems is the following weighted Sobolev-Hardy inequality due to Caffarelli, Kohn and Nirenberg [3], which is called the Caffarelli-Kohn-Nirenberg inequality.

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Let $1 < p < n$. For all $u \in C_0^\infty(\mathbb{R}^n)$, there is a constant $C_{a,b} > 0$ such that

$$\Big(\int_{\mathbb{R}^n} |x|^{-bq} |u|^q \, dx \Big)^{p/q} \le C_{a,b} \int_{\mathbb{R}^n} |x|^{-ap} |Du|^p \, dx, \tag{1.2}$$

where

$$\begin{gathered} -\infty < a < \frac{n-p}{p}, \quad a \le b \le a+1, \\ q = p^*(a,b) = \frac{np}{n-dp}, \quad d = 1 + a - b. \end{gathered} \tag{1.3}$$

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^1 boundary and $0 \in \Omega$, and let $\mathcal{D}_a^{1,p}(\Omega)$ be the completion of $C_0^\infty(\mathbb{R}^n)$, with respect to the norm $\|\cdot\|$ defined by

$$\|u\| = \Big(\int_\Omega |x|^{-ap} |Du|^p \, dx \Big)^{1/p}.$$

From the boundedness of Ω and the standard approximation argument, it is easy to see that (1.2) holds for any $u \in \mathcal{D}_a^{1,p}(\Omega)$ in the sense

$$\Big(\int_\Omega |x|^{-\alpha} |u|^r \, dx \Big)^{p/r} \le C \int_\Omega |x|^{-ap} |Du|^p \, dx, \tag{1.4}$$

for $1 \le r \le \frac{np}{n-p}$, $\alpha \le (1+a)r + n(1 - \frac{r}{p})$; that is, the imbedding $\mathcal{D}_a^{1,p}(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is continuous, where $L^r(\Omega, |x|^{-\alpha})$ is the weighted L^r space with norm

$$\|u\|_{r,\alpha} := \|u\|_{L^r(\Omega, |x|^{-\alpha})} = \Big(\int_\Omega |x|^{-\alpha} |u|^r \, dx \Big)^{1/r}.$$

In fact, we have the following compact imbedding result which is an extension of the classical Rellich-Kondrachov compactness theorem (cf. [4] for $p = 2$ and [15] for the general case). For the convenience of the reader, we include its proof here.

Theorem 1.1 (Compact imbedding theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with C^1 boundary and $0 \in \Omega$, $1 < p < n$, $-\infty < a < \frac{n-p}{p}$. The imbedding $\mathcal{D}_a^{1,p}(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is compact if $1 \le r < \frac{np}{n-p}$ and $\alpha < (1+a)r + n(1 - \frac{r}{p})$.*

Proof. The continuity of the imbedding is a direct consequence of the Caffarelli-Kohn-Nirenberg inequality (1.2) or (1.4). To prove the compactness, let $\{u_m\}$ be a bounded sequence in $\mathcal{D}_a^{1,p}(\Omega)$. For any $\rho > 0$ with $B_\rho(0) \subset \Omega$ is a ball centered at the origin with radius ρ, it follows that $\{u_m\} \subset W^{1,p}(\Omega \setminus B_\rho(0))$. Then the classical Rellich-Kondrachov compactness theorem guarantees the existence of a convergent subsequence of $\{u_m\}$ in $L^r(\Omega \setminus B_\rho(0))$. By taking a diagonal sequence, we can assume without loss of generality that $\{u_m\}$ converges in $L^r(\Omega \setminus B_\rho(0))$ for any $\rho > 0$.

On the other hand, for any $1 \le r < \frac{np}{n-p}$, there exists a $b \in (a, a+1]$ such that $r < q = p^*(a,b) = \frac{np}{n-dp}$, $d = 1 + a - b \in [0,\ 1)$. From the Caffarelli-Kohn-Nirenberg inequality (1.2) or (1.4), $\{u_m\}$ is also bounded in $L^q(\Omega, |x|^{-bq})$. By

