Electronic Journal of Differential Equations, Vol. 2004(2004), No. 33, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

INTEGRABILITY OF BLOW-UP SOLUTIONS TO SOME NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We investigate the integrability of solutions to the boundary blow-up problem

$$r^{-\lambda}\big(r^{\lambda}(u')^{p-1}\big)' = H(r,u), \quad u'(0) \geq 0, \quad u(R) = \infty$$

under some appropriate conditions on the non-linearity H.

1. Introduction

Let $\lambda \geq 0$, $p > 1$, $R > 0$. For $0 < r < R$ we consider solutions $u \in C^1([0,R))$ of the problem

$$\begin{gathered} r^{-\lambda}(r^{\lambda}|u'|^{p-2}u')' = H(r,u), \\ u(0) \geq 0, \quad u'(0) \geq 0, \quad \lim_{r\to R} u(r) = \infty. \end{gathered} \tag{1.1}$$

Here H satisfies the conditions

- (H1) $H : [0,R) \times [0,\infty) \to [0,\infty)$ is continuous,
- (H2) $H(\cdot, s)$ is non-decreasing,
- (H3) $H(0,s) > 0$ for all $s > 0$.

Further assumptions on H will be given as needed. In the literature, solutions of (1.1) are known as blow-up solutions, explosive solutions or large solutions.

These type of equations arise as radial solutions of the p-Laplace equation, as well as the Monge Ampére equation on balls. Radial solutions u of the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = J(|x|,u),$$

in the ball $B := B(0,R) \subseteq \mathbb{R}^N$ satisfy the first equation of (1.1) with $\lambda = N-1$, $H(r,u) = J(r,u)$. Likewise radial solutions of the Monge Ampére equation

$$\det(D^2u) = J(|x|,u),$$

in the ball B also satisfy the first equation of (1.1) with $\lambda = 0$, $p = N+1$ and $H(r,u) = Nr^{N-1}J(r,u)$.

1991 *Mathematics Subject Classification.* 34C11, 34B15, 35J65.
Key words and phrases. Blow-up solution, Keller-Osserman condition, integrability.

Submitted December 25, 2003. Published March 8, 2004.

Noting that u' is non-negative for any solution u of (1.1), we will find it convenient to rewrite equation (1.1) as

$$\begin{gathered}\big((u')^{p-1}\big)' + \frac{\lambda}{r}(u')^{p-1} = H(r,u),\\ u(0) \geq 0, \quad u'(0) \geq 0, \quad u(R) = \infty.\end{gathered} \tag{1.2}$$

A necessary and sufficient condition for the existence of a solution to problem (1.1) with $u'(0) = 0$, $H(r,u) = f(u)$, and $f(0) = 0$, is the (generalized) Keller-Osserman condition (see [5, 9, 8]).

$$\int_1^\infty \frac{ds}{F(s)^{1/p}} < \infty, \quad F(s) = \int_0^s f(t)dt. \tag{1.3}$$

If a nonnegative, non-decreasing continuous function F defined on $[0,\infty)$ satisfies the Keller-Osserman condition (1.3) for some $p > 1$, we will indicate this by writing $F \in KO(p)$.

When $H(r,s) = f(s)$, and $\lambda = N-1$, problem (1.1) has been studied extensively by several authors, (see [1, 2, 5, 6, 7, 8, 9] and the references therein). The questions of existence, uniqueness and asymptotic boundary estimates have received particular attention. The case when $p = 2$ and $H(r,s) = g(r)f(s)$ with $g \in C([0,R])$, possibly vanishing on a set of positive measure, has been considered in [6]. In all these cases, the Keller-Osserman condition on f remains the key condition for the existence of solutions. However, if g is allowed to be unbounded the situation is completely different and existence and boundary behavior of a blow-up solution depends on how fast g is allowed to grow near R. For such cases we refer the reader to [10] or [12]. For a discussion on solutions of (1.1) for general non-linearity H, we refer the reader to the paper [13].

In this paper we are interested in studying the integrability property of blow-up solutions to (1.1) for $F \in KO(p)$. A blow-up solution may not have any integrability property at all, as the following example, taken from [11], shows.

