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EXISTENCE OF SOLUTIONS TO NONLOCAL AND SINGULAR ELLIPTIC PROBLEMS VIA GALERKIN METHOD

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Abstract. We study the existence of solutions to the nonlocal elliptic equation

$$-M(\|u\|^2)\Delta u = f(x,u)$$

with zero Dirichlet boundary conditions on a bounded and smooth domain of $\mathbb{R}^n$. We consider the M-linear case with $f \in H^{-1}(\Omega)$, and the sub-linear case $f(u) = u^\alpha$, $0 < \alpha < 1$. Our main tool is the Galerkin method for both cases when M continuous and when M is discontinuous.

1. Introduction

In this paper we study some questions related to the existence of solutions for the nonlocal elliptic problem

$$\begin{gathered} -M(\|u\|^2)\Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $f \in H^{-1}(\Omega)$ and $M : \mathbb{R} \to \mathbb{R}$ is a function whose behavior will be stated later, and the norm in $H_0^1(\Omega)$ is $\|u\|^2 = \int_\Omega |\nabla u|^2$.

The main purpose of this work is establishing properties on M under which problem (1.1), and its nonlinear counterpart, possesses a solution. This equation has called our attention because the operator

$$Lu := M(\|u\|^2)\Delta u$$

contains the nonlocal term $M(\|u\|^2)$ which poses some interesting mathematical questions. Also the operator L appears in the Kirchhoff equation, which arises in nonlinear vibrations, namely

$$\begin{gathered} u_{tt} - M\big(\int_\Omega |\nabla u|^2 dx\big)\Delta u = f(x,u) \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{gathered}$$

For more details on physical motivation of this problem the interested reader is invited to consult Eisley, Limaco-Medeiros [6, 7] and the references therein. In a

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previous paper Alves-Corrêa [1] focused their attention on problem (1.1) in case $M(t) \geq m_0 > 0$, for all $t \geq 0$, where m_0 is a constant. Among other things they studied the above M-linear problem (1.1) where M, besides the strict positivity mentioned before, satisfies the following assumption:

The function $H : \mathbb{R} \to \mathbb{R}$ with

$$H(t) = M(t^2)t$$

is monotone and $H(\mathbb{R}) = \mathbb{R}$.

The above authors also studied the sublinear problem

$$\begin{gathered} -M(\|u\|^2)\Delta u = u^\alpha \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega\,, \end{gathered} \tag{1.2}$$

where $0 < \alpha < 1$, M is a non-increasing continuous function, H is increasing, $H(\mathbb{R}) = \mathbb{R}$ and

$$G(t) = t[M(t^2)]^{2/(1-\alpha)}$$

is injective. Under these assumptions it is proved that (1.2) possesses a unique solution. A straightforward computation shows that the function $M(t) = \exp(-t) + C$, with C a positive constant, satisfies the above assumptions.

In the present paper we prove similar results by allowing M to attain negative values and $M(t) \geq m_0 > 0$ only for t large enough.

This is possible thanks to a device explored by Alves-de Figueiredo [2], who use Galerkin method to attack a non-variational elliptic system. The technique can be conveniently adapted to problems such as (1.1) and (1.2). In this way we improve substantially the existence result on the above problems mainly because our assumptions on M are weakened. Indeed, we may also consider the case in which M possesses a singularity. The method we use rests heavily on the following result whose proof may be found in Lions [8, p.53], and it is a well known variant of Brouwer's Fixed Point Theorem.

Proposition 1.1. *Suppose that $F : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function such that $\langle F(\xi), \xi \rangle \geq 0$ on $|\xi| = r$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^m$ and $| \cdot |$ its related norm. Then, there exists $z_0 \in \overline{B_r}(0)$ such that $F(z_0) = 0$.*

We recall that by a solution of (1.1) we mean a weak solution, that is, a function $u \in H_0^1(\Omega)$ such that

$$M(\|u\|^2) \int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f(x, u)\varphi, \quad \text{for all } \varphi \in H_0^1(\Omega).$$

We point out that, depending on the regularity of $f(\cdot, u)$, a bootstrap argument may be used to show that a weak solution is a classical solution, i.e., a function in $C_0^2(\overline{\Omega})$. This happens, for instance, with the solution obtained in Theorem 4.1.

