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OSCILLATION OF SOLUTIONS TO SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this article, we establish conditions under which proper solutions of the second order linear differential equation

$$(r(t)y'(t))' + a(t)y(t) = 0$$

oscillate. The obtained results generalize, extend and improve the results in Džurina [1].

1. Introduction

Consider the second order linear differential equation

$$(r(t)y'(t))' + a(t)y(t) = 0, \quad t \geq t_0, \tag{1.1}$$

where a and r are continuous functions on $[t_0, \infty)$,

$$\begin{gathered} a(t) \geq 0 \quad \text{for } t \geq t_0\,, \\ r(t) > 0 \quad \text{for } t \geq t_0 \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{r(t)} \mathrm{d}t = \infty\,. \end{gathered} \tag{1.2}$$

Our results pertain only to the nontrivial continuable solutions x(t) of (1.1), i.e. x(t) is defined on an interval of the form $[t_x, \infty)$ and for every T in $[t_x, \infty)$ we have $\sup\{|x(t)| : t \geq T\} > 0$. Such a solution is called a proper solution of (1.1). A proper solution $x : [t_x, \infty) \to \mathbb{R}$ of (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$ and nonoscillatory otherwise. Equation (1.1) is said to be oscillatory if all its proper solutions are oscillatory, otherwise it is said to be nonoscillatory. If x(t) is a nonoscillatory solution of (1.1), in what follows, we will assume that it is positive on its interval of definition.

Among numerous papers dealing with the oscillatory character of the proper solutions of (1.1), we refer the reader to [1]. We concentrate on the following results, where $R(t) = \int_{t_0}^{t} \frac{1}{r(s)} \mathrm{d}s$ for $t \geq t_0$.

Theorem 1.1 ([4]). *Assume* (1.2) *and* $\int_{t_0}^{\infty} R^{\alpha}(t)a(t)\mathrm{d}t = \infty$ *for some* $\alpha \in (0, 1)$. *Then* (1.1) *is oscillatory.*

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Theorem 1.2 ([1]). *Assume* (1.2) *and*

$$\liminf_{t\to\infty} R(t)\int_t^\infty a(s)\mathrm{d}s > \frac{1}{4}\,. \tag{1.3}$$

Then (1.1) *is oscillatory.*

Theorem 1.3 ([3]). *Assume* (1.2) *and*

$$\int_{t_0}^\infty \big((1+R(t))a(t) - \frac{1}{4r(t)(1+R(t))}\big)\mathrm{d}t = \infty\,. \tag{1.4}$$

Then (1.1) *is oscillatory.*

This article consists of an introduction and two more sections. In Section 2 we show that (1.1) has a positive proper solution if and only if the Ricatti equation

$$x'(t) + \frac{1}{r(t)}x^2(t) + a(t) = 0$$

has a positive proper solution. We use this fact to establish our auxiliary results for existence or nonexistence of positive proper solutions. In Section 3, using the results of Section 2, we formulate our main results which generalize, extend and improve Theorems 1.1,1.2, 1.3. This fact is illustrated by an appropriate example.

2. Auxiliary results

The following proposition shows the relationship between (1.1) and the Ricatti equation

$$x'(t) + \frac{1}{r(t)}x^2(t) + a(t) = 0\,. \tag{2.1}$$

Lemma 2.1. *Let* (1.2) *hold. Then* (1.1) *has a positive solution if and only if (2.1) has a positive solution.*

Proof. If $a(t) = 0$ on an interval of the form $[T,\infty) \subset (t_0,\infty)$, then the function

$$x(t) = \frac{1}{1+\int_T^t \frac{1}{r(s)}\mathrm{d}s}$$

for $t \geq T$ is a positive solution of (2.1) on $[T,\infty)$. Let $\sup\{t \in [t_0,\infty) : a(t) > 0\} = \infty$ and let y be a positive solution of (1.1) on an interval of the form $[T,\infty) \subset (t_0,\infty)$. Then, by Lemma in Kiguradze [2], the function $x(t) = r(t)y'(t)/y(t)$ is positive. Taking x' and using (1.1), it is easy to check that $x(t)$ satisfies (2.1) for $t \geq T$.