Example 1.1. Let $u(r) = -1 + e^{(1-r)^{-1}}$. Then

$$\begin{gathered}u''(r) = f(u), \quad 0 < r < 1,\\ u'(0) \geq 0, \quad u(1) = \infty,\end{gathered}$$

where $f(s) = (s+1)[\log^4(s+1) + 2\log^3(s+1)],\ s \geq 0$. Notice that $u \notin L^\gamma(0,1)$ for any $\gamma > 0$. The antiderivative F of f that vanishes at zero is given by $F(s) = ((s+1)^2 \log^4(s+1))/2$, and observe that $F \in KO(2)$, but $F \notin KO(\alpha)$ for any $\alpha > 2$.

On the other extreme any positive power of a blow-up solution could be integrable. This can be seen from the following example.

Example 1.2. We fix $0 < R < 1/2$ and let

$$f(s) = e^s - 1, \quad s \in [0,\infty), \quad \text{and} \quad g(r) = \frac{1}{(r-R+1)(R-r)}, \quad r \in [0,R)$$

Then $u(r) = -\log(R-r)$ is a solution of

$$\begin{gathered}u''(r) = g(r)f(u), \quad 0 < r < R,\\ u'(0) \geq 0, \quad u(r) \to \infty \quad \text{as } r \to R.\end{gathered}$$

Note that $u \in L^\gamma(0,R)$ for all $\gamma > 0$. In this example the primitive F of f with $F(0) = 0$ satisfies $F \in KO(\alpha)$ for all $\alpha > 0$.

The outline of the paper is as follows. In Section 2 we compare solutions u of (1.2) with solutions of

$$\begin{gathered} ((w')^{p-1})' + \frac{\lambda}{r}(w')^{p-1} = H(0,w), \\ w(0) \geq 0, \quad w'(0) = 0, \quad w(R) = \infty, \end{gathered} \tag{1.4}$$

for $0 < r < R$.

The main result of Section 2, Theorem 2.4, is used in Section 3 to prove the following integrability result for solutions of (1.2).

Theorem 1.3. *Suppose in addition to (H1)–(H3), $H(r,\cdot)$ is non-decreasing on $[0,R)$ and for $f(s) = H(0,s)$, $f(0) = 0$ and $F \in KO(\alpha)$ for some $\alpha > p$. Then $u \in L^{(\alpha-p)/p}(0,R)$ for any solution u of (1.2).*

In Section 3, we also show that for $H(r,s) = g(r)f(s)$, the following result holds.

Theorem 1.4. *Let $H(r,s) = g(r)f(s)$ satisfy (H1)–(H3), with $g(0) > 0$ and g positive, non-decreasing near R. Suppose (1.2) has a blow-up solution u such that $u \in L^{(\alpha-p)/p}(0,R)$ for some $\alpha > p$. If $g \in L^{1/\sigma}(0,R)$ with $0 < \sigma < p(\alpha-p)/\alpha$, then $F \in KO(\gamma)$ for some $p < \gamma < \alpha$.*

Remark 1.5. When $H(r,s) = g(r)f(s)$, (H3) and the requirement that $g(0) > 0$ imply that $f(s) > 0$ for $s > 0$. Since $f(s) > 0$, it follows from (H1) that g is non-negative on $[0,R)$.

Finally, we give some corollaries to Theorem 1.4.

2. A Comparison Result

We will need the following comparison lemma (see [13] for a proof). For notational convenience in stating the lemma and in this section, we let L denote the differential operator on the left hand side of equation (1.1) above. In this lemma, we use the following notation: $u(a+) < w(a+)$ means there exists $\epsilon > 0$ such that $u < w$ in $(a, a+\epsilon)$.

Lemma 2.1. *Let $0 \leq a < b$, and suppose $u, w \in C^1([a,b])$ with $(u')^{p-1}, (w')^{p-1} \in C^1((a,b])$ satisfy*

$$\begin{gathered} Lu - G(r,u) \leq Lw - G(r,w) \quad \text{in } (a,b] \\ u(a+) < w(a+), \quad u'(a) \leq w'(a) \end{gathered}$$

for some function $G(r,s)$ which is non-decreasing in the second variable s. Then $u' \leq w'$ in $[a,b]$, which implies $u < w$ in $(a,b]$.

