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A NOTE ON A 3-DIMENSIONAL STATIONARY SCHRÖDINGER–POISSON SYSTEM

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ABSTRACT. In a previous paper we have proved existence of a ground state for a stationary Schrödinger–Poisson system in the whole space $\mathbb{R}^3$ under appropriate assumptions on the data, namely the dopant-density n^* and the effective potential $\widetilde{V}$. In this note we show that the same result remains true under less restrictive hypotheses.

1. INTRODUCTION

We are concerned with existence of standing waves (i.e. solutions of the form $u(t,x) = e^{i\omega t}u(x)$ with a real constant ω) for a time-dependent Schrödinger equation where the electric potential V satisfies a linear Poisson equation. This leads to solving the stationary Schrödinger–Poisson system

$$-\frac{1}{2}\Delta u + (V + \widetilde{V})u + \omega u = 0 \quad \text{in } \mathbb{R}^3 \tag{1.1}$$

$$-\Delta V = |u|^2 - n^* \quad \text{in } \mathbb{R}^3 \tag{1.2}$$

where the dopant-density n^* and the effective potential $\widetilde{V}$ are given reals functions. An existence result of a solution for (1.1)–(1.2) has been established by Lions [3] in the particular case where $\widetilde{V}(x) = -2/|x|$ and $n^* \equiv 0$, by Nier [4] under some assumptions on the data essentially when $\|\widetilde{V}\|_{L^2}$ and $\|n^*\|_{L^2}$ are small enough and also recently by the author [1] under appropriate assumptions on $\widetilde{V}$ and n^*.

In this note, we show existence of a ground state of (1.1)–(1.2) as in [1] but under less restrictive assumptions. More precisely, an adequate modification on the proof of the main result in [1, theorem 1.3] allows us to avoid the condition where $n^* \in L^1(\mathbb{R}^3)$.

Let us recall firstly the principal theorem and the several steps of its proof given in [1]: after solving explicitly the Poisson equation for any fixed $u \in H^1(\mathbb{R}^3)$, we substitute the unique solution then obtained $V = V(u)$ in the Schrödinger equation (1.1) and show existence of a ground state of

$$-\frac{1}{2}\Delta u + (V(u) + \widetilde{V})u + \omega u = 0 \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

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To this end, we show that the energy functional corresponding to (1.3) is exactly the expression

$$E(\varphi) := \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} |\nabla V(\varphi)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \widetilde{V}\varphi^2 dx + \frac{\omega}{2}\int_{\mathbb{R}^3} \varphi^2 dx \quad (1.4)$$

and a solution of (1.3) is obtained as a minimizer of E on $H^1(\mathbb{R}^3)$.

Before giving the assumptions imposed on $\widetilde{V}$ and n^* to solve the system (1.1)-(1.2), we recall the following concepts.

Definition 1.1. We say that g satisfies the decomposition (1.5) if:

(i) $g \in L^1_{\text{loc}}(\mathbb{R}^3)$,
(ii) $g \geq 0$, and
(iii) There exists $q_0 \in [3/2, \infty]$ such that for all $\lambda > 0$ there exists $g_{1\lambda} \in L^{q_0}(\mathbb{R}^3)$, $q_\lambda \in]3/2, \infty[$ and $g_{2\lambda} \in L^{q_\lambda}(\mathbb{R}^3)$ such that

$$g = g_{1\lambda} + g_{2\lambda} \quad \text{and} \quad \lim_{\lambda\to 0} \|g_{1\lambda}\|_{L^{q_0}} = 0. \quad (1.5)$$

As interesting examples of this definition we may consider $g(x) = 1/|x|^\alpha$ for some $0 < \alpha < 2$ or $g \in L^r(\mathbb{R}^3)$ for some $r > 3/2$ (taking $|g|$ if g is negative).

In what follows we will denote by $\|\cdot\|$ the norm $\|\cdot\|_{L^2}$ on $L^2(\mathbb{R}^3)$ and by $[E \leq c]$ the set $\{\varphi; E(\varphi) \leq c\}$.