Hölder inequality, for any $\delta > 0$, it follows that

$$
\begin{aligned}
&\int_{|x|<\delta} |x|^{-\alpha}|u_m - u_j|^r \, dx \\
&\le \Big(\int_{|x|<\delta} |x|^{-(\alpha - br)\frac{q}{q-r}} \, dx \Big)^{1-\frac{r}{q}} \Big(\int_\Omega |x|^{-br}|u_m - u_j|^r \, dx \Big)^{r/q} \\
&\le C\Big(\int_0^\delta r^{n-1-(\alpha-br)\frac{q}{q-r}} \, dr \Big)^{1-\frac{r}{q}} \\
&= C\delta^{n-(\alpha-br)\frac{q}{q-r}},
\end{aligned}
\tag{1.5}
$$

where $C > 0$ is a constant independent of m. Since $\alpha < (1 + a)r + n(1 - \frac{r}{p})$, it follows that $n - (\alpha - br)\frac{q}{q-r} > 0$. Therefore, for a given $\varepsilon > 0$, we first fix $\delta > 0$ such that

$$\int_{|x|<\delta} |x|^{-\alpha}|u_m - u_j|^r \, dx \le \frac{\varepsilon}{2}, \quad \forall\, m, j \in \mathbb{N}.$$

Then we choose $N \in \mathbb{N}$ such that

$$\int_{\Omega\setminus B_\delta(0)} |x|^{-\alpha}|u_m - u_j|^r \, dx \le C_\alpha \int_{\Omega\setminus B_\delta(0)} |u_m - u_j|^r \, dx \le \frac{\varepsilon}{2}, \quad \forall\, m, j \ge N,$$

where $C_\alpha = \delta^{-\alpha}$ if $\alpha \ge 0$ and $C_\alpha = (\operatorname{diam}(\Omega))^{-\alpha}$ if $\alpha < 0$. Thus

$$\int_\Omega |x|^{-\alpha}|u_m - u_j|^r \, dx \le \varepsilon, \quad \forall\, m, j \ge N,$$

that is, $\{u_m\}$ is a Cauchy sequence in $L^q(\Omega, |x|^{-bq})$. $\square$

For studying the eigenvalue problem (1.1), we introduce the following two functionals on $\mathcal{D}_a^{1,p}(\Omega)$:

$$\Phi(u) := \int_\Omega |x|^{-ap}|Du|^p \, dx, \quad J(u) := \int_\Omega |x|^{-(a+1)p+c}|u|^p \, dx.$$

For $c > 0$, J is well-defined. Furthermore, $\Phi, J \in C^1(\mathcal{D}_a^{1,p}(\Omega), \mathbb{R})$, and a real value λ is an eigenvalue of problem (1.1) if and only if there exists $u \in \mathcal{D}_a^{1,p}(\Omega) \setminus \{0\}$ such that $\Phi'(u) = \lambda J'(u)$. At this point, let us introduce the set

$$\mathcal{M} := \{u \in \mathcal{D}_a^{1,p}(\Omega) : J(u) = 1\}.$$

Then $\mathcal{M} \neq \emptyset$ and $\mathcal{M}$ is a C^1 manifold in $\mathcal{D}_a^{1,p}(\Omega)$. It follows from the standard Lagrange multiplier argument that eigenvalues of (1.1) correspond to critical values of $\Phi|_{\mathcal{M}}$. From Theorem 1.1, Φ satisfies the (PS) condition on $\mathcal{M}$. Thus a sequence of critical values of $\Phi|_{\mathcal{M}}$ comes from the Ljusternik-Schnirelman critical point theory on C^1 manifolds. Let $\gamma(A)$ denote the Krasnoselski genus on $\mathcal{D}_a^{1,p}(\Omega)$ and for any $k \in \mathbb{N}$, set

$$\Gamma_k := \{A \subset \mathcal{M} : A \text{ is compact, symmetric and } \gamma(A) \ge k\}.$$

Then the values

$$\lambda_k := \inf_{A\in\Gamma_k} \max_{u\in A} \Phi(u) \tag{1.6}$$

are critical values and hence are eigenvalues of problem (1.1). Moreover, $\lambda_1 \le \lambda_2 \le \dots \le \lambda_k \le \dots \to +\infty$.

One can also define another sequence of critical values minimaxing Φ along a smaller family of symmetric subsets of $\mathcal{M}$. Let us denote by S^k the unit sphere of $\mathbb{R}^{k+1}$ and

$$\mathcal{O}(S^k, \mathcal{M}) := \{h \in C(S^k, \mathcal{M}) : h \text{ is odd}\}.$$

Then for any $k \in \mathbb{N}$, the value

$$\mu_k := \inf_{h \in \mathcal{O}(S^{k-1}, \mathcal{M})} \max_{t \in S^{k-1}} \Phi(h(t)) \tag{1.7}$$

is an eigenvalue of (1.1). Moreover $\lambda_k \leq \mu_k$. This new sequence of eigenvalues was first introduced by [9] and later used in [7, 6] for $a = 0, c = p$. From the Caffarelli-Kohn-Nirenberg inequality (1.2) or (1.4), it is easy to see that

$$\lambda_1 = \mu_1 = \inf\{\Phi(u) : u \in \mathcal{D}_a^{1,p}(\Omega), J(u) = 1\} > 0,$$

and the corresponding eigenfunction $e_1 \geq 0$.