Another result we will need is the following, which is a consequence of Lemma 2.1 in [4] via L'Hôpital's Rule.

Lemma 2.2. *If $F \in KO(\alpha)$ for some $\alpha > 1$, then*

$$\lim_{t\to\infty} \frac{t^\alpha}{F(t)} = 0.$$

We need the following lemma, which shows that solutions of (1.2) with initial slope zero have non-decreasing slope for $r \in [0,R)$.

Lemma 2.3. *Suppose in addition to (H1)–(H3), $H(r,\cdot)$ is non-decreasing on $[0,R)$. If for $0<r<R$, w is a solution of*

$$((w')^{p-1})' + \frac{\lambda}{r}(w')^{p-1} = H(r,w), \quad w(0)\ge 0, \quad w'(0)=0, \quad w(R)=\infty, \tag{2.1}$$

then w' is non-decreasing on $[0,R)$.

Proof. Let w be a solution of (2.1). Integrating the equation $(r^\lambda (w')^{p-1})' = r^\lambda H(r,w)$ over the interval $(0,r)$ for any $r\in(0,R)$ and recalling that w' is non-negative, we obtain

$$\begin{aligned}(w')^{p-1} &= r^{-\lambda}\int_0^r s^\lambda H(s,w(s))\,ds\\ &\le r^{-\lambda} H(r,w(r))\int_0^r s^\lambda\,ds\\ &= \frac{r}{\lambda+1}H(r,w)\end{aligned}$$

Using this inequality back in the equation (2.1) we obtain

$$\begin{aligned}H(r,w) &= ((w')^{p-1})' + \frac{\lambda}{r}(w')^{p-1}\\ &\le ((w')^{p-1})' + \frac{\lambda}{r}\cdot\frac{r}{\lambda+1}H(r,w)\end{aligned}$$

so that

$$((w')^{p-1})' \ge \frac{1}{\lambda+1}H(r,w), \quad 0<r<R. \tag{2.2}$$

The fact that w' is non-decreasing on $(0,R)$ is a consequence of (2.2) as follows. Let $0<r_1<r_2<R$. Integrating (2.2) on (r_1,r_2) leads to

$$(w'(r_2))^{p-1} - (w'(r_1))^{p-1} \ge \frac{1}{\lambda+1}\int_{r_1}^{r_2} H(s,w(s))\,ds \ge 0.$$

□

We are now ready to state and prove the main result of this section.

Theorem 2.4. *Suppose in addition to (H1)–(H3), $H(r,\cdot)$ is non-decreasing on $[0,R)$ and for $f(s)=H(0,s)$, $f(0)=0$ and $F\in KO(p)$. Then there is a solution w of (1.4) such that for any solution u of (1.2),*

$$u(r)\le w(r), \quad 0\le r<R\,.$$

Proof. For each positive integer k, with $1/k<R$, let w_k be a solution, in $(0,R-1/k)$, of the problem

$$\begin{gathered}((w')^{p-1})' + \frac{\lambda}{r}(w')^{p-1} = H(0,w),\\ w(0)\ge 0, \quad w'(0)=0, \quad w(R-1/k)=\infty.\end{gathered} \tag{2.3}$$

This is possible, since $f(s)=H(0,s)$ satisfies the Keller-Osserman condition.

Since $H(0,u)\le H(r,u)$ for all $0\le r<R$, we first note that

$$Lw_k - H(0,w_k) \le Lu - H(0,u) \quad \text{on } (0,R-1/k).$$

Suppose that $w_k(0)<u(0)$. Then, since $0=w_k'(0)\le u'(0)$, by Lemma 2.1 we conclude that $w_k<u$ on $(0,R-1/k)$. But this is obviously not possible since w_k

blows up at $R-1/k$ and u does not. Thus we must have $u(0) \le w_k(0)$. Actually, we claim that

$$u(r) \le w_k(r), \quad \text{for all } r \text{ with } 0 \le r < R - \frac{1}{k}.$$

Suppose to the contrary that $u(r) > w_k(r)$ for some $0 < r < R - 1/k$. Since $u(0) \le w_k(0)$ the function $u - w_k$ takes on a positive maximum inside $[0, r_1]$ where r_1 is taken sufficiently close to $R - 1/k$. If r^* is such a maximum point, then we have

$$w_k(r^*) < u(r^*), \quad \text{and} \quad w_k'(r^*) = u'(r^*)\,.$$

By the comparison Lemma 2.1 we conclude that $w_k < u$ on $(r^*, R-1/k)$, which is impossible. Thus we must have $u(r) \le w_k(r)$, $r \in (0, R-1/k)$, as claimed.