This paper is organized as follows: Section 2 is devoted to the study of the M-linear problem in the continuous case. In Section 3 the M-linear is studied in case M possesses a discontinuity. In Section 4 we focus our attention on the sublinear problem. In Section 5 we analyze another type of nonlocal problem.

2. The M-linear Problem: Continuous Case

In this section we are concerned with the M-linear problem (1.1) where $f \in H^{-1}(\Omega)$ and $M : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

(M1) There are positive numbers t_∞ and m_0 such that $M(t) \geq m_0$, for all $t \geq t_\infty$.

Theorem 2.1. *Under assumption (M1), for each $0 \neq f \in H^{-1}(\Omega)$ problem (1.1) possesses a weak solution.*

Proof. Inspired by Alves-de Figueiredo [2] we use the Galerkin Method. First let us take $M^+ = \max\{M(t), 0\}$, the positive part of M, and consider the auxiliary problem

$$
\begin{gathered}
-M^+(\|u\|^2)\Delta u = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega.
\end{gathered}
\tag{2.1}
$$

We will prove that problem (2.1) possesses solution and such a solution also solves problem (1.1). We point out that M^+ also satisfies assumption (M1). We are ready to apply the Galerkin Method by using Proposition 1.1. Let $\sum = \{e_1, \dots, e_m, \dots\}$ be an orthonormal basis of the Hilbert space $H_0^1(\Omega)$. For each $m \in \mathbb{N}$ consider the finite dimensional Hilbert space

$$\mathbb{V}_m = \operatorname{span}\{e_1, \dots, e_m\}.$$

Since $(\mathbb{V}_m, \|\cdot\|)$ and $(\mathbb{R}^m, |\cdot|)$ are isometric and isomorphic, where $\|\cdot\|$ is the usual norm in $H_0^1(\Omega)$ and $|\cdot|$ is the Euclidian norm in $\mathbb{R}^m$, $\langle\cdot,\cdot\rangle$ its corresponding inner product, we make, with no additional comment, the identification

$$u = \sum_{j=1}^m \xi_j e_j \longleftrightarrow \xi = (\xi_1, \dots, \xi_m), \quad \|u\| = |\xi|.$$

We will show that for each m there is $u_m \in \mathbb{V}_m$, an approximate solution of (2.1), satisfying

$$M^+(\|u_m\|^2) \int_\Omega \nabla u_m \cdot \nabla e_i = \langle\langle f, e_i\rangle\rangle, \quad i = 1, \dots, m.$$

where $\langle\langle\ ,\ \rangle\rangle$ is the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. First we consider the function $F : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$
\begin{gathered}
F(\xi) = (F_1(\xi), \dots, F_m(\xi)), \\
F_i(\xi) = M^+(\|u\|) \int_\Omega \nabla u \cdot \nabla e_i - \langle\langle f, e_i\rangle\rangle,
\end{gathered}
$$

where $i = 1, \dots, m$ and $u = \sum_{j=1}^m \xi_j e_j$. So that

$$F_i(\xi) = M^+(\|u\|^2)\xi_i - \langle\langle f, e_i\rangle\rangle.$$

With the above identifications one has

$$\langle F(\xi), \xi\rangle = M^+(\|u\|^2)\|u\|^2 - \langle\langle f, u\rangle\rangle.$$

Using (M1), Hölder and Poincaré inequalities we get

$$\langle F(\xi), \xi\rangle \geq m_0\|u\|^2 - C\|f\|_{H^{-1}}\|u\| \geq 0,$$

if $\|u\| = r$, for r large enough, where $\|f\|_{H^{-1}}$ is the norm of the linear form f. Thus, because of Proposition 1.1, there is $u_m \in \mathbb{V}_m$, $\|u_m\| \leq r$, r does not depend on m, such that

$$M^+(\|u_m\|^2) \int_\Omega \nabla u_m \cdot \nabla e_i = \langle\langle f, e_i \rangle\rangle, \quad i = 1, \dots, m,$$

which implies that

$$M^+(\|u_m\|^2) \int_\Omega \nabla u_m \cdot \nabla \omega = \langle\langle f, \omega \rangle\rangle, \quad \text{for all } \omega \in \mathbb{V}_m. \tag{2.2}$$

