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OSCILLATION OF NONLINEAR IMPULSIVE HYPERBOLIC EQUATIONS WITH SEVERAL DELAYS

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Abstract. In this paper, we study oscillatory properties of solutions to nonlinear impulsive hyperbolic equations with several delays. Sufficient conditions for oscillations of the solutions are established

1. Introduction

The theory of partial functional differential equations can be applied to fields, such as to biology, population growth, engineering, medicine, physics and chemistry. In the last few years, a few of papers have been published on oscillation theory of partial differential equations. The qualitative theory of this class of equations, however, is still in an initial stage of development. We may easily visualize situations in nature where abrupt changes such as shock and disasters may occur. These phenomena are short-time perturbations. Consequently, it is natural to assume, in modelling these problems, that these perturbations act instantaneously, that is, in the form of impulses. In 1991, the first paper [8] on this class of equations was published. But for instance, on oscillation theory of impulsive partial differential equations only a few of papers have been published. Recently, Bainov, Minchev, Fu and Luo [4, 5, 9, 10, 19] investigated the oscillation of solutions of impulsive partial differential equations with or without deviating argument. But there is a scarcity in the study of oscillation theory of nonlinear impulsive hyperbolic equations with several delays. In this paper, we discuss the oscillatory properties of solutions for nonlinear impulsive hyperbolic equations with several delays (1.1), under the boundary condition (1.4). It should be noted that the equation we discuss here is nonlinear and that we could not find work for oscillations of this kind of problem.

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= a(t)h(u)\Delta u + \sum_{i=1}^{m} a_i(t)h_i(u(t-\tau_i,x))\Delta u(t-\tau_i,x) \\
&\quad - q(t,x)f(u(t,x)) - \sum_{j=1}^{n} g_j(t,x)f_j(u(t-\sigma_j,x)), \\
& t \neq t_k, \quad (t,x) \in \mathbb{R}_+ \times \Omega = G,
\end{aligned} \tag{1.1}
$$

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$$u(t_k^+, x) - u(t_k^-, x) = q_k u(t_k, x), \tag{1.2}$$

$$u_t(t_k^+, x) - u_t(t_k^-, x) = b_k u_t(t_k, x), \quad k = 1, 2, \dots. \tag{1.3}$$

with the boundary condition

$$\frac{\partial u}{\partial n} = 0, \quad (t, x) \in \mathbb{R}_+ \times \partial\Omega, \tag{1.4}$$

and the initial condition $u(t, x) = \Phi(t, x)$, $(t, x) \in [-\delta, 0] \times \Omega$. Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ smooth enough and n is a unit exterior normal vector of $\partial\Omega$, $\delta = \max\{\tau_i, \sigma_j\}$, $\Phi(t, x) \in C^2([-\delta, 0] \times \Omega, \mathbb{R})$.

Assume that the following conditions are fulfilled:

- (H1) $a(t), a_i(t) \in PC(\mathbb{R}_+, \mathbb{R}_+)$, $\tau_i = \text{const.} > 0$, $\sigma_j = \text{const.} > 0$, $i = 1, 2, \dots m$, $j = 1, 2, \dots n$, $q(t, x), g_j(t, x) \in C(\mathbb{R}_+ \times \overline{\Omega}, (0, \infty))$; where PC denote the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1, 2, \dots$ and left continuous at $t = t_k$, $k = 1, 2, \dots$.
- (H2) $h'(u), h_i'(u), f(u), f_j(u) \in C(\mathbb{R}, \mathbb{R})$; $f(u)/u \geq C = \text{const.} > 0$, $f_j(u)/u \geq C_j = \text{const.} > 0$, for $u \neq 0$; $q_k > -1$, $b_k > -1$, $b_k < q_k$, $0 < t_1 < t_2 < \dots < t_k < \dots$, $\lim_{t\to\infty} t_k = \infty$.
- (H3) $u(t, x)$ and their derivatives $u_t(t, x)$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k$, $k = 1, 2, \dots$ and left continuous at $t = t_k$, $u(t_k, x) = u(t_k^-, x)$, $u_t(t_k, x) = u_t(t_k^-, x)$, $k = 1, 2, \dots$.

Let us construct the sequence $\{\bar{t}_k\} = \{t_k\} \cup \{t_{k\tau_i}\} \cup \{t_{k\sigma_j}\}$, where $t_{k\tau_i} = t_k + \tau_i, t_{k\sigma_j} = t_k + \sigma_j$ and $\bar{t}_k < \bar{t}_{k+1}, i = 1, 2, \dots, m, k = 1, 2, \dots$.