Let x be a positive solution of equation (2.1) on an interval of the form $[T,\infty) \subset (t_0,\infty)$ and consider the function

$$y(t) = \exp\Big(\int_T^t \frac{x(s)}{r(s)}\mathrm{d}s\Big) \quad \text{for } t \geq T\,.$$

Then, using $y'(t)$ and (2.1), it is easy to see that $y(t)$ satisfies (1.1) for $t \geq T$. This completes the proof of the lemma. □

In view of the above result, it is enough to investigate the existence or nonexistence of positive solutions for equation

$$x'(t) + \frac{1}{\varphi(t)}x^2(t) + \psi(t) = 0, \tag{2.2}$$

where φ and ψ are continuous function on $[t_0, \infty)$

$$\begin{gathered} \psi(t) \geq 0 \quad \text{for } t \geq t_0\,, \\ \varphi(t) > 0 \quad \text{for } t \geq t_0 \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{\varphi(t)} \mathrm{d}t = \infty\,. \end{gathered} \tag{2.3}$$

For the proof of existence of positive solutions to (7), we will use of the following fixed point theorem [5]. This technique is convenient for our needs and uses the following notion.

Definition 2.2. Let X be a partially ordered set. Then it is said to be a *complete lattice*, if each its nonempty subset has a supremum and infimum in X.

Definition 2.3. Let X be a partially ordered set. Then the operator $\Theta : X \mapsto X$ is said to be an *isotonous one*, if $\Theta[x] \leq \Theta[y]$ for each $x, y \in X$ such that $x \leq y$.

Theorem 2.4. *Let X be a complete lattice and $\Theta : X \mapsto X$ be an isotonous operator. Then Θ has at least one fixed point in X.*

The proof of this theroem can be found in [5].

Lemma 2.5. *Let* (2.3) *hold and let x be a positive solution of (2.2) on an interval of the form $[T, \infty) \subset (t_0, \infty)$. Then*

$$x(t) = \int_t^{\infty} \frac{x^2(s)}{\varphi(s)} \mathrm{d}s + \int_t^{\infty} \psi(s) \mathrm{d}s \quad \textit{for } t \geq T\,. \tag{2.4}$$

Proof. By (2.2) and (2.3) the function x is nonincreasing on $[T, \infty)$ and

$$x(t) = x(s) + \int_t^s \frac{x^2(u)}{\varphi(u)} \mathrm{d}u + \int_t^s \psi(u) \mathrm{d}u \quad \text{for } T \leq t \leq s\,.$$

Hence for $s \to \infty$ we obtain

$$x(t) = \lim_{s \to \infty} x(s) + \int_t^{\infty} \frac{x^2(u)}{\varphi(u)} \mathrm{d}u + \int_t^{\infty} \psi(u) \mathrm{d}u \quad \text{for } t \geq T\,. \tag{2.5}$$

Let $t \geq T$ be arbitrary. If $\lim_{s \to \infty} x(s) > 0$, then, (2.3), it follows that

$$\int_t^{\infty} \frac{x^2(u)}{\varphi(u)} \mathrm{d}u = \infty.$$

However, this contradicts (2.5) and proves (2.4), which completes the proof. $\square$

To prove our next result, we set

$$\begin{gathered} \Phi(t) = 1 + \int_{t_0}^{t} \frac{1}{\varphi(s)} \mathrm{d}s\,, \tilde{\varphi}(t) = \phi(t)\Phi(t), \\ \tilde{\psi}(t) = \Phi(t)\psi(t) - \frac{1}{4\varphi(t)\Phi(t)} \ for \ t \geq t_0 \end{gathered}$$

and consider the equation

$$\tilde{x}'(t) + \frac{1}{\tilde{\varphi}(t)} \tilde{x}^2(t) + \tilde{\psi}(t) = 0\,. \tag{2.6}$$