Because $(\|u_m\|^2)$ is a bounded real sequence and M^+ is continuous one has

$$\|u_m\|^2 \to \tilde{t}_0,$$

for some $\tilde{t}_0 \geq 0$, and

$$u_m \rightharpoonup u \text{ in } H_0^1(\Omega), \quad u_m \to u \text{ in } L^2(\Omega), \quad M^+(\|u_m\|^2) \to M^+(\tilde{t}_0),$$

perhaps for a subsequence.

Take $k \leq m$, $\mathbb{V}_k \subset \mathbb{V}_m$. Fix k and let $m \to \infty$ in equation (2.2) to obtain

$$M^+(\tilde{t}_0) \int_\Omega \nabla u \cdot \nabla \omega = \langle\langle f, \omega \rangle\rangle, \quad \text{for all } \omega \in \mathbb{V}_k.$$

Since k is arbitrary we will have that the last equality remains true for all $\omega \in H_0^1(\Omega)$. If $M^+(\tilde{t}_0) = 0$ we would have $\langle\langle f, \omega \rangle\rangle = 0$ for all $\omega \in H_0^1$ and so $f = 0$ in $H^{-1}(\Omega)$ which is a contradiction. Consequently $M^+(\tilde{t}_0) > 0$ and so $M(\tilde{t}_0) = M^+(\tilde{t}_0)$.

We now take $\omega = u_m$ in (2.2) to obtain

$$M(\|u_m\|^2)\|u_m\|^2 = \langle\langle f, u_m \rangle\rangle$$

and so $M(\tilde{t}_0)\tilde{t}_0 = \langle\langle f, u \rangle\rangle$. From this equality and

$$M(\tilde{t}_0)\|u\|^2 = \langle\langle f, u \rangle\rangle$$

we have $\|u\|^2 = \tilde{t}_0$ which shows that the function u is a weak solution of problem (1.1) and the proof of Theorem 2.1 is complete. $\square$

Remark 2.2. It follows from the proof of Theorem 2.1 that the solution u obtained there satisfies $M(\|u\|^2) > 0$ (of course, if we had used another device in order to obtain a solution of (1.1) such a property might not be true).

We claim that there is only one solution to (1.1) satisfying this property. This may be proved as follows. Let u and v be solutions of (1.1) obtained as before. Since u and v are weak solutions of (1.1) one has

$$M(\|u\|^2) \int_\Omega \nabla u \cdot \nabla \omega = M(\|v\|^2) \int_\Omega \nabla v \cdot \nabla \omega, \quad \text{for all } \omega \in H_0^1(\Omega).$$

Hence $M(\|u\|^2)u$ and $M(\|v\|^2)v$ are both solutions of the problem

$$\begin{aligned} -\Delta U &= f \quad \text{in } \Omega, \\ U &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

By the uniqueness one has $M(\|u\|^2)u = M(\|v\|^2)v$ in Ω and so $M(\|u\|^2)\|u\| = M(\|v\|^2)\|v\|$. Supposing that the function $t \to M(t^2)t$ is increasing for $t > 0$ one obtains that $\|u\| = \|v\|$. Consequently

$$\begin{gathered} -\Delta u = -\Delta v \quad \text{in } \Omega, \\ u = v \quad \text{on } \partial\Omega. \end{gathered}$$

and then $u = v$ in Ω. Hence, we have proved that problem (1.1) possesses only one solution u if $t \to M(t^2)t$ is increasing for $t > 0$.