Definition. By a solution of problem (1.1), (1.4) with initial condition, we mean that any function $u(t, x)$ for which the following conditions are valid:

1. If $-\delta \leq t \leq 0$, then $u(t, x) = \Phi(t, x)$.
2. If $0 \leq t \leq \bar{t}_1 = t_1$, then $u(t, x)$ coincides with the solution of the problem (1.1)–(1.4) with initial condition.
3. If $\bar{t}_k < t \leq \bar{t}_{k+1}$, $\bar{t}_k \in \{t_k\} \setminus (\{t_{ki}\} \bigcup \{t_{kj}\})$, then $u(t, x)$ coincides with the solution of the problem (1.1)–(1.4).
4. If $\bar{t}_k < t \leq \bar{t}_{k+1}$, $\bar{t}_k \in \{t_{ki}\} \bigcup \{t_{kj}\}$, then $u(t, x)$ coincides with the solution of the problem (1.4) and the following equations

$$\begin{gathered}
\frac{\partial^2 u}{\partial t^2} = a(t)h(u(t^+, x))\Delta u(t^+, x) + \sum_{i=1}^{m} a_i(t)h_i(u((t - \tau_i)^+, x))\Delta u((t - \tau_i)^+, x) \\
-q(t, x)f(u(t^+, x)) - \sum_{j=1}^{n} g_j(t, x)f_j(u((t - \sigma_j^+), x)) \quad (t, x) \in \mathbb{R}_+ \times \Omega = G \\
u(\bar{t}_k^+, x) = u(\bar{t}_k, x), \quad u_t(\bar{t}_k^+, x) = u_t(\bar{t}_k, x), \\
\text{for } \bar{t}_k \in (\{t_{k\tau_i}\} \cup \{t_{k\sigma_j}\}) \setminus \{t_k\},
\end{gathered}$$

or

$$\begin{gathered}
u(\bar{t}_k^+, x) = (1 + q_{k_i})u(\bar{t}_k, x), \quad u_t(\bar{t}_k^+, x) = (1 + b_{k_i})u_t(\bar{t}_k, x) \\
\text{for } \bar{t}_k \in (\{t_{k\tau_i}\} \cup \{t_{k\sigma_j}\}) \cap \{t_k\}.
\end{gathered}$$

Where the number k_i is determined by the equality $\bar{t}_k = t_{k_i}$.

We introduce the notation: $\Gamma_k = \{(t,x) : t \in (t_k, t_{k+1}), x \in \Omega\}$, $\Gamma = \bigcup_{k=0}^{\infty} \Gamma_k$, $\bar{\Gamma}_k = \{(t,x) : t \in (t_k, t_{k+1}), x \in \bar{\Omega}\}$, $\bar{\Gamma} = \bigcup_{k=0}^{\infty} \bar{\Gamma}_k$, $v(t) = \int_\Omega u(t,x)dx$ and $p(t) = minq(t,x)$, $p_j(t) = ming_j(t,x), x \in \bar{\Omega}$.

Definition. The solution $u \in C^2(\Gamma) \cap C^1(\bar{\Gamma})$ of problem (1.1), (1.4) is called nonoscillatory in the domain G if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

2. Oscillation properties of the problem (1.1), (1.4)

For the main theorem of this paper, we need following lemmas.

Lemma 2.1. *Let $u \in C^2(\Gamma) \cap C^1(\bar{\Gamma})$ be a positive solution of (1.1), (1.4) in G, then function $v(t)$ satisfies the impulsive differential inequality*

$$v''(t) + Cp(t)v(t) + \sum_{j=1}^{n} C_j p_j(t) v(t - \sigma_j) \leq 0, \ , t \neq t_k, \tag{2.1}$$

$$v(t_k^+) = (1 + q_k) v(t_k) \ \ k = 1, 2, \dots. \tag{2.2}$$

$$v'(t_k^+) = (1 + b_k) v'(t_k) \ \ k = 1, 2, \dots. \tag{2.3}$$

Proof. Let $u(t,x)$ be a positive solution of the problem (1.1), (1.4) in G. Without loss of generality, we may assume that $u(t,x) > 0$, $u(t - \tau_i, x) > 0$, $i = 1, 2, \dots, m, u(t - \sigma_j, x) > 0$, $j = 1, 2, \dots, n$, for any $(t,x) \in [t_0, \infty) \times \Omega$.