Lemma 2.6. *Let* (2.3) *hold and let $\tilde{\psi}(t) \geq 0$ for $t \geq t_0$. Then (2.2) has a positive solution solution if and only if equation* (2.6) *has a positive solution.*

Proof. Remark that the functions $\tilde\varphi$ and $\tilde\psi$ satisfy the conditions (2.3). If $\tilde\psi(t) = 0$ on an interval of the form $[T, \infty) \subset (t_0, \infty)$, then the function

$$x(t) = \frac{1}{1 + \int_T^t \frac{1}{\tilde\varphi(s)} \mathrm{d}s}$$

for $t \geq T$ is a positive solution of equation (2.6) on the interval $(T_1, \infty), T_1 \geq T$.

Now, let $\sup\{t \in [t_0, \infty) : \tilde\psi(t) > 0\} = \infty$ and let x be a positive solution of equation (2.2) on an interval of the form $[T, \infty) \subset (t_0, \infty)$. Define the function $\tilde x(t) = \Phi(t)x(t) - \frac{1}{2}$. Taking $\tilde x'(t)$ and using (2.2), it is easy to see that $\tilde x(t)$ is a solution of equation (2.6) on the interval $[T, \infty)$. Moreover, from (2.6) it follows that $\tilde x'(t) \leq 0$, while $\tilde x(t)$ is nonincreasing for $t \geq T$. Thus, there exists (proper or improper) $\lim_{t\to\infty} \tilde x(t)$. If $\lim_{t\to\infty} \tilde x(t) = \alpha \neq 0$, then there exist $\beta > 0$ and $T_2 > T$ such that $\tilde x^2(t) > \beta$ and from (2.6),

$$\tilde x'(t) \leq -\frac{\beta}{\tilde\varphi(t)} \quad \text{for } t \geq T_2\,.$$

Integrating this inequality from T_2 to t, $t > T_2$, and using (2.3), we find that $\lim_{t\to\infty} \tilde x(t) = -\infty$. Hence, there is a $T_3 > T$ such that $\tilde x(t) < -\frac{1}{2}$ and $x(t) < 0$ for $t > T_3$. This contradicts the positivity of x. Therefore, we see that $\lim_{t\to\infty} \tilde x(t) = 0$. Since $\tilde x$ is nonincreasing and $\tilde\psi$ is not eventually trivial, the function $\tilde x$ must be positive on $[T, \infty)$.

On the other hand, if (2.6) has a positive solution $\tilde x$ on some interval of the form $[T, \infty) \subset (t_0, \infty)$, then the function $x(t) = \frac{2\tilde x(t)+1}{2\Phi(t)}$ is a positive solution of equation (2.2). This completes the proof of lemma. ☐

Lemma 2.7. *Let* (2.3) *hold and let $T > t_0$ be such that*

$$\begin{gathered} \int_T^\infty \psi(t)\mathrm{d}t < \infty\,, \\ \int_t^\infty \frac{\Psi^2(s)}{\varphi(s)}\mathrm{d}s \leq \frac{1}{4}\Psi(t) \quad \textit{for } t \geq T, \end{gathered} \tag{2.7}$$

where $\Psi(t) = \int_t^\infty \psi(s)\mathrm{d}s$ for $t > T$. Then (2.2) *has at least one positive solution.*

Proof. If $\psi(t) = 0$ on some $[T_0, \infty) \subset [T, \infty)$, then the function

$$x(t) = \frac{1}{1 + \int_S^t \frac{1}{\varphi(s)} \mathrm{d}s}$$

for $t \geq T_0$ is a positive solution of (2.2) on $[T_0, \infty)$.