Remark 2.3. If $M(t_0) = 0$ for some $t_0 > 0$ and $f = 0$ in $H^{-1}(\Omega)$ then we lose uniqueness. In fact, let $u \neq 0$ be a function in $C_0^2(\overline{\Omega})$ and set $v = \sqrt{t_0}\, u/\|u\|$. In this case $\|v\|^2 = t_0$ and so

$$\begin{gathered} -M(\|v\|^2)\Delta v = 0 \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.3}$$

that is, for each nonzero function $u \in C_0^2(\overline{\Omega})$ the function v defined above is a nontrivial solution of (2.3).

Remark 2.4 (A Dual Problem)**.** Suppose that $M : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

- ($\tilde{\text{M}}$1) There are positive numbers $\tilde{t}_\infty$ and $\tilde{m}_0$ such that $M(t) \leq -\tilde{m}_0$ for all $t \geq \tilde{t}_\infty$.

In this case (1.1) possesses a solution. Indeed, suppose $f \neq 0$ in $H^{-1}(\Omega)$ and consider the problem

$$\begin{gathered} -\widetilde{M}(\|u\|^2)\Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.4}$$

where $\widetilde{M}(t) = -M(t)$. Clearly $\widetilde{M}$ satisfies (M1) and so problem (2.4) possesses a solution $v \in H_0^1(\Omega)$ with $\widetilde{M}(\|v\|^2) > 0$. Hence $u = -v$ is a solution of (1.1) with $M(\|u\|^2) < 0$.

3. The M-linear Problem: A Discontinuous Case

In this section we concentrate our interest on problem (1.1) when M possesses a discontinuity. More precisely, we study problem (1.1) with $M : \mathbb{R}/\{\theta\} \to \mathbb{R}$ continuous such that

- (M2) $\lim_{t\to\theta^+} M(t) = \lim_{t\to\theta^-} M(t) = +\infty$
- (M3) $\limsup_{t\to+\infty} M(t^2)t = +\infty$ and (M1) is satisfied for some $t_\infty > \theta$.

Theorem 3.1. *If M satisfies (M1)–(M3) problem* (1.1) *possesses a solution $u \in H_0^1(\Omega)$, for each $0 \neq f \in H^{-1}(\Omega)$.*

Proof. We first consider the sequence of functions $M_n : \mathbb{R} \to \mathbb{R}$ given by

$$M_n(t) = \begin{cases} n, & \theta - \epsilon_n' \leq t \leq \theta + \epsilon_n'', \\ M(t), & t \leq \theta - \epsilon_n' \text{ or } t \geq \theta + \epsilon_n'', \end{cases}$$

for $n > m_0$, where $\theta - \epsilon_n'$ and $\theta + \epsilon_n''$, ϵ_n', $\epsilon_n'' > 0$, are, respectively, the points closest to θ, at left and at right, so that

$$M(\theta - \epsilon_n') = M(\theta + \epsilon_n'') = n.$$

We point out that, in this case, $\epsilon_n', \epsilon_n'' \to 0$ as $n \to \infty$.

Take $n > m_0$ and observe that the horizontal lines $y = n$ cross the graph of M. Hence M_n is continuous and satisfies (M1), for each $n > m_0$. In view of this, for each n like above, there is $u_n \in H_0^1(\Omega)$ satisfying

$$M_n(\|u_n\|^2) \int \nabla u_n \cdot \nabla \omega = \langle\langle f, \omega \rangle\rangle, \quad \text{for all } \omega \in H_0^1(\Omega).$$

Taking $\omega = u_n$ in the above equation one has

$$M_n(\|u_n\|^2)\|u_n\|^2 = \langle\langle f, u_n \rangle\rangle,$$

and so

$$M_n(\|u_n\|^2)\|u_n\| \leq \|f\|_{H^{-1}}$$

Because of (M3) the sequence $(\|u_n\|)$ must be bounded. Hence

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega), \\ u_n \to u \quad \text{in } L^2(\Omega), \\ \|u_n\|^2 \to \theta_0, \quad \text{for some } \theta_0, \end{gathered}$$

perhaps for subsequences.