By a similar argument as above, and using w_{k+1} instead of u, we also conclude that

$$w_{k+1}(r) \le w_k(r), \quad 0 \le r < R - \frac{1}{k}\,.$$

Using this and the fact that w_k and w_{k+1} satisfy equation (2.3) we obtain

$$\begin{aligned} (w_{k+1}'(r))^{p-1} &= r^{-\lambda} \int_0^r s^\lambda H(0, w_{k+1}(s))\, ds \\ &\le r^{-\lambda} \int_0^r s^\lambda H(0, w_k(s))\, ds \\ &= (w_k'(r))^{p-1}, \quad 0 < r < R - 1/k. \end{aligned}$$

This shows that $w_{k+1}'(r) \le w_k'(r)$, $0 \le r < R - 1/k$. Therefore, we have

$$w_n'(r) \le w_m'(r), \quad 0 \le r < R - 1/m, \tag{2.4}$$

whenever $n \ge m > 1/R$.

For $t, r \in (0, R-1/k)$, and $n > k$ we have

$$|w_n(r) - w_n(t)| = \Big| \int_t^r w_n'(s)\, ds \Big| \le w_n'(\zeta)|r-t| \le w_{k+1}'(R-1/k)|r-t|,$$

where $\zeta = \max\{r, t\}$. The fact that w_{k+1}' is non-decreasing, by Lemma 2.3, has been exploited in the last inequality.

Thus $\{w_n\}_{n=k+1}^\infty$ is a bounded equicontinuous family in $C([0, R-1/k])$, and hence has a uniformly convergent subsequence. Let w be the limit. For $r \in [0, R-1/k]$ and $n > k$ the solution w_n satisfies the integral equation

$$w_n(r) = w_n(0) + \int_0^r \Big(\int_0^t \Big(\frac{s}{t}\Big)^\lambda H(0, w_n(s))\, ds \Big)^{1/(p-1)} dt\,.$$

Letting $n \to \infty$ we see that w satisfies the same integral equation. Since k is arbitrary we conclude that w satisfies equation (1.4). Since $u \le w_n$ on $(0, R-1/k)$ for each $n \ge k$ we conclude that $u \le w$ on $(0, R)$. □

3. Proofs of Main Results and Some Corollaries

Proof of Theorem 1.3. By Theorem 2.4 we take a solution w of (1.4) such that $u(r) \le w(r)$ for $0 \le r < R$. Using $f(w) := H(0, w)$ in place of $H(r, w)$ in inequality (2.2), we note that w satisfies

$$((w')^{p-1})' > \frac{1}{\lambda+1} f(w), \quad 0 < r < R.$$

Multiplying both sides of the above inequality by w' and integrating on $(0, r)$, we find that for r close to R,

$$\begin{aligned}\frac{p-1}{p}(w'(r))^p &\geq \frac{1}{\lambda+1}[F(w(r)) - F(w(0))] \\ &= \frac{1}{\lambda+1}F(w(r))\big[1 - \frac{F(w(0))}{F(w(r))}\big].\end{aligned}$$

Thus, for some positive constants C and τ, which may change in each line below, but depend only on the constants λ and the primitive F, we see that

$$\frac{p-1}{p}(w'(r))^p \geq CF(w(r)), \quad \tau < r < R,$$

or

$$F(w(r))^{1/p} \leq Cw'(r), \quad \tau < r < R. \tag{3.1}$$

From Lemma 2.2, it follows that

$$w(r) \leq F(w(r))^{\frac{1}{\alpha}}, \quad \tau < r < R. \tag{3.2}$$

Using (3.1) and (3.2), we obtain

$$\begin{aligned}\int_\tau^R w(r)^{\frac{\alpha-p}{p}}\,dr &\leq \int_\tau^R F(w(r))^{\frac{1}{p}-\frac{1}{\alpha}}\,dr \\ &\leq C\int_\tau^R \frac{w'(r)}{F(w(r))^{\frac{1}{\alpha}}}\,dr \\ &= C\int_{w(\tau)}^\infty \frac{1}{F(t)^{\frac{1}{\alpha}}}\,dt < \infty\end{aligned}$$

