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CLASSIFICATION AND EXISTENCE OF POSITIVE SOLUTIONS TO NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. A classification scheme is given for the eventually positive solutions to a class of second order nonlinear dynamic equations, in terms of their asymptotic magnitudes. Also we provide necessary and/or sufficient conditions for the existence of positive solutions.

1. Introduction

The study of dynamic equations on time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis [4]. Since Hilger formed the definition of derivatives and integrals on time scales, several authors have expended on various aspects of the new theory, see the paper by Agarwal et al. [2] and the book by Bohner and Peterson [3].

In recent years there has been much research activity concerning the oscillation and nonoscillation of some different equations on time scales, we refer the reader to the papers [13, 14, 15]. However, to the best of our knowledge, there have not appeared any results concerned with asymptotic behavior and existence of solutions of dynamic equations on time scales. In this paper, we classify positive solutions of the second order nonlinear dynamic equation

$$y^{\Delta\Delta}(t) + r(t)f(y^\sigma(t)) = 0 \quad \text{for } t \in \mathbb{T} \tag{1.1}$$

according to the limiting behavior and then provide sufficient and/or necessary conditions for their existence, where $r \in C_{rd}([t_0, +\infty) \cap \mathbb{T},\ [0, \infty)$, $r \not\equiv 0$ for $t \in \mathbb{T}$, $t_0 > 0$ and $f(y) > 0$ is nondecreasing for any $y \in \mathbb{R} - \{0\}$.

We note that if $\mathbb{T} = \mathbb{R}$, then (1.1) becomes the differential equation

$$y''(t) + r(t)f(y(t)) = 0 \quad \text{for } t \in \mathbb{R}. \tag{1.2}$$

The asymptotic behavior of solutions of (1.2) has been studied by several authors under different conditions, see Naito [10, 11]. If $\mathbb{T} = \mathbb{Z}$, then (1.1) becomes the

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difference equation

$$\Delta^2(y_n) + r_n f(y_{n+1}) = 0 \quad \text{for } n \in \mathbb{Z}, \tag{1.3}$$

which has been discussed in detail by many authors, one can refer to [5, 6, 7, 8, 9].

Since we are interested in asymptotic behavior of positive solutions of (1.1), we suppose that the time scale $\mathbb{T}$ is a time scale interval of the form $[t_0, \infty) \cap \mathbb{T}$ and shall employ $t \geq t_0$ to denote $[t_0, \infty) \cap \mathbb{T}$ unless otherwise stated. Here a solution of (1.1) means that $y \in C^2_{rd}$ and satisfies (1.1) for $t \geq t_0$.

2. Some preliminaries

In this section, we give a short introduction to the time scale calculus. For the explanation and results we refer to the book by Bohner and Peterson [3].

Definition. The forward jump operator $\sigma(t)$ and $\rho(t)$ are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The graininess μ of the time scale is defined by $\mu(t) = \sigma(t) - t$. Naturally, $\sigma(t) = t + \mu(t) \geq t$.

Definition. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left-hand sided limit at all left-dense points.

Definition. For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative is defined by

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \begin{cases} \lim_{s\to t} \frac{f(\sigma(t))-f(s)}{t-s}, & \text{if } \mu(t) = 0, \\ \frac{f(\sigma(t))-f(t)}{\sigma(t)-t}, & \text{if } \mu(t) > 0. \end{cases}$$

The function f is said to be differentiable if its derivative exists.

The derivative of f and the jump operator σ are related by the formula

$$f(\sigma(t)) = f^\sigma(t) = f(t) + f^\Delta(t)\mu(t) \quad \text{for } t \in \mathbb{T}. \tag{2.1}$$

Lemma 2.1 ([3]). *The function $f(t)$ is increasing in t if $f^\Delta(t) > 0$.*

From this lemma, we can say that f is decreasing, nondecreasing and nonincreasing if $f^\Delta(t) < 0$, $f^\Delta(t) \geq 0$ and $f^\Delta(t) \leq 0$ for $t \in \mathbb{T}$, respectively.