Assume now that $\sup\{t \in [t_0, \infty) : \psi(t) > 0\} = \infty$. Then $\Psi(t) > 0$ for $t \geq t_0$. Let X be the set of all nonincreasing functions x defined on $[T, \infty)$ and such that

$$\Psi(t) \leq x(t) \leq 2\Psi(t) \quad \text{for } t \geq T\,. \tag{2.8}$$

Then X is a , partially ordered set with the usual point-wise ordering, i.e. $x \leq y$, if $x(t) \leq y(t)$ for any $t \geq T$. It is obvious that for any nonempty subset Y of X the function $x(t) = \sup\{y(t) : y \in Y\}$ for $t \geq T$ is nonincreasing and satisfies (2.8). The same is valid for the infimum. Thus, the set X is a complete lattice. Define the operator Θ, acting on the set X, by the formula

$$\Theta[x](t) = \int_t^\infty \frac{x^2(s)}{\varphi(s)}\mathrm{d}s + \Psi(t) \quad \text{for } t \geq T\,. \tag{2.9}$$

In view of (2.3) and (2.7), $\Theta[x]$ is well-defined, $\Theta[X] \subset X$, and Θ is an isotonous operator. Therefore, by Theorem 2.4, there exists an $x \in X$ such that $x = \Theta[x]$. Using (2.9) and (2.7)(iii), it is not difficult to show that this fixed point x is the desired positive solution of equation (2.2) on $[T, \infty)$. The proof of the lemma is complete. □

Lemma 2.8. *Let* (2.3) *hold and let* $T > t_0$ *be such that*

$$\begin{gathered} \int_T^\infty \psi(t) \mathrm{d}t < \infty \,, \\ \Phi(t) \int_t^\infty \psi(s) \mathrm{d}s \leq \frac{1}{4} \quad \text{for } t \geq T \,. \end{gathered} \tag{2.10}$$

Then (2.2) has at least one positive solution.

Proof. The proof of this lemma is similar to the proof of the Lemma 2.7. We use the same operator Θ, acting on the set X of all nonincreasing functions x defined on an interval of the form $[T, \infty) \subset (t_0, \infty)$ and such that

$$\frac{1}{4\Phi(t)} \leq x(t) \leq \frac{1}{2\Phi(t)} \quad \text{for } t \geq T \,.$$

□

Lemma 2.9. *Let* (2.3) *hold and let*

$$\int_{t_0}^\infty \psi(t) \mathrm{d}t = \infty \,. \tag{2.11}$$

Then (2.2) *has no positive solution.*

Proof. Assume, for the sake of contradiction, that (2.2) has a positive solution x on some interval of the form $[T, \infty) \subset (t_0, \infty)$. Then, in view of (2.2), we see that $x'(t) \leq -\psi(t)$ for $t \geq T$. An integration of this inequality from T to t, $t > T$, and the assumption (2.11) lead to a contradiction which proves the lemma. □

Lemma 2.10. *Let* (2.3) *hold and let*

$$\int_{t_0}^\infty \psi(t) \mathrm{d}t < \infty, \quad \liminf_{t \to \infty} \Phi(t) \int_t^\infty \psi(s) \mathrm{d}s > \frac{1}{4} \,. \tag{2.12}$$

Then (2.2) *has no positive solution.*

Proof. Assume, for the sake of contradiction, that (2.2) has a positive solution x on an interval of the form $[T, \infty) \subset (t_0, \infty)$. Then, by Lemma 2.5, (2.4) holds. Moreover, by (2.12), there exist an $\alpha > \frac{1}{4}$ and a $T_1 \geq T$ such that

$$\int_t^\infty \psi(s) \mathrm{d}s \geq \frac{\alpha}{\Phi(t)} \quad \text{for } t \geq T_1 \,.$$