To obtain the properties of the eigenvalues of problem (1.1), first we need some boundedness and regularity results of the eigenfunctions of problem (1.1). In section 2, based on the Moser's iteration technique, we shall deduce the L^∞ boundedness and $C^{1,\alpha}(\Omega \setminus \{0\})$ regularity results. In section 3, we shall obtain the simplicity of the first eigenvalue λ_1. In section 4, we shall prove that the first eigenvalue λ_1 is isolated. Section 5 is concerned with the properties of the second eigenvalue λ_2.

2. Regularity results

In this section, we will prove the L^∞ boundedness and $C^{1,\alpha}(\Omega \setminus \{0\})$ regularity results of the weak solution to problem (1.1) (cf. [4, 5] for the case $p = 2$).

Theorem 2.1. *Assume that $1 < p < n$, $0 \leq a < \frac{n-p}{p}$, $c > 0$, and $u \in \mathcal{D}_a^{1,p}(\Omega)$ is a solution of (1.1). Then $u \in L^\infty(\Omega, |x|^{-\alpha})$ and $u \in C^{1,\alpha}(\Omega \setminus \{0\})$ for some $\alpha \geq 0$.*

Proof. By the standard elliptic regularity theory (e.g. [13]), it suffices to show the L^∞ boundedness of u. To do this, we apply the Moser's iteration as in [10] and [5].

For $k > 0, q \geq 1$, we define two C^1 functions on $\mathbb{R}$, h and H by

$$h(t) = \begin{cases} \operatorname{sign}(t)|t|^q, & \text{if } |t| \leq k, \\ \operatorname{sign}(t)\{qk^{q-1}|t| + (1-q)k^q\}, & \text{if } |t| > k, \end{cases} \tag{2.1}$$

and $H(t) = \int_0^t (h'(s))^p \, ds$. Thus, it is easy to see that $h'(t) \geq 0$ for all $t \in \mathbb{R}$ and $H(u(x)) \in \mathcal{D}_a^{1,p}(\Omega)$ if $u \in \mathcal{D}_a^{1,p}(\Omega)$. In fact, a simple calculation shows that

$$h'(t) = \begin{cases} q|t|^{q-1}, & \text{if } |t| \leq k, \\ qk^{q-1}, & \text{if } |t| > k \end{cases} \tag{2.2}$$

and

$$H(t) = \begin{cases} \dfrac{q^p}{p(q-1)+1}|t|^{p(q-1)+1} \operatorname{sign}(t), & \text{if } |t| \leq k, \\ q^p\big(\dfrac{1}{p(q-1)+1}k^{p(q-1)+1} + k^{p(q-1)}(|t| - k)\big) \operatorname{sign}(t), & \text{if } |t| > k. \end{cases} \tag{2.3}$$

It is trivial to verify that

$$|H(t)| \leq q|h(t)|(h'(t))^{p-1}, \quad |H(t)||t|^{p-1} \leq q^p|h(t)|^p, \tag{2.4}$$

for all $t \in \mathbb{R}$. In fact, for all $|t| \le k,\ q \ge 1$, we see that

$$|H(t)| = \frac{q^p}{p(q-1)+1}|t|^{p(q-1)+1} \le q|h(t)|(h'(t))^{p-1} = q^p|t|^{p(q-1)+1}$$

and

$$|H(t)||t|^{p-1} = \frac{q^p}{p(q-1)+1}|t|^{pq} \le q^p|h(t)|^p = q^p|t|^{pq}.$$

For $|t| > k$, $q \ge 1$, a direct calculation shows that

$$\begin{aligned}&|H(t)| - q|h(t)|(h'(t))^{p-1}\\&= q^p\big((\frac{1}{p(q-1)+1} - 1)k^{p(q-1)+1} + (1-q)k^{p(q-1)}(|t|-k)\big) \le 0\end{aligned}$$

and

$$\begin{aligned}|H(t)||t|^{p-1} - q^p|h(t)|^p = q^p\big((&\frac{1}{p(q-1)+1} - 1)k^{p(q-1)+1}|t|^{p-1}\\&- q^pk^{p(q-1)}(|t|-k)^p + k^{p(q-1)}(|t|^p - k^p)\big) \le 0\end{aligned}$$