For $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \dots$, integrating (1.1) with respect to x over Ω yields

$$\begin{aligned} \frac{d^2}{dt^2}[\int_\Omega u dx] &= a(t) \int_\Omega h(u) \Delta u dx + \sum_{i=1}^{m} a_i(t) \int_\Omega h_i(u(t - \tau_i, x)) \Delta u(t - \tau_i, x) dx \\ &\quad - \int_\Omega q(t,x) f(u(t,x)) dx - \sum_{j=1}^{n} \int_\Omega g_j(t,x) f_j(u(t - \sigma_j, x)). \end{aligned} \tag{2.4}$$

By Green's formula and the boundary condition, we have

$$\int_\Omega h(u) \Delta u dx = \int_{\partial\Omega} h(u) \frac{\partial u}{\partial n} ds - \int_\Omega h'(u) |gradu|^2 dx \leq - \int_\Omega h'(u) |gradu|^2 dx \leq 0,$$

$$\int_\Omega h_i(u(t - \tau_i, x)) \Delta u(t - \tau_i, x) dx \leq 0.$$

From condition (H2), we can easily obtain

$$\int_\Omega q(t,x) f(u(t,x)) dx \geq Cp(t) \int_\Omega u(t,x) dx,$$

$$\int_\Omega q_j(t,x) f_j(u(t - \sigma_j, x)) dx \geq C_j p_j(t) \int_\Omega u(t - \sigma_j, x) dx.$$

It follows that from above that

$$v'' + Cp(t)v(t) + \sum_{j=1}^{n} C_j p_j(t) v(t - \sigma_j) \leq 0, \quad (t \geq t_0, \ t \neq t_k). \tag{2.5}$$

Where $v(t) > 0$. For $t > t_0$, $t = t_k$, $k = 1, 2, \dots$, we have

$$\int_\Omega u(t_k^+, x) dx - \int_\Omega u(t_k^-, x) dx = q_k \int_\Omega u(t_k, x) dx,$$

$$\int_\Omega u_t(t_k^+,x)dx - \int_\Omega u_t(t_k^-,x)dx = b_k \int_\Omega u_t(t_k,x)dx.$$

This implies

$$v(t_k^+) = (1+q_k)v(t_k), \tag{2.6}$$

$$v'(t_k^+) = (1+b_k)v'(t_k) \quad k = 1,2,\dots. \tag{2.7}$$

Hence we obtain that $v(t) > 0$ is a positive solution of differential inequality (2.1)–(2.3). This completes the proof. □

Definition. The solution $v(t)$ of differential inequality (2.1)–(2.3) is called eventually positive (negative) if there exists a number t^* such that $v(t) > 0 (v(t) < 0)$ for $t \ge t^*$.

Lemma 2.2 ([2, Theorem 1.4.1]). *Assume that*

(i) *$m(t) \in PC^1[\mathbb{R}^+, \mathbb{R}]$ is left continuous at t_k for $k = 1,2,\dots,$*

(ii) *For $k = 1,2,\dots$ and $t \ge t_0$,*

$$m'(t) \le p(t)m(t) + q(t) \quad (t \ne t_k),$$

$$m(t_k^+) \le d_k m(t_k) + e_k.$$

where $p(t), q(t) \in C(\mathbb{R}^+, R), d_k \ge 0$ and e_k are real constants, $PC^1[\mathbb{R}^+, \mathbb{R}] = \{x : \mathbb{R}^+ \to \mathbb{R}; x(t)$ is continuous and continuously differentiable everywhere except some t_k at which $x(t_k^+), x(t_k^-), x'(t_k^+)$ and $x'(t_k^-)$ exist and $x(t_k) = x(t_k^-), x'(t_k) = x'(t_k^-)\}$. Then

$$m(t) \le m(t_0) \prod_{t_0<t_k<t} d_k \exp(\int_{t_0}^t p(s)ds) + \int_{t_0}^t \prod_{s<t_k<t} d_k exp(\int_s^t p(r)dr)q(s)ds$$
$$+ \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} d_j exp(\int_{t_k}^t p(s)ds)e_k.$$

From the above lemma, we can obtain the following result; see also [19].

Lemma 2.3. *Let $v(t)$ be eventually positive (negative) solution of differential inequality (2.1)–(2.3). Assume that there exists $T \ge t_0$ such that $v(t) > 0 (v(t) < 0)$ for $t \ge T$. If the following condition holds,*

$$\lim_{t\to+\infty} \int_{t_0}^t \prod_{t_0<t_k<s} \frac{1+b_k}{1+q_k} ds = +\infty, \tag{2.8}$$

then $v'(t) \ge 0 (v'(t) \le 0)$ for $t \in [T, t_l] \bigcup (\bigcup_{k=l}^{+\infty} (t_k, t_{k+1}])$, where $l = \min\{k : t_k \ge T\}$.