Thus, recalling that $u \leq w$ on $(0, R)$ we get

$$\int_\tau^R u(r)^{\frac{\alpha-p}{p}}\,dr \leq C\int_{w(\tau)}^\infty \frac{1}{F(t)^{\frac{1}{\alpha}}}\,dt < \infty,$$

giving the desired result. □

Proof of Theorem 1.4. Suppose that g is positive and non-decreasing on (r_*, R) for some $0 < r_* < R$. Observe that from (1.2) we obtain

$$((u')^{p-1})' \leq g(r)f(u),$$

and multiplying both sides of this by u' and integrating on (r_*, r) we find that

$$\frac{u'}{F(u)^{1/p}} \leq (qg(r))^{1/p}\big[1 + \frac{u'(r_*)^p}{qg(r)F(u(r))}\big]^{1/p}, \quad r_* < r < R,$$

where q is the Hölder conjugate exponent of p. From this we conclude that, for some positive constants C and r_0,

$$\frac{u'}{F(u)^{1/p}} \leq Cg(r)^{1/p}, \quad r_0 < r < R. \tag{3.3}$$

Let $\gamma = \alpha(1-\sigma/p)$. The hypothesis $0 < \sigma < p(\alpha-p)/\alpha$ implies that $p < \gamma < \alpha$. Using (3.3) and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{u(r_0)}^{\infty} \frac{1}{F(t)^{1/\gamma}}\,dt \\
&= \int_{u(r_0)}^{\infty} \frac{t^{(\alpha-p)/\alpha}}{F(t)^{1/\gamma}} \cdot \frac{1}{t^{(\alpha-p)/\alpha}}\,dt \\
&\le \Big(\int_{u(r_0)}^{\infty} \frac{t^{[(\alpha-p)/\alpha]\cdot[\gamma/p]}}{F(t)^{1/p}}\,dt\Big)^{p/\gamma} \Big(\int_{u(r_0)}^{\infty} \frac{1}{t^{[(\alpha-p)/\alpha]\cdot[\gamma/(\gamma-p)]}}\,dt\Big)^{(\gamma-p)/\gamma} \\
&\le C\Big(\frac{\alpha}{p}\cdot\frac{\gamma-p}{\alpha-\gamma}\Big)^{(\gamma-p)/\gamma} \Big(\int_{r_0}^{R} \frac{u'(r)u(r)^{[(\alpha-p)/p]\cdot[\gamma/\alpha]}}{F(u(r))^{1/p}}\,dr\Big)^{p/\gamma} \\
&\le C\Big(\frac{\alpha}{p}\cdot\frac{\gamma-p}{\alpha-\gamma}\Big)^{(\gamma-p)/\gamma} \Big(\int_{0}^{R} g(r)^{1/p} u(r)^{[(\alpha-p)/p]\cdot[\gamma/\alpha]}\,dr\Big)^{p/\gamma} \\
&\le C\Big(\frac{\alpha}{p}\cdot\frac{\gamma-p}{\alpha-\gamma}\Big)^{(\gamma-p)/\gamma} \Big(\int_{0}^{R} g(r)^{1/p\cdot\alpha/(\alpha-\gamma)}\,dr\Big)^{\frac{p}{\gamma}\cdot\frac{\alpha-\gamma}{\alpha}} \Big(\int_{0}^{R} u(r)^{(\alpha-p)/p}\,dr\Big)^{p/\alpha}.
\end{aligned}$$

Recalling that $1/p\cdot\alpha/(\alpha-\gamma) = 1/\sigma$, by hypothesis the right hand side of the last inequality is finite and this proves the claim. □

Note that if g is bounded on $[0,R)$, but not necessarily non-decreasing near R, the right hand side of (3.3) can be replaced by a constant. The proof of Theorem 1.4 shows that $F \in KO(\gamma)$ for any $0 < \gamma < \alpha$. We record this as follows.

