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NONLINEAR TRIPLE-POINT PROBLEMS ON TIME SCALES

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Abstract. We establish the existence of multiple positive solutions to the nonlinear second-order triple-point boundary-value problem on time scales,

$$u^{\Delta\nabla}(t) + h(t)f(t, u(t)) = 0,$$
$$u(a) = \alpha u(b) + \delta u^{\Delta}(a), \quad \beta u(c) + \gamma u^{\Delta}(c) = 0$$

for $t \in [a, c] \subset \mathbb{T}$, where $\mathbb{T}$ is a time scale, $\beta, \gamma, \delta \geq 0$ with $\beta + \gamma > 0$, $0 < \alpha < \frac{c-a}{c-b}$ and $b \in (a, c) \subset \mathbb{T}$.

1. Introduction to the boundary-value problem

We are concerned with proving the existence of multiple positive solutions to the second-order triple-point nonlinear boundary-value problem on a time scale $\mathbb{T}$ given by the time-scale dynamic equation

$$u^{\Delta\nabla}(t) + h(t)f(t, u(t)) = 0, \quad t \in (a, c) \subset \mathbb{T} \tag{1.1}$$

with boundary conditions

$$u(a) = \alpha u(b) + \delta u^{\Delta}(a), \quad \beta u(c) + \gamma u^{\Delta}(c) = 0, \tag{1.2}$$

where $\beta, \gamma, \delta \geq 0$ with $\beta+\gamma > 0$, $0 < \alpha < \frac{c-a}{c-b}$ and $b \in (a, c) \subset \mathbb{T}$ for $a \in \mathbb{T}_{\kappa}$, $c \in \mathbb{T}^{\kappa}$. The function $h \in C_{ld}[a, c]$ is nonnegative with $h(t_0) > 0$ for at least one $t_0 \in (a, b]$, and the nonlinearity $f : [a, c] \times [0, \infty) \to [0, \infty)$ is continuous such that $f(t, \cdot) > 0$ on any subset of $\mathbb{T}$ containing t_0. This problem is related to that first studied in the case $\mathbb{T} = \mathbb{R}$ on the unit interval by He and Ge [14] and Ma [19, 20, 21],

$$u'' + f(t, u) = 0, \quad t \in (0, 1) \quad u(0) = 0, \quad \alpha u(\eta) = u(1) \tag{1.3}$$

where $0 < \eta < 1$ and $0 < \alpha < 1/\eta$. The boundary-value problem (1.3) has since been extended to general time scales in Anderson [2] and Kaufmann [17] as

$$u^{\Delta\nabla}(t) + f(t, u(t)) = 0, \quad u(0) = 0, \ \alpha u(\eta) = u(T),$$

and in a slightly different way by Sun and Li [22]

$$u^{\Delta\nabla}(t) + a(t)f(t, u(t)) = 0, \quad u^{\Delta}(0) = 0, \ \alpha u(\eta) = u(T).$$

In this paper there is a nexus between the boundary conditions at a and b instead of at $b = \eta$ and $c = T$, and we add the $\delta \geq 0$ term as well. Consequently these results

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are new for differential and difference equations as well as for dynamic equations on general time scales.

For more on time scales and positive solutions, see the books by Bohner and Peterson [8, 9] and the following articles: [1, 3, 4, 5, 6, 10, 11, 12, 15, 16].

2. Preliminary results

To prove the main existence results we will employ several straightforward lemmas. These lemmas are based on the linear boundary-value problem

$$u^{\Delta\nabla}(t) + y(t) = 0 \quad t \in (a, c) \subset \mathbb{T} \tag{2.1}$$

$$u(a) = \alpha u(b) + \delta u^{\Delta}(a), \quad \beta u(c) + \gamma u^{\Delta}(c) = 0. \tag{2.2}$$

Lemma 2.1. *Let*

$$d := \gamma(1-\alpha) + \beta\left[(c-a) - \alpha(c-b) + \delta\right]. \tag{2.3}$$

If $d \neq 0$, then for $y \in C_{ld}[a, c]$ the boundary-value problem (2.1), (2.2) *has the unique solution*

$$\begin{aligned} u(t) = &\frac{1}{d}\left(\gamma + \beta(c-t)\right)\Big[\int_a^c (s-a+\delta)y(s)\nabla s \\ &- \alpha\int_b^c (s-b)y(s)\nabla s\Big] - \int_t^c (s-t)y(s)\nabla s. \end{aligned} \tag{2.4}$$