Hence, in view of (2.4), we find that $x(t) \geq \frac{\alpha}{\Phi(t)}$ for $t \geq T_1$. Applying the same step, we get

$$x(t) \geq \int_t^\infty \frac{\alpha^2}{\varphi(s)\Phi^2(s)} \mathrm{d}s + \int_t^\infty \psi(s) \mathrm{d}s \geq \frac{\alpha^2 + \alpha}{\Phi(t)} \quad \text{for } t \geq T_1.$$

Repeating the above procedure n-times, we conclude that

$$x(t) \geq \frac{\beta_n}{\Phi(t)} \quad \text{for } t \geq S$$

where $\beta_1 = \alpha$ and $\beta_{n+1} = \beta_n^2 + \alpha$ for $n = 1, 2, \dots$. As we see, the sequence $\{\beta_n\}_{n=1}^{\infty}$ is nondecreasing and bounded, while $\lim_{n\to\infty} \beta_n = \beta$ is a solution to the quadratic equation $\beta^2 - \beta + \alpha = 0$. This implies that $1 - 4\alpha \geq 0$, which contradicts $\alpha > \frac{1}{4}$. The proof of the lemma is complete. $\square$

Lemma 2.11. *Let* (2.3) *hold and let*

$$\int_{t_0}^{\infty} \psi(t)\mathrm{d}t < \infty, \quad \int_{t_0}^{\infty} \Phi^{\alpha}(t)\psi(t)\mathrm{d}t = \infty \quad \text{for some } \alpha \in (0,1)\,. \tag{2.13}$$

Then (2.2) *has no positive solution.*

Proof. Assume, for the sake of contradiction , that (2.2) has a positive solution x on an interval of the form $[T, \infty) \subset (t_0, \infty)$. Let $z(t) = \Phi^{\alpha}(t)x(t)$ for $t \geq T$. Then, by (2.2), for $t \geq T$,

$$\begin{aligned} z'(t) &= -\Phi^{\alpha}(t)\psi(t) - \frac{\Phi^{\alpha}(t)}{\varphi(t)}[x(t) - \frac{\alpha}{2\Phi(t)}]^2 + \frac{\alpha^2\Phi^{\alpha-2}(t)}{4\varphi(t)} \\ &\leq -\Phi^{\alpha}(t)\psi(t) + \frac{\alpha^2\Phi^{\alpha-2}(t)}{4\varphi(t)}\,. \end{aligned}$$

Integrating this inequality and using (2.13), we lead to a contradiction. The proof of the lemma is complete. $\square$

Lemma 2.12. *Let* (2.3) *hold and let*

$$\int_{t_0}^{\infty} \psi(t)\mathrm{d}t < \infty, \quad \limsup_{t\to\infty} \int_{t_0}^{t} \big[\Phi(s)\psi(s) - \frac{1}{4\varphi(s)\Phi(s)}\big]\mathrm{d}s = \infty\,. \tag{2.14}$$

Then equation (2.2) *has no positive solution.*

Proof. Assume, for the sake of contradiction , that (2.2) has a positive solution x on an interval of the form $[T, \infty) \subset (t_0, \infty)$. Set $z(t) = \Phi(t)x(t) - \frac{1}{2}$ for $t \geq T$. Then, by (2.2), for $t \geq T$,

$$\begin{aligned} z'(t) &= -\frac{z^2(t)}{\varphi(t)\Phi(t)} - \Phi(t)\psi(t) + \frac{1}{4\varphi(t)\Phi(t)} \\ &\leq -\Phi(t)\psi(t) + \frac{1}{4\varphi(t)\Phi(t)} \end{aligned}$$

Integrating this inequality and using (2.14), we conclude that there is a $T_1 \geq T$ such that $z(t) < -1$ for $t \geq T_1$. This implies $x(t) < 0$ for $t \geq T_1$, which is a contradiction. The proof of the lemma is complete. $\square$