Theorem 2.4. *If condition (2.8) hold and, for some j_0,*

$$\lim_{t\to+\infty} \int_{t_0}^t \prod_{t_0<t_k<s} \frac{1+q_k}{1+b_k} p_{j_0}(s) ds = +\infty, \tag{2.9}$$

then each solution of (1.1)–(1.4) oscillates in G.

Proof. Let $u(t,x)$ be a nonoscillatory solution of (1.1), (1.4). Without loss of generality, we can assume that $u(t,x) > 0$, $u(t-\tau_i, x) > 0$, $i = 1,2,\dots,m$, $u(t-\sigma_j, x) > 0$, $j = 1,2,\dots,n$ for any $(t,x) \in [t_0,\infty) \times \Omega$. From Lemma 2.1, we know that $v(t)$ is a positive solution of (2.1)–(2.3). Thus from Lemma 2.3, we can find that $v'(t) \geq 0$ for $t \geq t_0$.

For $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \dots$, define

$$w(t) = \frac{v'(t)}{v(t)}, \quad t \geq t_0.$$

Then we have $w(t) > 0$, $t \geq t_0$, $v'(t) - w(t)v(t) = 0$. We may assume that $v(t_0) = 1$, thus in view of (2.1)–(2.3) we have that for $t \geq t_0$,

$$v(t) = \exp(\int_{t_0}^{t} w(s)ds) \tag{2.10}$$

$$v'(t) = w(t)\exp(\int_{t_0}^{t} w(s)ds) \tag{2.11}$$

$$v''(t) = w^2(t)\exp(\int_{t_0}^{t} w(s)ds) + w'(t)\exp(\int_{t_0}^{t} w(s)ds) \tag{2.12}$$

We substitute (2.10)–(2.12) into (2.1) and can obtain

$$w^2(t)\exp(\int_{t_0}^{t} w(s)ds) + w'(t)\exp(\int_{t_0}^{t} w(s)ds) + C_{j_0}p_{j_0}(t)exp(\int_{t_0}^{t-\sigma_{j_0}} w(s)ds) \leq 0.$$

Hence, we have

$$w^2(t) + w'(t) + C_{j_0}p_{j_0}(t)exp(-\int_{t-\sigma_{j_0}}^{t} w(s)ds) \leq 0\,.$$

From above and condition $b_k < q_k$, it is easy to see that the function $w(t)$ is non-increasing for $t \geq t_1 \geq \delta + t_0$. Thus $w(t) \leq w(t_1)$, for $t \geq t_1$, which implies

$$w'(t) + C_{j_0}p_{j_0}(t)\exp(-\delta w(t_1)ds) \leq 0, \quad t \geq t_1.$$

From (2.2), (2.3) we get

$$w(t_k^+) = \frac{v'(t_k^+)}{v(t_k^+)} = \frac{1+b_k}{1+q_k}w(t_k), \quad k = 1, 2, \dots.$$

It follows that

$$w'(t) \leq -C_{j_0}p_{j_0}(t)exp(-\delta w(t_1)ds) \quad (t \neq t_k), \tag{2.13}$$

$$w(t_k^+) = \frac{1+b_k}{1+q_k}w(t_k) \quad (t = t_k). \tag{2.14}$$

Using Lemma 2.2, we obtain

$$\begin{aligned} w(t) &\leq w(t_0)\prod_{t_0<t_k<t}\frac{1+b_k}{1+q_k} + \int_{t_0}^{t}\prod_{s<t_k<t}\frac{1+b_k}{1+q_k}(-C_{j_0}p_{j_0}(s)\exp(-\delta w(t_1)))ds \\ &= \prod_{t_0<t_k<t}\frac{1+b_k}{1+q_k}\{w(t_0) - \int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{1+q_k}{1+b_k}C_{j_0}p_{j_0}(s)\exp(-\delta w(t_1))ds\}. \end{aligned}$$

Since $w(t) > 0$, the last inequality contradicts (2.9). Therefore, the proof is complete. $\square$

It should be noted that obviously all solutions of problem (1.1)–(1.2) are oscillatory if there exists a subsequence n_k of n such that $q_{n_k} < -1$, for $k = 1, 2, \dots$. So we discuss only the case of $q_k > -1$.

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