Consider now the following hypotheses:

$$\widetilde{V}^+ \in L^1_{\text{loc}}(\mathbb{R}^3) \quad \text{and} \quad \widetilde{V}^- \text{satisfies the decomposition (1.5)} \quad (1.6)$$

$$n^* \in L^1 \cap L^{6/5}(\mathbb{R}^3) \quad (1.7)$$

$$\inf\Big\{\int_{\mathbb{R}^3} \big(|\nabla\varphi|^2 + \varrho(x)\varphi^2\big)\,dx, \int_{\mathbb{R}^3} |\varphi|^2 = 1\Big\} < 0 \quad (1.8)$$

where $\varrho(x) := 2\widetilde{V}(x) - \frac{1}{2\pi}\int_{\mathbb{R}^3} \frac{n^*(y)}{|x-y|} dy$.

The main result in [1] is as follows.

Theorem 1.2. *Assuming* (1.6), (1.7) *and* (1.8) *there exists* $\omega_* > 0$ *such that for all* $0 < \omega < \omega_*$ *the equation* (1.3) *has a nonnegative solution* $u \not\equiv 0$ *which minimizes the functional* E *given by* (1.4):

$$E(u) = \min_{\varphi\in H^1(\mathbb{R}^3)} E(\varphi).$$

The proof of this theorem is divided into the four following Lemmas.

Lemma 1.3. *Let* $\omega \geq 0$ *and* $c \in \mathbb{R}$. *If the set* $[E \leq c]$ *is bounded in* $L^2(\mathbb{R}^3)$ *then it is also bounded in* $H^1(\mathbb{R}^3)$.

Lemma 1.4. *For all* $\omega > 0$ *and* $c \in \mathbb{R}$ *the set* $[E \leq c]$ *is bounded in* $L^2(\mathbb{R}^3)$.

Lemma 1.5. *For any* $\omega > 0$ *the functional* E *is weakly lower semicontinuous on* $H^1(\mathbb{R}^3)$ *and attains its minimum on* $H^1(\mathbb{R}^3)$ *at* $u \geq 0$.

Lemma 1.6. *There exists* $\omega_* > 0$ *such that if* $0 < \omega < \omega_*$ *then* $E(u) < E(0)$ *and thus* $u \not\equiv 0$.

After analyzing the proofs of the four Lemmas above given in [1], we remark that theorem 1.2 remains true even if we replace the condition (1.7) by

$$n^* \in L^{6/5}(\mathbb{R}^3). \quad (1.9)$$

In the sequel we shall minimize the energy functional E on the space

$$H := \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \widetilde{V}^+ u^2 dx < \infty\}$$

which is a Hilbert space, continuously embedded in $H^1(\mathbb{R}^3)$, when endowed it with its natural scalar product and norm

$$(\varphi|\psi) := \int_{\mathbb{R}^3} \Big(\nabla\varphi \cdot \nabla\psi + \varphi\psi + \widetilde{V}^+\varphi\psi\Big)\, dx, \quad \|\varphi\|_H := (\varphi|\varphi)^{1/2}.$$

Consequently Theorem 1.2 becomes

Theorem 1.7. *Assuming* (1.6), (1.8) *and* (1.9) *there exists* $\omega_* > 0$ *such that for all* $0 < \omega < \omega_*$ *the equation* (1.3) *has a nonnegative solution* $u \not\equiv 0$ *which minimizes on the space* H *the functional* E*:*

$$E(u) = \min_{\varphi \in H} E(\varphi).$$

2. Preliminaries

Here we recall the three following Lemmas which will be useful in the sequel.

Lemma 2.1. *Let* $n^* \in L^{6/5}(\mathbb{R}^3)$*. For all* $\varphi \in H^1(\mathbb{R}^3)$ *the Poisson equation*

$$-\Delta V = |\varphi|^2 - n^* \quad \text{in } \mathbb{R}^3 \tag{2.1}$$

has a unique solution $V := V(\varphi) \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ *given by*

$$V(\varphi)(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(|\varphi|^2 - n^*)(y)}{|x-y|} dy. \tag{2.2}$$

Moreover if we denote by

$$I(\varphi) := \frac{1}{4}\int_{\mathbb{R}^3} |\nabla V(\varphi)|^2 dx,$$

then I *is* C^1 *on* $H^1(\mathbb{R}^3)$ *and its derivative satisfies*

$$\langle I'(\varphi), \psi\rangle = \int_{\mathbb{R}^3} V(\varphi)\varphi\psi dx \quad \forall \psi \in H^1(\mathbb{R}^3).$$

For the proof of this lemma see [1, Lemma 2.1, Lemma 2.2].