Lemma 2.13. *Let* (2.3) *hold,* $\sup\{t \in [t_0, \infty) : \psi(t) > 0\} = \infty$*, and let*

$$\int_{t_0}^{\infty} \psi(t)\mathrm{d}t < \infty, \quad \liminf_{t\to\infty} \frac{\int_t^{\infty} \frac{\Psi^2(s)}{\varphi(s)}\mathrm{d}s}{\Psi(t)} > \frac{1}{4}\,, \tag{2.15}$$

where $\Psi(t) = \int_t^{\infty} \psi(s)\mathrm{d}s$*. Then* (2.2) *has no positive solution.*

Proof. Let, for the sake of contradiction, (2.2) has a positive solution x on an interval of the form $[T, \infty) \subset (t_0, \infty)$. In view of (2.15), there exist an $\alpha > \frac{1}{4}$ and a $T_0 \geq T$ such that

$$\int_t^{\infty} \frac{\Psi^2(s)}{\varphi(s)}\mathrm{d}s \geq \alpha\Psi(t) \quad \text{for } t \geq T_0\,.$$

Using this inequality and (2.4), as in the proof of the Lemma 2.10, we get

$$x(t) \ge \beta_n \Psi(t) \quad \text{for } t \ge S$$

where $\beta_1 = 1$ and $\beta_{n+1} = \alpha\beta_n^2 + 1$ for $n = 1, 2, \dots$. It is easy to see that the sequence $\{\beta_n\}_{n=1}^{\infty}$ is nondecreasing and bounded, while $\lim_{n\to\infty} \beta_n = \beta$ satisfies the quadratic equation $\alpha\beta^2 - \beta + 1 = 0$. This implies that $1 - 4\alpha \ge 0$, which contradicts $\alpha > \frac{1}{4}$. The proof of the lemma is complete. □

3. Main results

Now we turn our attention to equation (1.1). Set

$$\begin{gathered} \varphi_1(t) = r(t), \psi_1(t) = a(t), \quad \Phi_1(t) = 1 + \int_{t_0}^{t} \frac{1}{\varphi_1(s)} \mathrm{d}s\,, \\ \varphi_{i+1}(t) = \varphi_i(t)\Phi_i(t), \quad \psi_{i+1}(t) = \Phi_i(t)\psi_i(t) - \frac{1}{4\varphi_i(t)\Phi_i(t)}, \\ \Phi_{i+1}(t) = 1 + \int_{t_0}^{t} \frac{1}{\varphi_{i+1}(s)} \mathrm{d}s \end{gathered} \tag{3.1}$$

for $t \ge t_0$ and $i = 1, 2, \dots$. Consider the Ricatti equation

$$x'(t) + \frac{x^2(t)}{\varphi_i(t)} + \psi_i(t) = 0\,. \tag{3.2}$$

In what follows we shall assume that for $i \in \{1, 2, \dots\}$ and $T > t_0$,

$$\psi_i(t) \ge 0 \quad \text{for } t \ge T. \tag{3.3}$$

First, we state the following very important result.

Theorem 3.1. *Let* (1.2) *and* (3.3) *hold. Then* (1.1) *has a positive solution if and only if equation* (3.2) *has a positive solution.*

Proof. Note that the functions $\varphi = \varphi_j$ and $\psi = \psi_j$ satisfy the conditions (2.3) for $j = 1, 2, \dots, i$. If $i = 1$, the proof of the lemma follows by applying Lemma 2.1 with $\varphi = \varphi_1$, $\psi = \psi_1$, while, if $i > 1$, then the proof follows by applying Lemma 2.6 with $\varphi = \varphi_{j-1}$, $\psi = \psi_{j-1}$, $\tilde{\varphi} = \varphi_j$, $\tilde{\psi} = \psi_j$ for $j = 2, 3, \dots, i$. □

Then, using Theorem 3.1 and the results of the previous section with $\varphi = \varphi_i$, $\psi = \psi_i$, we formulate the following statements.