Proof. Let u be as in (2.4). Then the delta derivative of u is given by

$$u^{\Delta}(t) = -\frac{\beta}{d}\left[\int_a^c (s-a+\delta)y(s)\nabla s - \alpha\int_b^c (s-b)y(s)\nabla s\right] + \int_t^c y(s)\nabla s$$

and

$$u^{\Delta\nabla}(t) = -y(t),$$

so that u given in (2.4) is a solution of (2.1). It is routine to check that the boundary conditions (2.2) are met by u in (2.4) as well. □

Lemma 2.2. *If $d > 0$ and $y \in C_{ld}[a, c]$ with $y \geq 0$, the unique solution u of* (2.1), (2.2) *given in* (2.4) *satisfies*

$$u(t) \geq 0, \quad t \in [a, c] \subset \mathbb{T}.$$

Proof. From the fact that $u^{\Delta\nabla}(t) = -y(t) \leq 0$, we know that if $u(a) \geq 0$ and $u(c) \geq 0$, then $u(t) \geq 0$ for $t \in [a, c]$. For $0 < \alpha \leq 1$,

$$\begin{aligned} u(c) &= \frac{\gamma}{d}\left[\int_a^c (s-a+\delta)y(s)\nabla s - \alpha\int_b^c (s-b)y(s)\nabla s\right] \\ &\geq \frac{\gamma}{d}\left[(1-\alpha)\int_b^c (s-b)y(s)\nabla s + \delta\int_a^c y(s)\nabla s\right] \\ &\geq 0. \end{aligned}$$

For $1 < \alpha < \frac{c-a}{c-b}$,

$$\begin{aligned} u(c) &= \frac{\gamma}{d}\left[\int_a^b (s-a+\delta)y(s)\nabla s + \int_b^c (s-a-\alpha(s-b)+\delta)y(s)\nabla s\right] \\ &\geq \frac{\gamma}{d}\left[\int_a^b (s-a+\delta)y(s)\nabla s + \int_b^c (c-a-\alpha(c-b)+\delta)y(s)\nabla s\right] \\ &\geq 0 \end{aligned}$$

since $\alpha < (c-a)/(c-b)$. Finally,

$$\begin{aligned} u(a) &= \frac{1}{d}\left(\gamma+\beta(c-a)\right)\Big[\int_a^c (s-a+\delta)y(s)\nabla s - \alpha\int_b^c (s-b)y(s)\nabla s\Big] - \frac{d}{d}\int_a^c (s-a)y(s)\nabla s \\ &= \frac{\alpha}{d}(\gamma+\beta(c-b))\int_a^b (s-a)y(s)\nabla s + \frac{\alpha}{d}(b-a)\int_b^c (\gamma+\beta(c-s))y(s)\nabla s \\ &\quad + \frac{\delta}{d}\int_a^c (\gamma+\beta(c-s))y(s)\nabla s \\ &\geq 0. \end{aligned}$$

□

Lemma 2.3. *Let $d > 0$. If $y \in C_{ld}[a,c]$ with y nonnegative but nontrivial, then the unique solution u as in (2.4) of (2.1), (2.2) satisfies*

$$\inf_{t\in[a,b]} u(t) \geq k\|u\|, \quad \|u\| := \sup_{t\in[a,c]} |u(t)|, \tag{2.5}$$

where

$$k := \min\left\{\frac{\alpha(c-b)}{c-a}, \frac{c-b}{c-a}, \frac{\alpha(b-a)}{c-a-\alpha(c-b)}\right\} \in (0,1). \tag{2.6}$$

Proof. Note that $u^{\Delta\nabla}(t) = -y(t) \leq 0$ for all $t \in (a,c)$, so that

$$\min_{t\in[a,b]} u(t) = \min\{u(a), u(b)\}.$$

Then for any $\tau \in [a,c)$,

$$\eta(t) := u(t) - \big(\frac{c-t}{c-\tau}\big)u(\tau)$$

satisfies $\eta(\tau) = 0$, $\eta(c) = u(c) \geq 0$, and $\eta^{\Delta\nabla}(t) = u^{\Delta\nabla}(t) \leq 0$ on $[\tau, c)$. In particular,

$$\frac{u(t)}{c-t} \geq \frac{u(\tau)}{c-\tau}$$

for all $t \in [\tau, c)$. Now fix $\tau \in [a,c)$ such that $u(\tau) = \|u\|$. If $a \leq \tau \leq b$ and $u(a) \leq u(b)$, then