If $M_n(\|u_n\|^2) \to 0$, then $\langle\langle f, \omega \rangle\rangle = 0$, for all $\omega \in H_0^1(\Omega)$, which is impossible because $0 \neq f \in H^{-1}(\Omega)$. Thus if $(M_n(\|u_n\|^2)$ converges its limit is different of zero. Suppose that $\|u_n\|^2 \to \theta$.

If $\|u_n\|^2 > \theta + \epsilon''_n$ or $\|u_n\|^2 < \theta - \epsilon'_n$, for infinitely many n, we would get $M_n(\|u_n\|^2) = M(\|u_n\|^2)$, for such n, and so

$$M(\|u_n\|^2)\|u_n\|^2 = \langle\langle f, u_n \rangle\rangle \;\Rightarrow\; +\infty = \langle\langle f, u \rangle\rangle$$

which is a contradiction. On the other hand, if there are infinitely many n so that $\theta - \epsilon'_n \leq \|u_n\|^2 \leq \theta + \epsilon''_n \;\Rightarrow\; M_n(\|u_n\|^2) = n$ and so $n\|u_n\|^2 = \langle\langle f, u_n \rangle\rangle \;\Rightarrow\; \infty = \langle\langle f, u \rangle\rangle$ and we arrive again in a contradiction.

Consequently $\|u_n\|^2 \to \theta_0 \neq \theta$ which implies that for n large enough

$$\|u_n\|^2 < \theta - \epsilon'_n \quad \text{or} \quad \|u_n\|^2 > \theta + \epsilon''_n$$

and so $M_n(\|u_n\|^2) = M(\|u_n\|^2)$ which yields

$$M(\|u_n\|^2) \int_\Omega \nabla u_n \cdot \nabla \omega = \langle\langle f, \omega \rangle\rangle, \quad \forall \omega \in H_0^1(\Omega).$$

Consequently $M(\theta_0) \int_\Omega \nabla u \cdot \nabla \omega = \langle\langle f, \omega \rangle\rangle$, for all $\omega \in H_0^1(\Omega)$ which implies $M(\|u_n\|^2)\|u_n\|^2 = \langle\langle f, u_n \rangle\rangle$ and taking limits

$$M(\theta_0)\theta_0 = \langle\langle f, u \rangle\rangle$$

Hence $M(\theta_0)\|u\|^2 = M(\theta_0)\theta_0$. Reasoning as before we conclude that $M(\theta_0) \neq 0$ and so $\|u\|^2 = \theta_0$ and the proof of the theorem is complete. □

4. A Sublinear Problem

In this section we focus our attention on problem (1.2). More precisely, we have the following result:

Theorem 4.1. *If M satisfies assumption (M1), $M(t) \leq m_\infty$, for some positive constant m_∞ and all $t \geq 0$, and $\lim_{t\to\infty} M(t^2)t^{1-\alpha} = +\infty$, then problem* (1.2) *possesses a solution.*

Since the proof of this theorem is quite similar to the one in Alves-de Figueiredo [2] we omit it and make only some remarks giving some directions on how to proceed. First of all we have to consider the problem

$$\begin{gathered} -M(\|u\|^2)\Delta u = (u^+)^\alpha + \lambda\phi(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega, \end{gathered} \tag{4.1}$$

where $\lambda > 0$ is a parameter, $\phi > 0$ is a function in $H_0^1(\Omega)$, and $u^+ = \max\{u, 0\}$ is the positive part of u. Proceeding as in the proof of Theorem 2.1 we found, for each $\lambda \in (0, \tilde{\lambda})$, a solution u_λ of equation (4.1) and, in view of $M(\|u_\lambda\|^2) > 0$- we can prove that $u_\lambda \geq 0$-, using the maximum principle to conclude that $u_\lambda > 0$. Hence

$$\begin{gathered} -M(\|u_\lambda\|^2)\Delta u_\lambda = (u_\lambda)^\alpha + \lambda\phi(x) \geq u_\lambda^\alpha \quad \text{in } \Omega, \\ u_\lambda = 0 \quad \text{on } \partial\Omega, \\ u_\lambda > 0 \quad \text{in } \Omega, \end{gathered} \tag{4.2}$$