Let $\psi(x) = \eta^pH(u(x))$ be a test function defined in Ω, where η is a non-negative smooth function in Ω to be specified later. Then from (1.1), it follows that

$$\int_\Omega |x|^{-ap}|Du|^{p-2}Du \cdot D\psi\,dx = \lambda\int_\Omega |x|^{-(a+1)p+c}|u|^{p-2}u\psi\,dx. \tag{2.5}$$

From the definitions of h, H, ψ, (2.4) implies that

$$\begin{aligned}&\int_\Omega |x|^{-ap}|Du|^{p-2}Du \cdot D\psi\,dx\\&= \int_\Omega |x|^{-ap}\eta^p|Du|^{p-2}Du \cdot DH(u)\,dx + p\int_\Omega |x|^{-ap}\eta^{p-1}H(u)|Du|^{p-2}Du \cdot D\eta\,dx\\&\ge \int_\Omega |x|^{-ap}\eta^p|Dh(u)|^p\,dx - pq\int_\Omega |x|^{-ap}\eta^{p-1}|Du|^{p-1}|D\eta||h(u)|(h'(u))^{p-1}\,dx.\end{aligned} \tag{2.6}$$

By the Hölder inequality, it follows that

$$\begin{aligned}&pq\int_\Omega |x|^{-ap}\eta^{p-1}|Du|^{p-1}|D\eta||h(u)|(h'(u))^{p-1}\,dx\\&\le \frac{1}{2}\int_\Omega |x|^{-ap}\eta^p|Dh(u)|^p\,dx + Cq^p\int_\Omega |x|^{-ap}|h(u)|^p|D\eta|^p\,dx,\end{aligned} \tag{2.7}$$

where and hereafter C is a universal positive constant independent of k, q. Inserting (2.7) into (2.6), we see that

$$\begin{aligned}&\int_\Omega |x|^{-ap}|Du|^{p-2}Du \cdot D\psi\,dx\\&\ge \frac{1}{2}\int_\Omega |x|^{-ap}\eta^p|Dh(u)|^p\,dx - Cq^p\int_\Omega |x|^{-ap}|h(u)|^p|D\eta|^p\,dx\,.\end{aligned} \tag{2.8}$$

Equation (2.4) also implies

$$\begin{aligned}\lambda\int_\Omega |x|^{-(a+1)p+c}|u|^{p-2}u\psi\,dx &= \lambda\int_\Omega |x|^{-(a+1)p+c}|u|^{p-2}u\eta^pH(u)\,dx\\&\le \lambda q^p\int_\Omega |x|^{-(a+1)p+c}\eta^p|h(u)|^p\,dx.\end{aligned} \tag{2.9}$$

For any $r \in (p, \dfrac{np}{n-p})$, let $\alpha = n + (a+1)r - \dfrac{nr}{p} \in (ar, (a+1)r)$, from the Caffarelli-Kohn-Nirenberg inequality (1.4), it follows that

$$\Big(\int_\Omega |x|^{-\alpha} |\eta h(u)|^r\,dx\Big)^{p/r} \le C \int_\Omega |x|^{-ap} |D(\eta h(u))|^p\,dx. \tag{2.10}$$

Thus, substituting (2.8)–(2.10) into (2.5), it is easy to show that

$$\begin{aligned} &\Big(\int_\Omega |x|^{-\alpha} |\eta h(u)|^r\,dx\Big)^{p/r} \\ &\le q^p C \Big\{ \int_\Omega |x|^{-ap} |h(u)|^p |D\eta|^p\,dx + \int_\Omega |x|^{-(a+1)p+c} \eta^p |h(u)|^p\,dx \Big\}. \end{aligned} \tag{2.11}$$

For each $x_0 \in \bar\Omega$, let $\eta \in C_0^\infty(B_{2R}(x_0)), R < 1$, such that

$$0 \le \eta \le 1, \eta \equiv 1 \text{ in } B_R(x_0), \quad |D\eta| < 2/R.$$

Then (2.11) implies that

$$\begin{aligned} &\Big(\int_{B_R(x_0)} |x|^{-\alpha} |h(u)|^r\,dx\Big)^{p/r} \\ &\le q^p C \Big\{ \int_{B_{2R}(x_0)} \frac{|x|^{-ap}}{R^p} |h(u)|^p\,dx + \int_{B_{2R}(x_0)} |x|^{-(a+1)p+c} |h(u)|^p\,dx \Big\}. \end{aligned} \tag{2.12}$$