$$\frac{\alpha u(b)}{c-b} \geq \frac{\alpha u(\tau)}{c-\tau} \geq \frac{\alpha u(\tau)}{c-a},$$

rewritten with the boundary condition at b as

$$u(a) - \delta u^{\Delta}(a) \geq \big(\frac{c-b}{c-a}\big)\alpha\|u\|.$$

Therefore,

$$\min_{t\in[a,b]} u(t) = u(a) \geq \big(\frac{c-b}{c-a}\big)\alpha\|u\|,$$

since $u^\Delta(a) \geq 0$ in this case. If $a \leq \tau \leq b$ and $u(b) \leq u(a)$, then

$$u(b) \geq \big(\frac{c-b}{c-\tau}\big)u(\tau) \geq \big(\frac{c-b}{c-a}\big)u(\tau),$$

so that

$$\min_{t\in[a,b]} u(t) = u(b) \geq \big(\frac{c-b}{c-a}\big)\|u\|.$$

If $b < \tau < c$, then $u(a) = \min_{t\in[a,b]} u(t)$, and by the concavity of u and a secant line we have

$$\begin{aligned} u(\tau) &\leq u(a) + \Big(\frac{u(b)-u(a)}{b-a}\Big)(c-a) \\ &= \frac{[c-a-\alpha(c-b)]\,u(a) - \delta(c-a)u^\Delta(a)}{\alpha(b-a)} \\ &\leq \frac{[c-a-\alpha(c-b)]\,u(a)}{\alpha(b-a)}, \end{aligned}$$

since again $u^\Delta(a) \geq 0$. Consequently,

$$\min_{t\in[a,b]} u(t) = u(a) \geq \frac{\alpha(b-a)}{c-a-\alpha(c-b)}\|u\|.$$

□

3. Existence of at Least Two Positive Solutions

We apply the Avery-Henderson Fixed Point Theorem [7] to prove the existence of at least two positive solutions to the nonlinear boundary-value problem (1.1), (1.2), where $h \in C_{ld}[a,c]$ is nonnegative with $h(t_0) > 0$ for at least one $t_0 \in (a,b]$,and the nonlinearity $f : [a,c]\times[0,\infty) \to [0,\infty)$ is continuous such that $f(t,\cdot) > 0$ on any subset of $\mathbb{T}$ containing t_0. The solutions are the fixed points of the operator A defined by

$$\begin{aligned} Au(t) = \frac{1}{d}\,(\gamma+\beta(c-t))\,\Big[\int_a^c (s-a+\delta)h(s)f(s,u(s))\nabla s \\ - \alpha\int_b^c (s-b)h(s)f(s,u(s))\nabla s\Big] - \int_t^c (s-t)h(s)f(s,u(s))\nabla s \end{aligned}$$

by Lemma 2.1. Notationally, the cone P has subsets of the form $P(\phi,r) := \{u \in P : \phi(u) < r\}$ for a given functional ϕ.

Theorem 3.1. [7] *Let P be a cone in a real Banach space $\mathcal{S}$. If η and ϕ are increasing, nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive constants r and M,*

$$\phi(u) \leq \theta(u) \leq \eta(u) \quad \textit{and} \quad ||u|| \leq M\phi(u)$$

for all $u \in \overline{P(\phi,r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \leq \lambda\theta(u), \qquad \textit{for all } 0 \leq \lambda \leq 1 \textit{ and } u \in \partial P(\theta,q).$$

If $A : \overline{P(\phi,r)} \to P$ is a completely continuous operator satisfying

(i) *$\phi(Au) > r$ for all $u \in \partial P(\phi,r)$,*
(ii) *$\theta(Au) < q$ for all $u \in \partial P(\theta,q)$,*
(iii) *$P(\eta,p) \neq \emptyset$ and $\eta(Au) > p$ for all $u \in \partial P(\eta,p)$,*

then A has at least two fixed points u_1 and u_2 such that

$$p < \eta(u_1) \text{ with } \theta(u_1) < q \quad \text{and} \quad q < \theta(u_2) \text{ with } \phi(u_2) < r.$$