which implies

$$\begin{gathered} -\Delta u_\lambda \geq m_\infty^{-1} u_\lambda^\alpha \quad \text{in } \Omega, \\ u_\lambda = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Thanks to a result by Ambrosetti-Brézis-Cerami [3], one has

$$u_\lambda \geq m_\infty^{-1}\omega_1,$$

where $\omega_1 > 0$ in Ω is the only positive solution of

$$\begin{gathered} -\Delta\omega_1 = \omega_1^\alpha \quad \text{in } \Omega, \\ \omega_1 = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

As in the proof of Theorem 2.1 one has that $\|u_\lambda\| \leq r_\lambda$ where r_λ is a positive constant that depends on λ.

Let us consider $\lambda \in (0, \overline{\lambda})$ and make $\lambda \to 0^+$. For we have to guarantee that $(\|u_\lambda\|)$ is bounded for all $\lambda \in (0, \overline{\lambda})$. First observe that

$$M(\|u_\lambda\|^2)\|u_\lambda\|^2 = \int_\Omega u_\lambda^{\alpha+1} + \lambda \int_\Omega \phi u_\lambda$$

Because $0 < \alpha < 1$ and using some standard arguments we have

$$M(\|u_\lambda)\|^2\|u_\lambda\|^{1-\alpha} \leq C + \frac{C}{\|u_\lambda\|}$$

Since $M(t^2)t^{1-\alpha} \to +\infty$ as $t \to \infty$ we have that $(\|u_\lambda\|)$ is bounded for all $\lambda \in (0, \overline{\lambda})$. Finally, we may take $\lambda \to 0$ to obtain a solution u of problem (1.2).

5. Another Nonlocal Problem

Next, we make some remarks on a nonlocal problem which is a slight generalization of one studied by Chipot-Lovat [4] and Chipot-Rodrigues [5]. More precisely, the above authors studied the problem

$$\begin{gathered} -a\Big(\int_\Omega u\Big)\Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 1$, and $a : \mathbb{R} \to (0, +\infty)$ is a given function. Equation (5.1) is the stationary version of the parabolic problem

$$\begin{gathered} u_t - a\Big(\int_\Omega u(x,t)dx\Big)\Delta u = f \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{gathered}$$

Here T is some arbitrary time and u represents, for instance, the density of a population subject to spreading. See [4, 5] for more details. In particular, [4] studies problem (5.1), with $f \in H^{-1}(\Omega)$, and proves the following result.

Proposition 5.1. *Let $a : \mathbb{R} \to (0, +\infty)$ be a positive function, $f \in H^{-1}(\Omega)$. Then problem (5.1) has as many solutions μ as the equation*

$$a(\mu)\mu = \langle\langle f, \varphi\rangle\rangle,$$

where φ is the function(unique) satisfying

$$\begin{gathered} -\Delta\varphi = 1 \quad \textit{in } \Omega, \\ \varphi = 0 \quad \textit{on } \partial\Omega, \end{gathered}$$

Now, we study the nonlocal problem

$$\begin{gathered} -a\Big(\int_\Omega |u|^q\Big)\Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{5.2}$$

where Ω and f are as before and $1 < q < 2N/(N-2), N \geq 3$. When $q = 2$ we have the well known Carrier equation.

Theorem 5.2. *If $t \mapsto a(t)$ is a decreasing and continuous function, for $t \geq 0$, $\lim_{t\to+\infty} a(t^q)t = +\infty$ and $t \mapsto a(t^q)t$ is injective, for $t \geq 0$, then, for each $0 \neq f \in H^{-1}(\Omega)$, problem (5.2) possesses a unique weak solution.*