Letting $k \to \infty$ in (2.11), the Hölder inequality implies

$$\begin{aligned} &\Big(\int_{B_R(x_0)} |x|^{-\alpha} |u|^{qr}\,dx\Big)^{p/r} \\ &\le q^p C \Big\{ \int_{B_{2R}(x_0)} \frac{|x|^{-ap}}{R^p} |u|^{pq}\,dx + \int_{B_{2R}(x_0)} |x|^{-(a+1)p+c} |u|^{pq}\,dx \Big\} \\ &\le q^p C \Big(\int_{B_{2R}(x_0)} |x|^{-\alpha} |u|^{pqs}\,dx \Big)^{1/s}, \end{aligned} \tag{2.13}$$

where

$$s \in \Big(\max\Big\{1, \frac{n-\alpha}{n-(a+1)p+c}, \frac{n-\alpha}{n-ap}\Big\}, \frac{r}{p}\Big).$$

A simple covering argument yields that

$$\Big(\int_\Omega |x|^{-\alpha} |u|^{qr}\,dx\Big)^{p/r} \le q^p C \Big(\int_\Omega |x|^{-\alpha} |u|^{pqs}\,dx \Big)^{1/s}, \tag{2.14}$$

that is,

$$\|u\|_{L^{qr}(\Omega,|x|^{-\alpha})} \le (Cq)^{1/q} \|u\|_{L^{pqs}(\Omega,|x|^{-\alpha})},$$

which is a reversed Hölder inequality and implies that $u \in L^{qr}(\Omega, |x|^{-\alpha})$ for all $q > 1$. Then letting $q = \chi^m, m = 0, 1, 2, \cdots, \chi = \frac{r}{ps} > 1$, the Moser's iteration technique (cf. [12]) implies

$$\begin{aligned} \|u\|_{L^{ps\chi^N}(\Omega,|x|^{-\alpha})} &\le \prod_{m=0}^{N-1} (C\chi^m)^{\chi^{-m}} \|u\|_{L^{ps}(\Omega,|x|^{-\alpha})} \\ &\le C^\sigma \chi^\tau \|u\|_{L^{ps}(\Omega,|x|^{-\alpha})} \\ &\le C \|u\|_{L^{ps}(\Omega,|x|^{-\alpha})}, \end{aligned}$$

where $\sigma = \sum_{m=0}^{N-1} \chi^{-m}$, $\tau = \sum_{m=0}^{N-1} m\chi^{-m}$. Letting $N \to \infty$, we therefore obtain $\|u\|_{L^\infty(\Omega,|x|^{-\alpha})} < \infty$. $\square$

Based on the above regularity result, the strong maximum principle due to Vazquez [14] implies the following positivity of nonnegative eigenfunction.

Corollary 2.2. *Suppose that $1 < p < n$, $0 \le a < \frac{n-p}{p}$, $c > 0$, $u \ge 0$ is an eigenfunction corresponding to $\lambda > 0$. Then $u > 0$ in $\Omega \setminus \{0\}$.*

3. Simplicity of λ_1

In this section, we prove the simplicity of λ_1, that is, any two eigenfunctions both corresponding to λ_1 are proportional.

Theorem 3.1. *Suppose that $1 < p < n$, $0 \le a < \frac{n-p}{p}$ $c > 0$, $u \ge 0$ and $v \ge 0$ are eigenfunctions both corresponding to λ_1. Then u and v are proportional.*