Corollary 3.1. *Let $H(r,s) = g(r)f(s)$ satisfy (H1)–(H3), with $g(0) > 0$. Suppose (1.2) has a blow-up solution that belongs to $L^{(\alpha-p)/p}(0,R)$ for some $\alpha > p$. If g is bounded, then $F \in KO(\gamma)$ for any $0 < \gamma < \alpha$.*

Remark 3.2. The conclusion of Corollary 3.1 is false when g is unbounded near R as the following example shows.

The function $u(r) = (1-r)^{-1}$ is a solution of

$$\begin{gathered}
u''(r) = g(r)f(u), \\
u(0) \ge 0, \quad u'(0) \ge 0, \quad u(1) = \infty,
\end{gathered}$$

where

$$g(r) := 2/(1-r), \quad \text{and} \quad f(s) := s^2.$$

Observe that $u \in L^{(\alpha-2)/2}(0,1)$ for $2 < \alpha < 4$. However note that $F \notin KO(3)$.

Corollary 3.3. *Suppose $H(r,s) = g(r)f(s)$ satisfies (H1)–(H3), with $g(0) > 0$, g non-decreasing on $[0,R)$, and $f(0) = 0$. Further, let g be bounded on $[0,R)$, and let $F \in KO(p)$. Then a blow up solution u of (1.1) belongs to $L^q(0,R)$ for some $q > 0$ if and only if $F \in KO(\gamma)$ for some $\gamma > p$.*

Proof. Suppose $F \in KO(\gamma)$ for some $\gamma > p$. Then by Theorem 1.3, we see that $u \in L^q(0,R)$ for $q = (\gamma-p)/p$. For the converse, suppose that $u \in L^q(0,R)$ for some $q > 0$. Then for $\alpha = p(q+1)$ we see that $q = (\alpha-p)/p$ so that by the above corollary, $F \in KO(\gamma)$ for some $p < \gamma < p(q+1)$. □

References

[1] C. Bandle, M. Marcus, *Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour*, J. d'Anal. Math., **58** (1992), 9-24.

[2] C. Bandle, M. Marcus, *Asymptotic behavour of solutions and their derivatives for semilinear elliptic problems with blow-up on the boundary*, Ann. Inst. H. Poincare **12**, (1995), 155-171.

[3] C. Bandle, Y. Cheng, G. Porru, *Boundary blow-up in semilinear elliptic problems with singular weights at the boundary*, Institut Mitag-Leffler, Report No. 39, (1999/2000),1-14.

[4] F. Gladiali, G. Porru, *Estimates for explosive solutions to p-Laplace equations*, Progress in Partial Differential Equations, (Pont-á-Mousson 1997), Vol. 1, Pitman Res. Notes Math. Series, Longman **383** (1998), 117-127.

[5] J. B. Keller, *On solutions of* $\Delta u = f(u)$, Comm. Pure Appl. Math., **10** (1957), 503-510.

[6] A. V. Lair, *A Necessary and Sufficient Condition for Existence of Large Solutions to semilinear Elliptic Equations*, J. Math. Anal. Appl. **240** (1999),205-218.

[7] A.C. Lazer, P.J. McKenna, *Asymptotic behaviour of solutions of boundary blow up problems*, Differential and Integral Equations, **7** (1994), 1001-1019.

[8] J. Matero, *Quasilinear Elliptic Equations with Boundary Blow-up*, J. d'Anal. Math., **69** (1996) 229-247

[9] R. Osserman, *On the inequality* $\Delta u \geq f(u)$, Pacific J. Math. **7** (1957), 1641-1647.

[10] A. Mohammed, *Boundary behavior of blow-up solutions to some weighted non-linear differential equations*, Electron. J. Diff. Eqns., **2002** (2002), No. 78, 1-15.

[11] A. Mohammed, G. Porru, *Boundary Behaviour and Integrability of Large solutions to p-Laplace equations*, Mat. Contemp. **19** (2000), 31-40

[12] A. Mohammed, G. Porcu, G. Porru, *Large solutions to some non-linear O.D.E. with singular coefficients*, Non-linear Analysis, TMA. **47(1)** (2001) 513-524

[13] W. Reichel, W. Walter, *Radial solutions of equations and inequalities involving the p-Laplacian*, J. Ineq. Appl. **1** (1997), 47-71

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