Definition (Antiderivative). A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^\Delta(t) = f(t) \quad \text{for all } t \in \mathbb{T}.$$

If a function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, its antiderivative exists, denoted by

$$F(t) = \int_{t_0}^t f(s)\Delta s \quad \text{for } t_0 \text{ and } t \in \mathbb{T},$$

where

$$\int_{t_0}^t f(s)\Delta s = \begin{cases} \int_{t_0}^t f(s)ds, & \text{if } \mathbb{T} = \mathbb{R}; \\ \sum_{s\in[t_0,t)} \mu(s)f(s), & \text{if } [t_0, t] \text{ consists of only isolated points.} \end{cases}$$

Obviously, if $f(t)$ is differentiable, its antiderivative exists, and

$$f(t) - f(t_0) = \int_{t_0}^t f^\Delta(s)\Delta s. \tag{2.2}$$

For the sake of convenience, we still call the antiderivative $F(t)$ of $f(t)$ as the integral of $f(t)$, antiderivative calculus as integral calculus, respectively. Similarly, infinite integral is defined as

$$\int_{t_0}^{\infty} f^{\Delta}(s)\Delta s = \lim_{t\to\infty}\int_{t_0}^{t} f^{\Delta}(s)\Delta s.$$

If $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=\rho(t)=t$, $\mu(t)\equiv 0$, $f^{\Delta}=f'$ and

$$\int_a^b f(s)\Delta s = \int_a^b f(s)ds.$$

If $\mathbb{T}=\mathbb{Z}$, then $\sigma(t)=t+1$, $\rho(t)=t-1$, $\mu(t)\equiv 1$, $f^{\Delta}=\Delta f$, and

$$\int_a^b f^{\Delta}(s)\Delta s = \sum_{t=a}^{b-1} f(t) \quad \text{for } a<b.$$

3. Main Results

Let $y(t)$ be a positive solution of (1.1). Then from (1.1) we have

$$y^{\Delta\Delta}(t) = -r(t)f(y^{\sigma}(t)) \le 0,$$

which, by Lemma 2.1, implies that $y^{\Delta}(t)$ is nonincreasing. Thus we claim that

$$y^{\Delta}(t)\ge 0 \quad \text{for } t\ge t_0.$$

If not, there exists a sufficiently large $t_1\ge t_0$, such that $y^{\Delta}(t)<-c$, where $c>0$ is a constant. Hence, for $t>t_1$, we obtain

$$y(t)-y(t_1)=\int_{t_1}^{t} y^{\Delta}(s)\Delta s < \int_{t_1}^{t}(-c)\Delta s = -c(t-t_1).$$

This means that $\lim y(t)=-\infty$, which contradicts $y(t)>0$.

Since $y^{\Delta}(t)$ is nonincreasing and $y^{\Delta}(t)\ge 0$ for $t\ge t_0$, then there are positive constants α and β such that

$$\alpha\le y(t)\le \beta t \quad \text{for } t\ge t_0.$$

In view of the above considerations, we may now make the following classifications. Let Ω be the set of all non-oscillatory solutions of (1.1) and Ω^+ be the subset of Ω containing those which are ultimately positive. Then any non-oscillatory solution in Ω^+ must belong to one of the following three sets:

$$C[\max]=\big\{y\in\Omega^+ : \lim_{t\to+\infty} y^{\Delta}(t)=\alpha>0\big\};$$
$$C[\mathrm{int}]=\big\{y\in\Omega^+ : \lim_{t\to+\infty} y(t)=\infty \text{ and } \lim_{t\to+\infty} y^{\Delta}(t)=0\big\};$$
$$C[\min]=\big\{y\in\Omega^+ : \lim_{t\to+\infty} y(t)=\beta>0\big\}.$$

In the following, we will give several necessary and/or sufficient conditions for the existence of positive solutions of (1.1).