Proof. As in the proof of Theorem 2.1, let $F : \mathbb{R}^m \to \mathbb{R}^m$ be the function $F(\xi) = (F_1(\xi), \dots, F_m(\xi))$, where

$$F_i(\xi) = a(\|u\|_q^q)\int_\Omega \nabla u \cdot \nabla e_i - \langle\langle f, e_i\rangle\rangle, \quad i = 1, \dots, m$$

with $u = \sum_{j=1}^m \xi_j e_j$ and the identifications of $\mathbb{R}^m$ and $\mathbb{V}_m$ mentioned before. So

$$F_i(\xi) = a(\|u\|_q^q)\xi_i - \langle\langle f, e_i\rangle\rangle, \quad i = 1, \dots, m$$

and then

$$< F(\xi), \xi >= a(\|u\|_q^q)\|u\|^2 - \langle\langle f, u\rangle\rangle$$

We have to show that there is $r > 0$ so that $\langle F(\xi), \xi\rangle \geq 0$, for all $|\xi| = r$ in $\mathbb{V}_m$. Suppose, on the contrary, that for each $r > 0$ there is $u_r \in \mathbb{V}_m$ such that $\|u_r\| = r$ and

$$\langle F(\xi_r), \xi_r\rangle < 0, \quad \xi_r \leftrightarrow u_r.$$

Taking $r = n \in \mathbb{N}$ we obtain a sequence (u_n), $\|u_n\| = n$, $u_n \in \mathbb{V}_m$ and

$$\langle F(u_n), u_n\rangle = a(\|u_n\|_q^q)\|u_n\|^2 - \langle\langle f, u_n\rangle\rangle < 0$$

and so

$$a(\|u_n\|_q^q)\|u_n\| < C\|f\|, \quad \forall n = 1, 2, \dots.$$

Because of the continuous immersion $H_0^1(\Omega) \subset L^q(\Omega)$ one gets $\|u\|_q \leq C\|u\|$ and the monotonicity of a yields $a(\|u\|_q^q) \geq a(C\|u\|^q)$ and so

$$a(C\|u_n\|^q)\|u_n\| < C\|f\|.$$

In view of $\lim_{t\to+\infty} a(t^q)t = +\infty$ one has that $\|u_n\| \leq C, \forall n \in \mathbb{N}$, which contradicts $\|u_n\| = n$. So, there is $r_m > 0$ such that $\langle F(\xi), \xi\rangle \geq 0$, for all $|\xi| = r_m$. In view of Proposition 1.1 there is $u_m \in \mathbb{V}_m$, $\|u_m\| \leq r_m$ such that $F_i(u_m) = 0, i = 1, \dots, m$, that is,

$$a\left(\|u_m\|_q^q\right) \int_\Omega \nabla u_m \cdot \nabla \omega = \langle\langle f, \omega\rangle\rangle, \quad \forall \omega \in \mathbb{V}_m. \tag{5.3}$$

Reasoning as before, by using the facts that $t \to a(t)$ is decreasing for $t \geq 0$ and $\lim_{t\to+\infty} a(t^q)t = +\infty$, we conclude that $\|u_m\| \leq C, \forall m = 1, 2\dots$ for some constant C that does not depend on m. Hence, $u_m \rightharpoonup u$ in $H_0^1(\Omega)$, $u_m \to u$ in $L^q(\Omega)$, $1 < q < \frac{2N}{N-2}$, and so $\|u_m\|_q \to \|u\|_q$. Taking limits on both sides of equation (5.3) we conclude that the function u is a weak solution of problem (5.2). Since $a(t^q)t$ is injective on $t \geq 0$ such a solution is unique. □

Remark 5.3. The function

$$a(t) = \frac{1}{t^{2\beta} + 1},$$

where β and q are related by $2\beta q < 1$, satisfies the assumptions of Theorem 5.2.

Remark 5.4. Following the same steps of the proof of Theorem 4.1 we can prove that the problem

$$\begin{gathered} -a\Big(\int_\Omega |u|^q\Big)\Delta u = u^\alpha \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u > 0 \quad \text{in } \Omega. \end{gathered} \tag{5.4}$$

where $0 < \alpha < 1$, and a satisfies the assumptions in Theorem 5.2 possesses a solution.

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