Proof. From Theorem 2.1, u, v are bounded. We use the modified test-functions as in [11]:

$$\eta = \frac{(u+\varepsilon)^p - (v+\varepsilon)^p}{(u+\varepsilon)^{p-1}} \quad \text{and} \quad \frac{(v+\varepsilon)^p - (u+\varepsilon)^p}{(v+\varepsilon)^{p-1}} \tag{3.1}$$

in the corresponding equations for u and v, respectively, where ε is a positive parameter. Direct calculation implies

$$D\eta = \Big\{1 + (p-1)\big(\frac{v+\varepsilon}{u+\varepsilon}\big)^p\Big\}Du - p\big(\frac{v+\varepsilon}{u+\varepsilon}\big)^{p-1}Dv, \tag{3.2}$$

and, by symmetry, the gradient of the test-function in the corresponding equations for v has a similar expression with u and v interchanged. Set

$$u_\varepsilon = u + \varepsilon, \quad v_\varepsilon = v + \varepsilon.$$

Inserting the chosen test-functions into their respective equations and adding these, it follows that

$$\begin{aligned}
&\lambda_1 \int_\Omega |x|^{-(a+1)p+c}\big[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\big](u_\varepsilon^p - v_\varepsilon^p)\,dx \\
&= \int_\Omega |x|^{-ap}\Big\{1 + (p-1)\big(\frac{v_\varepsilon}{u_\varepsilon}\big)^p\Big\}|Du_\varepsilon|^p\,dx \\
&\quad + \int_\Omega |x|^{-ap}\Big\{1 + (p-1)\big(\frac{u_\varepsilon}{v_\varepsilon}\big)^p\Big\}|Dv_\varepsilon|^p\,dx \\
&\quad - \int_\Omega |x|^{-ap}p\big(\frac{v_\varepsilon}{u_\varepsilon}\big)^{p-1}|Du_\varepsilon|^{p-2}Du_\varepsilon \cdot Dv_\varepsilon\,dx \\
&\quad - \int_\Omega |x|^{-ap}p\big(\frac{u_\varepsilon}{v_\varepsilon}\big)^{p-1}|Dv_\varepsilon|^{p-2}Dv_\varepsilon \cdot Du_\varepsilon\,dx \\
&= \int_\Omega |x|^{-ap}(u_\varepsilon^p - v_\varepsilon^p)(|D\log u_\varepsilon|^p - |D\log v_\varepsilon|^p)\,dx \\
&\quad - \int_\Omega |x|^{-ap}pv_\varepsilon^p|D\log u_\varepsilon|^{p-2}D\log u_\varepsilon \cdot (D\log v_\varepsilon - D\log u_\varepsilon)\,dx \\
&\quad - \int_\Omega |x|^{-ap}pu_\varepsilon^p|D\log v_\varepsilon|^{p-2}D\log v_\varepsilon \cdot (D\log u_\varepsilon - D\log v_\varepsilon)\,dx \\
&\ge 0,
\end{aligned} \tag{3.3}$$

where the last inequality is a consequence of the following simple calculus inequality (cf. [11]):

$$|w_2|^p > |w_1|^p + p|w_1|^{p-2}w_1 \cdot (w_2 - w_1) \tag{3.4}$$

for points in $\mathbb{R}^n, w_1 \neq w_2, p > 1$. By the Lebesgue's Dominated Convergence Theorem, it follows that

$$\lim_{\varepsilon \to 0^+} \int_\Omega |x|^{-(a+1)p+c} \big[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\big](u_\varepsilon^p - v_\varepsilon^p)\, dx = 0. \tag{3.5}$$

The same argument as in [11] implies that $vDu = uDv$ a.e. in Ω, which implies that u and v are proportional. $\square$

4. Isolation of λ_1

In this section, we prove the isolation of λ_1. First, we show that only the first eigenfunctions are non-negative.

Theorem 4.1. *Suppose that $1 < p < n$, $0 \le a < \frac{n-p}{p}$, $c > 0$. If $v \ge 0$ is any eigenfunction corresponding to the eigenvalue λ, then $\lambda = \lambda_1$.*

Proof. Let $u \ge 0$ denote a first eigenfunction, then the same procedure as in Section 3 yields

$$\int_\Omega |x|^{-(a+1)p+c} \big[\lambda_1 \frac{u^{p-1}}{u_\varepsilon^{p-1}} - \lambda \frac{v^{p-1}}{v_\varepsilon^{p-1}}\big](u_\varepsilon^p - v_\varepsilon^p)\, dx \ge 0, \tag{4.1}$$

and arguing as before, it follows that

$$(\lambda_1 - \lambda) \int_\Omega |x|^{-(a+1)p+c}(u^p - v^p)\, dx \ge 0. \tag{4.2}$$

This leads to a contradiction, if $\lambda > \lambda_1$, since u can be replaced by any of the functions $2u, 3u, 4u, \cdots$. Thus $\lambda = \lambda_1$. $\square$

From Theorem 4.1, for any eigenvalue $\lambda > \lambda_1$, the corresponding eigenfunction v must change sign. Next, we need an estimate of the measure of the nodal domains of an eigenfunction v. We recall that a nodal domain of v is a connected component of $\Omega \setminus \{x \in \Omega : u = 0\}$.