Theorem 3.1. *Equation* (1.1) *has a positive solution of class* $C[\max]$ *if and only if*

$$\int_{t_0}^{+\infty} r(s)f(b\sigma(s))<\infty \quad \text{for some } b>0. \tag{3.1}$$

Proof. Let $y(t) \in C[\max]$ of (1.1), then

$$\lim_{t\to+\infty} y^{\Delta}(t) = \alpha > 0 \quad \text{for } t \geq t_0.$$

Hence there exist a sufficiently large t_1 such that

$$\frac{1}{2}\alpha < y^{\Delta}(t) < \frac{3}{2}\alpha \quad \text{for } t \geq t_1,$$

so that

$$\frac{1}{2}\alpha t < y(t) < \frac{3}{2}\alpha t \quad \text{for } t > t_2 > t_1.$$

Set $b = \alpha/2$. Then the nondecreasing property of f implies

$$f(y(t)) \geq f(bt) \quad \text{and} \quad f(y^{\sigma}(t)) \geq f(b\sigma(t)). \tag{3.2}$$

Integrating both sides of (1.1) from t_1 to t, we see

$$y^{\Delta}(t_1) - y^{\Delta}(t) = \int_{t_1}^{t} r(s) f(y^{\sigma}(s))\Delta s.$$

Taking limits on both sides of the above equality, we get

$$\lim_{t\to\infty} \int_{t_2}^{t} r(s) f(y^{\sigma}(s))\Delta s = y^{\Delta}(t_1) - \alpha,$$

which implies

$$\int_{t_1}^{\infty} r(s) f(y^{\sigma}(s))\Delta s < \infty. \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\int_{t_2}^{\infty} r(s) f(b\sigma(s))\Delta s < \infty.$$

Conversely, assume that (3.1) holds. Then there exists a large number T such that

$$\int_{t}^{\infty} r(s)(f(b\sigma(s)))\Delta s < \frac{b}{2} \quad \text{for } t \geq T. \tag{3.4}$$

Consider the sequence $\{x_n\}_0^{\infty}$ defined by $x_0(t) = b/2$ for $t \geq T$ and

$$x_{n+1}(t) = \frac{b}{2} + \frac{1}{t}\int_{T}^{t}\int_{\tau}^{\infty} r(s) f(\sigma(s)x_n^{\sigma}(s))\Delta s\Delta\tau \tag{3.5}$$

for $t \geq T$, $n = 0, 1, 2, \cdots$. In view of (3.4), the sequence $\{x_n(t)\}_0^{\infty}$ is well defined. In fact,

$$\begin{aligned}
x_1(t) &= \frac{b}{2} + \frac{1}{t}\int_{T}^{t}\int_{\tau}^{\infty} r(s) f(\frac{b}{2}\sigma(s))\Delta s\Delta\tau \\
&\leq \frac{b}{2} + \frac{t-T}{t}\int_{T}^{\infty} r(s) f(b\sigma(s))\Delta s\Delta\tau \\
&\leq \frac{b}{2} + \int_{T}^{\infty} r(s) f(b\sigma(s))\Delta s < \frac{b}{2} + \frac{b}{2} = b
\end{aligned}$$

and $x_1(t) \geq x_0(t)$ for $t \geq T$. By induction and the nondecreasing property of f, we have

$$x_{n+1}(t) \geq x_n(t) \quad \text{for } t \geq T,\ n = 0,\ 1, 2, \cdots. \tag{3.6}$$

Next, we prove $\{x_n(t)\}_0^\infty$ is bounded for $t \geq T$. Since

$$x_0(t) = \frac{b}{2} < b \quad \text{and} \quad x_1(t) < b,$$

so if we assume $x_n(t) < b$ for $t \geq T$, then $\sigma(s)x_n^\sigma(s) < b\sigma(s)$ and

$$\begin{aligned} x_{n+1}(t) &= \frac{b}{2} + \frac{1}{t}\int_T^t \int_\tau^\infty r(s)f(\sigma(s)x_n^\sigma(s))\Delta s\Delta\tau \\ &\leq \frac{b}{2} + \frac{1}{t}\int_T^t \int_\tau^\infty r(s)f(b\sigma(s))\Delta s\Delta\tau \\ &\leq \frac{b}{2} + \int_T^\infty r(s)f(b\sigma(s))\Delta s < b \quad \text{for } t \geq T, \end{aligned} \tag{3.7}$$