This Lemma shows in particular that the energy functional corresponding to (1.3) is exactly the expression given in (1.4), namely

$$E(\varphi) := \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx + I(\varphi) + \frac{1}{2}\int_{\mathbb{R}^3} \widetilde{V}\varphi^2 dx + \frac{\omega}{2}\int_{\mathbb{R}^3} \varphi^2 dx.$$

Lemma 2.2. *Let* $\theta \in L^r(\mathbb{R}^3)$ *for some* $r \geq 3/2$ *then for all* $\delta > 0$ *there exists* $C_\delta > 0$ *such that*

$$\int_{\mathbb{R}^3} \theta(x)|\varphi(x)|^2 dx \leq \delta\|\nabla\varphi\|^2 + C_\delta\|\varphi\|^2 \quad \forall \varphi \in H^1(\mathbb{R}^3).$$

For the proof of this lemma see [1] or [2].

Remark that since $\widetilde{V}^-$ satisfies the decomposition (1.5) then for any fixed $\lambda > 0$ we have $\widetilde{V}^- = \widetilde{V}^-_{1\lambda} + \widetilde{V}^-_{2\lambda}$ where for $i = 1, 2$, $\widetilde{V}^-_{i\lambda} \in L^s(\mathbb{R}^3)$ for some $s \in [3/2, \infty]$

($s = q_0$ or $s = q_\lambda$). Hence taking $\theta := \widetilde{V}_{i\lambda}^-$ the inequality of Lemma 2.2 holds for $i = 1, 2$ and consequently for all $\delta > 0$ there exists $C_\delta > 0$ so that

$$\int_{\mathbb{R}^3} \widetilde{V}^-(x)|\varphi(x)|^2 dx \le \delta \|\nabla\varphi\|^2 + C_\delta \|\varphi\|^2 \quad \forall \varphi \in H^1(\mathbb{R}^3). \tag{2.3}$$

Lemma 2.3. *Let $\psi \in L^r(\mathbb{R}^3)$ for some $r > 3/2$. If $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$ then*

$$\int_{\mathbb{R}^3} \psi(x) v_n^2(x) dx \to 0 \quad as \quad n \to +\infty$$

For the proof of this lemma see [1, Lemma 2.5].

3. Proof of Theorem 1.7

We will use once again the same steps as in [1]. Remark at first that the proofs given in [1] for Lemma 1.5 and Lemma 1.6 do not require the hypothesis $n^* \in L^1(\mathbb{R}^3)$ and consequently remain valid assuming (1.9) instead of (1.7).

Proof of Lemma 1.3. We show here that if the set

$$[E \le c] := \{\varphi \in H; E(\varphi) \le c\}$$

is bounded in $L^2(\mathbb{R}^3)$ then it is also bounded in H. Indeed since $I(\varphi)$ and ω are both nonnegative, the inequality $E(\varphi) \le c$ gives in particular

$$\frac{1}{4}\|\nabla\varphi\|^2 + \frac{1}{2}\int \widetilde{V}^+\varphi^2 dx - \frac{1}{2}\int \widetilde{V}^-\varphi^2 dx \le c.$$