Theorem 4.2. *Suppose that $1 < p < n$, $0 \le a < \frac{n-p}{p}$, $c > 0$. If v is any eigenfunction corresponding to the eigenvalue $\lambda > \lambda_1 > 0$ and $\mathcal{N}$ is a nodal domain of v, then*

$$|\mathcal{N}| \ge (C\lambda)^{-1/\sigma} \tag{4.3}$$

for some positive constant $C > 0$, where $\sigma = 1 - \frac{p}{r} - \frac{1}{s}$, $r \in (p, \frac{n-p}{np})$, $s > \frac{r}{r-p}$ if $c \ge n - \frac{np}{r}$, $s \in (\frac{r}{r-p}, \frac{nr}{nr-cr-np})$ if $0 < c < n - \frac{np}{r}$.

Proof. Assume that $v > 0$ in $\mathcal{N}$, the case $v < 0$ being completely analogous. Since $v \in \mathcal{D}_a^{1,p}(\Omega)$, then $v|_{\mathcal{N}} \in \mathcal{D}_a^{1,p}(\mathcal{N})$. Hence the function $w(x) = v(x)$ if $x \in \mathcal{N}$ and $w(x) = 0$ if $\Omega \setminus \mathcal{N}$ belongs to $\mathcal{D}_a^{1,p}(\Omega)$. Using w as a test function in the weak equation satisfied by v yields

$$\int_{\mathcal{N}} |x|^{-ap}|Dv|^p\, dx = \lambda \int_{\mathcal{N}} |x|^{-(a+1)p+c}|v|^p\, dx. \tag{4.4}$$

For $r \in (p, \frac{n-p}{np})$, let $\alpha = (1+a)r + n(1 - \dfrac{r}{p})$. Then the Hölder inequality implies that

$$\begin{aligned} \int_{\mathcal{N}} |x|^{-(a+1)p+c}|v|^p\,dx &\le |\mathcal{N}|^\sigma \Big(\int_{\mathcal{N}} |x|^{(-(a+1)p+c+\frac{\alpha p}{r})s}\,dx\Big)^{1/s}\Big(\int_{\mathcal{N}} |x|^{-\alpha}|v|^r\Big)^{p/r} \\ &\le C|\mathcal{N}|^\sigma\Big(\int_{\mathcal{N}} |x|^{-\alpha}|v|^r\Big)^{p/r}, \end{aligned} \tag{4.5}$$

since the choice of s implies that $(-(a+1)p + c + \frac{\alpha p}{r})s > -n$. On the other hand, the Caffarelli-Kohn-Nirenberg inequality (1.2) or (1.4) implies that

$$\Big(\int_{\mathcal{N}} |x|^{-\alpha}|v|^r\Big)^{p/r} \le C\int_{\mathcal{N}} |x|^{-ap}|Dv|^p\,dx, \tag{4.6}$$

where $C = C(a, \alpha)$. Thus (4.4)–(4.6) imply (4.3). □

Corollary 4.3. *Each eigenfunction has a finite number of nodal domains.*

Proof. Let $\mathcal{N}_j$ be a nodal domain of an eigenfunction associated to some positive eigenvalue λ. It follows from (4.3) that

$$|\Omega| \ge \sum_j |\mathcal{N}_j| \ge (C\lambda)^{-1/\sigma}\sum_j 1$$

and the claim follows. □

Theorem 4.4. *Suppose that* $1 < p < n,\ 0 \le a < \frac{n-p}{p},\ c > 0$. λ_1 *is isolated.*

Proof. Suppose, on the contrary, there exists a sequence of eigenvalues $\{\nu_m\}$ such that $\nu_m \ne \lambda_1$ and $\nu_m \to \lambda_1$ as $m \to \infty$. Let u_m be an eigenfunction associated to ν_m such that $\|u_m\|_{\mathcal{D}_a^{1,p}(\Omega)} = 1$. Thus, up to a subsequence, $\{u_m\}$ converge weakly in $\mathcal{D}_a^{1,p}(\Omega)$ and strongly in $L^p(\Omega, |x|^{-(a+1)p+c})$ to a function $u \in \mathcal{D}_a^{1,p}(\Omega)$. Furthermore, the limit function u is an eigenfunction associated to the first eigenvalue λ_1. Without loss of generality, assume that $u \ge 0$. Then for any $\delta > 0$, by the Egorov theorem, u_m converges uniformly to u on a subset $\Omega_\delta \subset \Omega$, with $|\Omega \setminus \Omega_\delta| < \delta$. Let $\mathcal{N}_m$ be a nodal domain of u_m such that $u_m < 0$ in $\mathcal{N}_m$, then $|\mathcal{N}_m| \to 0$ as $m \to \infty$, which contradicts (4.3). □