which, by induction implies $\{x_n(t)\}_0^\infty$ is bounded for $t \geq T$. In view of (3.6), we know $\{x_n(t)\}_0^\infty$ is pointwise convergent to some function $x^*(t)$. By means of the Lebesgue's dominated convergence theorem, see [1, 12], we obtain

$$x^*(t) = \frac{b}{2} + \frac{1}{t}\int_T^t \int_\tau^\infty r(s)f(\sigma(s)x^\sigma(s))\Delta s\Delta\tau \quad \text{for } t \geq T$$

and $\frac{b}{2} \leq x^*(t) < b$. Setting $y(t) = tx^*(t)$, i.e.,

$$y(t) = \frac{b}{2}t + \int_T^t \int_\tau^\infty r(s)f(y^\sigma(s))\Delta s\Delta\tau \text{ for } t \geq T,$$

obviously, $y(t)$ is a solution of (3.1) and belongs to $C[\max]$. The proof is complete. □

Theorem 3.2. *Equation* (1.1) *has a positive solution of class* $C[\min]$ *if and only if*

$$\int_{t_0}^{+\infty}\int_\tau^{+\infty} r(s)f(d)\Delta s\Delta\tau < \infty \quad \textit{for some } d > 0. \tag{3.8}$$

Proof. Let $y(t) \in C[\min]$ for (1.1), then

$$\lim_{t\to+\infty} y(t) = \beta > 0 \quad \text{and} \quad \lim_{t\to+\infty} y^\Delta(t) = 0 \quad \text{for } t \geq t_0.$$

Hence there exists a sufficient large t_1 such that

$$\frac{1}{2}\beta < y(t) < \frac{3}{2}\beta \quad \text{for } t \geq t_1.$$

Set $d = \beta/2$, then the nondecreasing property implies

$$f(y(t)) > f(d) \quad \text{and} \quad f(y^\sigma(t)) > f(d) \quad \text{for } t > t_1.$$

Integrating both sides of (1.1) from t to ∞ for $t > t_1$, we obtain

$$\beta - y(t) = \int_t^\infty \int_\tau^\infty r(s)f(y^\sigma(s))\Delta s\Delta\tau, \tag{3.9}$$

which implies

$$\int_{t_1}^\infty \int_\tau^\infty r(s)f(d)\Delta s\Delta\tau < \infty.$$

i.e., (3.8) holds. The rest of the proof is similar to that of Theorem 3.1, we omit it here. □

Theorem 3.3. *If* (1.1) *has a positive solution in* $C[\text{int}]$, *then*

$$\int_{t_0}^{\infty} r(s)f(a)\Delta s < \infty \quad \textit{for some } a > 0\,, \tag{3.10}$$

$$\int_{t_0}^{\infty}\int_{\tau}^{\infty} r(s)f(b\sigma(s))\Delta s\Delta\tau = \infty \quad \textit{for every } b > 0. \tag{3.11}$$

Proof. Let $y \in C[\text{int}]$, be a solution of (1.1), then $\lim_{t\to+\infty} y(t) = \infty$ and $\lim_{t\to+\infty} y^{\Delta}(t) = 0$. Hence there exist two positive constants a, b and a sufficient large $t_1 > t_0$ such that $a < y(t) < bt$ for $t > t_1$, which, in view of the nondecreasing property of f, implies that

$$\begin{gathered} f(y(t)) \geq f(a) \quad \text{and} \quad f(y^{\sigma}(t)) \leq f(a), \\ f(y(t)) \leq f(bt) \quad \text{and} \quad f(y^{\sigma}(t)) \leq f(b\sigma(t)) \quad \text{for } t > t_1. \end{gathered} \tag{3.12}$$

From equation (1.1), we have

$$y^{\Delta}(t) + \int_{t_1}^{t} r(s)f(y^{\sigma}(s))\Delta s = y^{\Delta}(t_1) \text{ for } t > t_1. \tag{3.13}$$

In view of $\lim_{t\to+\infty} y^{\Delta}(t) = 0$, (3.13) yields

$$\int_{t_1}^{\infty} r(s)f(y^{\sigma}(s))\Delta s = y^{\Delta}(t_1),$$

and so $\int_{t_1}^{\infty} r(s)f(a)\Delta s < \infty$, which implies (3.10) holds.