Now using the estimate (2.3) with $\delta = 1/4$ we get

$$\frac{1}{8}\|\nabla\varphi\|^2 + \frac{1}{2}\int \widetilde{V}^+\varphi^2 dx \le K_0\|\varphi\|^2 + c.$$

for some constant $K_0 > 0$. ☐

Let us recall that in [1] we have decomposed the expression of $E(\varphi)$ as

$$E(\varphi) = E_1(\varphi) - E_2(\varphi) + E_3(\varphi) + E(0)$$

where

$$\begin{gathered} E_1(\varphi) := \frac{1}{4}\int |\nabla\varphi|^2\, dx + \frac{1}{2}\int \widetilde{V}^+\varphi^2 dx + \frac{\omega}{2}\int \varphi^2\, dx \\ E_2(\varphi) := \frac{1}{2}\int \widetilde{V}^-\varphi^2\, dx + \frac{1}{8\pi}\iint \frac{n^*(y)}{|x-y|}\varphi^2(x)\, dx\, dy \\ E_3(\varphi) := \frac{1}{16\pi}\iint \frac{\varphi^2(x)\varphi^2(y)}{|x-y|}\, dx\, dy \\ E(0) := \frac{1}{16\pi}\iint \frac{n^*(x)n^*(y)}{|x-y|}\, dx\, dy. \end{gathered}$$

Indeed, for the term $I(\varphi)$ it suffices to multiply the equation (2.1) by $V(\varphi)$, integrate by parts and use the formula (2.2).

In the proof of the similar lemma [1, Lemma 3.1] we have estimated $E_2(\varphi)$ instead of $\int \widetilde{V}^-\varphi^2 dx$. More precisely we have estimated the second term of $E_2(\varphi)$ by using a certain inequality of type *Hardy* and the fact that $n^* \in L^1(\mathbb{R}^3)$.

We point out finally that the decomposition of $E(\varphi)$ as above remains useful for the rest of proofs.

Proof of Lemma 1.4. Assume by contradiction that there exists a sequence $(u_j)_j \subset H$ such that $E(u_j) \leq c$ and $\|u_j\| \longrightarrow +\infty$ as $j \to +\infty$. Let $v_j := u_j/\|u_j\|$ then $\|v_j\| = 1$ and from $E(u_j) \leq c$ we get

$$\frac{1}{4}\int |\nabla v_j|^2 dx - E_2(v_j) + E_3(v_j)\|u_j\|^2 + \frac{\omega}{2} \leq \frac{c_0}{\|u_j\|^2} \tag{3.1}$$

where $c_0 = c - E(0)$. To estimate $E_2(v_j)$ it suffices to use (2.3) for the first term $\int \widetilde{V}^- v_j^2 dx$. As to the second, unlike the proof in [1] we do not require here the assumption $n^* \in L^1(\mathbb{R}^3)$. Indeed, setting

$$V^*(x) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{n^*(y)}{|x-y|}dy = -V(0)(x) \tag{3.2}$$

as denoted in Lemma 2.1 we may write

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{n^*(y)}{|x-y|} v_j^2(x)dxdy = \int_{\mathbb{R}^3} V^*(x)v_j^2(x)dx.$$

Knowing that $V(0) \in L^6(\mathbb{R}^3)$ we can use once more Lemma 2.2 with $\theta := V^*$.

On the whole, we obtain in particular

$$E_2(v_j) \leq \frac{1}{8}\|\nabla v_j\|^2 + K_0$$

for some positive constant K_0 and consequently we infer from the inequality (3.1) that

$$\frac{1}{8}\|\nabla v_j\|^2 + E_3(v_j)\|u_j\|^2 + \frac{\omega}{2} \leq \frac{c_0}{\|u_j\|^2} + K_0.$$

For the remainder of the proof, we conclude exactly as in of [1, Lemma 3.2]. Precisely we show first that, up to a subsequence, $v_j \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$. Next, from (3.1) it follows in particular that

$$\frac{\omega}{2} - E_2(v_j) \leq \frac{c_0}{\|u_j\|^2}. \tag{3.3}$$

Using the decomposition $\widetilde{V}^- = \widetilde{V}^-_{1\lambda} + \widetilde{V}^-_{2\lambda}$ and (3.2), we show according to Lemma 2.3 that $E_2(v_j) \longrightarrow 0$ as $j \to \infty$. Finally, letting j go to infinity in (3.3) we obtain a contradiction since ω is positive. Consequently, all $(u_j)_j \subset H$ such that $E(u_j) \leq c$ is bounded in $L^2(\mathbb{R}^3)$. □

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