Theorem 3.2. *Let* (1.2) *and* (3.3) *hold, and let*

$$\begin{gathered} \int_T^{\infty} \psi_i(t)\mathrm{d}t < \infty\,, \\ \int_t^{\infty} \frac{\Psi_i^2(s)}{\varphi_i(s)} \mathrm{d}s \le \frac{1}{4}\Psi_i(t) \quad \text{for } t \ge T \end{gathered} \tag{3.4}$$

where $\Psi_i(t) = \int_t^{\infty} \psi_i(s)\mathrm{d}s$ *for* $t > T$. *Then* (1.1) *is nonoscillatory.*

Theorem 3.3. *Let* (1.2) *and* (3.3) *hold, and let*

$$\begin{gathered} \int_T^{\infty} \psi_i(t)\mathrm{d}t < \infty\,, \\ \Phi_i(t) \int_t^{\infty} \psi_i(s)\mathrm{d}s \le \frac{1}{4} \quad \text{for } t \ge T\,. \end{gathered} \tag{3.5}$$

Then (1.1) *is nonoscillatory.*

Theorem 3.4. *Let* (1.2) *and* (3.3) *hold, and let*

$$\int_{t_0}^{\infty} \psi_i(t)\mathrm{d}t = \infty\,. \tag{3.6}$$

Then (1.1) *is oscillatory.*

Theorem 3.5. *Let* (1.2) *and* (3.3) *hold, and let*

$$\int_{t_0}^{\infty} \psi_i(t)\mathrm{d}t < \infty, \quad \liminf_{t\to\infty} \Phi_i(t) \int_{t}^{\infty} \psi_i(s)\mathrm{d}s > \frac{1}{4}\,. \tag{3.7}$$

Then (1.1) *is oscillatory.*

Theorem 3.6. *Let* (1.2) *and* (3.3) *hold, and let*

$$\int_{t_0}^{\infty} \psi_i(t)\mathrm{d}t < \infty, \quad \int_{t_0}^{\infty} \Phi_i^{\alpha}(t)\psi(t)\mathrm{d}t = \infty \quad \textit{for some } \alpha \in (0,1)\,. \tag{3.8}$$

Then (1.1) *is oscillatory.*

Theorem 3.7. *Let* (1.2) *and* (3.3) *hold, and let*

$$\int_{t_0}^{\infty} \psi_i(t)\mathrm{d}t < \infty, \quad \limsup_{t\to\infty} \int_{t_0}^{t} \Big[\Phi_i(s)\psi_i(s) - \frac{1}{4\varphi_i(s)\Phi_i(s)}\Big]\mathrm{d}s = \infty\,. \tag{3.9}$$

Then (1.1) *is oscillatory.*

Theorem 3.8. *Let* (1.2) *and* (3.3) *hold,* $\sup\{t \in [t_0,\infty) : \psi(t) > 0\} = \infty$*, and let*

$$\int_{t_0}^{\infty} \psi_i(t)\mathrm{d}t < \infty, \quad \liminf_{t\to\infty} \frac{\int_t^{\infty} \frac{\Psi_i^2(s)}{\varphi_i(s)}\mathrm{d}s}{\Psi_i(t)} > \frac{1}{4}\,, \tag{3.10}$$

where $\Psi_i(t) = \int_t^{\infty} \psi_i(s)\mathrm{d}s$*. Then* (1.1) *is oscillatory.*

Remark 3.9. The above results for $i = 1$ lead to Theorems 1.1, 1.2, and 1.3.

Example 3.10. To illustrate our results, consider (1.1) with $r(t) = 1$ and for $t \geq 1$,

$$a(t) = \frac{1}{4t^2} + \frac{1}{4t^2(1+\ln t)^2} + \frac{1}{4t^2(1+\ln t)^2(1+\ln(1+\ln t))}\,.$$

This equation satisfies the conditions (3.9) of the Theorem 3.7 for $i = 3$, but not for $i < 3$. On the other hand, it does not satisfy the conditions of Theorems 1.1-1.3. In a similar way we can construct examples when $i > 3$.

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