Furthermore, since $\lim_{t\to+\infty} y^{\Delta}(t) = 0$, we obtain

$$\int_{s}^{\infty} r(s)f(y^{\sigma}(s))\Delta s = y^{\Delta}(s) \quad \text{for } s > t_1. \tag{3.14}$$

Integrating on both sides of (3.14) from t_1 to t, we obtain

$$\begin{aligned} y(t) - y(t_1) &= \int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f(y^{\sigma}(s))\Delta s\Delta\tau \\ &\leq \int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f(b\sigma(s))\Delta s\Delta\tau \text{ for } t > t_1. \end{aligned}$$

Hence, (3.12) and $\lim_{t\to+\infty} y(t) = \infty$ imply

$$\int_{t_1}^{\infty}\int_{\tau}^{\infty} r(s)f(b\sigma(s))\Delta s\Delta\tau = \infty.$$

The proof is complete. □

Theorem 3.4. *Equation* (1.1) *has a positive solution in* $C[\text{int}]$ *provided that*

$$\int_{t_0}^{\infty} r(s)f(a\sigma(s))\Delta s < \infty \quad \textit{for some } a > 0 \tag{3.15}$$

$$\int_{t_0}^{\infty}\int_{\tau}^{\infty} r(s)f(b)\Delta s\Delta\tau = \infty \quad \textit{for every } b > 0. \tag{3.16}$$

Proof. In view of (3.15) and (3.16), there exist two positive constants a, b and a sufficiently large t_1 such that

$$\frac{b}{t} < a, \quad \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f(a\sigma(s))\Delta s\Delta\tau < a \quad \text{for } t \geq t_1. \tag{3.17}$$

Consider the sequence $\{x_n(t)\}_0^\infty$ defined by $x_0(t) = 0$ and

$$x_{n+1}(t) = Px_n(t) = \frac{b}{t}+\frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f(\sigma(s)x_n^\sigma(s))\Delta s\Delta\tau \quad \text{for } t \ge t_1, n = 0, 1, \cdots .$$

It is easy to see that $\{x_n(t)\}_0^\infty$ is well defined. In fact,

$$x_1(t) = \frac{b}{t} < a,\ x_1^\sigma(s) < a \quad \text{for } t \ge t_1$$

and

$$\begin{aligned} x_2(t) &= \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f(\sigma(s)x_1^\sigma(s))\Delta s\Delta\tau \\ &\le \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f(a\sigma(s))\Delta s\Delta\tau < a \quad \text{for } t \ge t_1. \end{aligned}$$

Furthermore, if we assume that $x_n(t) < a$ for $t \ge t_1$, then $x_n^\sigma(t) = x_n(\sigma(t)) < a$ and

$$\begin{aligned} x_{n+1}(t) &= \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f(\sigma(s)x_n^\sigma(s))\Delta s\Delta\tau \\ &\le \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f(a\sigma(s))\Delta s\Delta\tau < a \quad \text{for } t \ge t_1, \end{aligned}$$

which, by induction, shows that $\{x_n(t)\}_0^\infty$ is bounded, i.e.,

$$0 \le x_n(t) < a \quad \text{for } t \ge t_1,\ n = 0, 1, \cdots . \tag{3.18}$$

In view of $x_0(t) \le x_1(t)$ and the nondecreasing property of f, we have

$$x_{n+1}(t) \ge x_n(t) \quad \text{for } t \ge t_1,\ n = 0, 1, \cdots . \tag{3.19}$$

Hence, Lebesgue's dominated convergence theorem [1, 12] implies that

$$x^*(t) = \frac{a}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f(\sigma(s)x^*(\sigma(s)))\Delta s\Delta\tau \quad \text{for } t \ge t_1.$$

Set $y(t) = tx^*(t)$. Then

$$y(t) = a + \int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f(y^\sigma(s))\Delta s\Delta\tau \quad \text{for } t \ge t_1.$$

It is easily verified that $y(t)$ is a solution of (1.1) and belongs to $C[\text{int}]$. The proof is complete. $\square$

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