5. Variational property of the second eigenvalue

Since λ_1 is isolated in the spectrum and there exist eigenvalues different from λ_1, it makes sense to define the second eigenvalue of (1.1) as

$$\underline{\lambda}_2 := \inf\{\lambda \in \mathbb{R} : \lambda \text{ is eigenvalue and } \lambda > \lambda_1\} > \lambda_1.$$

It follows from the closure of the set of eigenvalues of (1.1) that $\underline{\lambda}_2$ is a different eigenvalue of (1.1) from λ_1.

Theorem 5.1. $\underline{\lambda}_2 = \lambda_2$, *where* λ_2 *is defined by (1.6).*

Proof. It is trivial that $\underline{\lambda}_2 \le \lambda_2$. It suffices to show that $\lambda_2 \le \underline{\lambda}_2$. Suppose that v is the eigenfunction associated to $\underline{\lambda}_2$, then from Theorem 4.1 and Corollary 4.3, let $\mathcal{N}_1, \cdots, \mathcal{N}_r, r \ge 2$ denote the nodal domains of v. For $i = 1, \cdots, r$, set

$$v_i(x) = \begin{cases} \dfrac{v(x)}{\big[\int_{\mathcal{N}_i} |x|^{-(a+1)p+c}|v|^p\,dx\big]^{1/p}}, & \text{if } x \in \mathcal{N}_i \\ 0, & \text{if } x \in \Omega \setminus \mathcal{N}_i. \end{cases}$$

It is easy to see that $v_i \in \mathcal{D}_a^{1,p}(\Omega)$. Let $\mathcal{F}_r$ denote the subspace of $\mathcal{D}_a^{1,p}(\Omega)$ spanned by $\{v_1, \cdots, v_r\}$ and $A_r = \{u \in \mathcal{F}_r : J(u) = 1\}$. For each $u \in \mathcal{F}_r,\ u = \sum_{i=1}^r \alpha_i v_i$, it follows that

$$J(u) = \sum_{i=1}^r |\alpha_i|^p J(v_i) = \sum_{i=1}^r |\alpha_i|^p.$$

Thus the set A_r can also be represented as

$$A_r = \big\{ \sum_{i=1}^r \alpha_i v_i : \sum_{i=1}^r |\alpha_i|^p = 1 \big\}.$$

It is easy to see that A_r is compact, symmetric and $\gamma(A_r) = r \geq 2$, that is, $A_r \in \Gamma_2$.

On the other hand, inserting v_i into the corresponding equation of v yields

$$\int_{\mathcal{N}_i} |Dv_i|^p \, dx = \underline{\lambda}_2 \int_{\mathcal{N}_i} |x|^{-(a+1)p+c} |v_i|^p \, dx. \tag{5.1}$$

Then for any $u = \sum_{i=1}^r \alpha_i v_i \in A_r$, it follows that

$$\Phi(u) = \sum_{i=1}^r |\alpha_i|^p \Phi(v_i) = \underline{\lambda}_2 \sum_{i=1}^r |\alpha_i|^p J(v_i) = \underline{\lambda}_2. \tag{5.2}$$

Thus, it follows that

$$\lambda_2 := \inf_{A \in \Gamma_2} \max_{u \in A} \Phi(u) \leq \max_{u \in A_r} \Phi(u) = \underline{\lambda}_2\,, \tag{5.3}$$

which implies the conclusion. $\square$

The above argument implies the following further variation characterization of the second eigenvalue (cf. [9, 8, 2] for $a = 0, c = p$).

Theorem 5.2. $\underline{\lambda}_2 = \lambda_2 = \mu_2 = \inf_{h \in\, athcal F} \max_{u \in h([-1,1])} \Phi(u)$, *where* $\mathcal{F} := \{h \in C([-1,1], \mathcal{M}) : h(\pm 1) = \pm e_1\}$ *and* $e_1 \in \mathcal{M}$ *is the positive eigenfunction associated to* λ_1.

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