Let $\mathcal{S}$ denote the Banach space $C[\rho(a), \sigma(c)]$ with the supremum norm. Define the cone $P \subset \mathcal{S}$ by

$$P = \{u \in \mathcal{S} : u(t) \geq 0, u \text{ is concave, and } \min_{t\in[a,b]} u(t) \geq k\|u\|\}, \tag{3.1}$$

where k is given in (2.6). Finally, let the nonnegative, increasing, continuous functionals ϕ, θ, and η be defined on the cone P by

$$\phi(u) := \min_{t\in[a,b]} u(t), \quad \theta(u) := \max_{t\in[b,c]} u(t), \quad \eta(u) := \max_{t\in[a,c]} u(t).$$

Observe that, for each $u \in P$, the concavity of u implies

$$\phi(u) \leq \theta(u) \leq \eta(u), \tag{3.2}$$

$$\|u\| =\leq \frac{1}{k} \min_{t\in[a,b]} u(t) = \frac{1}{k}\phi(u) \leq \frac{1}{k}\theta(u) \leq \frac{1}{k}\eta(u). \tag{3.3}$$

Theorem 3.2. *Let $d > 0$ and k be as in* (2.6). *Suppose there exist positive numbers $0 < p < q < r$ such that the function f satisfies the following conditions:*

(i) *$f(s, u) > pM$ for $s \in [a, b]$ and $u \in [kp, p]$,*
(ii) *$f(s, u) < qm$ for $s \in [a, c]$ and $u \in [0, q/k]$,*
(iii) *$f(s, u) > rM$ for $s \in [a, b]$ and $u \in [r, r/k]$*

for some positive constants m and M. Then the second-order boundary-value problem (1.1), (1.2), *has at least two positive solutions u_1 and u_2 such that*

$$p < \max_{t\in[a,c]} u_1(t) \quad \text{with} \quad \max_{t\in[b,c]} u_1(t) < q,$$
$$q < \max_{t\in[b,c]} u_2(t) \quad \text{with} \quad \min_{t\in[a,b]} u_2(t) < r.$$

Proof. For $u \in P$, $u(t) \geq k\|u\|$ for all $t \in [a, b]$. By Lemma 2.3, $A(P) \subset P$. Standard arguments show that $A : P \to P$ is completely continuous. For any $u \in P$, (3.2) and (3.3) imply

$$\phi(u) \leq \theta(u) \leq \eta(u), \quad \|u\| \leq \frac{1}{k}\phi(u).$$

It is clear that $\theta(0) = 0$, and for all $u \in P$, $\lambda \in [0, 1]$ we have

$$\theta(\lambda u) = \max_{t\in[b,c]} (\lambda u)(t) = \lambda \max_{t\in[b,c]} u(t) = \lambda\theta(u).$$

Since $0 \in P$ and $p > 0$, $P(\eta, p) \neq \emptyset$.

In the following claims, we verify the remaining conditions of Theorem 3.1.

Claim 1: If $u \in \partial P(\eta, p)$, then $\eta(Au) > p$: Since $u \in \partial P(\eta, p)$, $kp \leq u(t) \leq \|u\| = p$ for $t \in [a, b]$. Define

$$M := d\Big(\min\{\alpha, 1\}(\gamma + \beta(c - b)) \int_a^b (s - a + \delta)h(s)\nabla s\Big)^{-1}. \tag{3.4}$$

Thus

$$\begin{aligned}
\eta(Au) &= \max_{t\in[a,c]} Au(t)\\
&\geq Au(b)\\
&= \frac{1}{d}\left(\gamma+\beta(c-b)\right)\Big[\int_a^c (s-a+\delta)h(s)f(s,u(s))\nabla s\\
&\quad -\alpha\int_b^c (s-b)h(s)f(s,u(s))\nabla s\Big] - \int_b^c (s-b)h(s)f(s,u(s))\nabla s\\
&= \frac{\gamma+\beta(c-b)}{d}\int_a^b (s-a+\delta)h(s)f(s,u(s))\nabla s\\
&\quad + \frac{b-a+\delta}{d}\int_b^c (\gamma+\beta(c-s))h(s)f(s,u(s))\nabla s\\
&\geq \frac{\gamma+\beta(c-b)}{d}\int_a^b (s-a+\delta)h(s)f(s,u(s))\nabla s\\
&> pM\frac{\gamma+\beta(c-b)}{d}\int_a^b (s-a+\delta)h(s)\nabla s\\
&\geq p
\end{aligned}$$

by hypothesis (i) and (3.4).

Claim 2: If $u\in\partial P(\theta,q)$, then $\theta(Au)<q$: Note that $u\in\partial P(\theta,q)$ and (3.3) imply that $0\leq u(t)\leq\|u\|\leq q/k$ for $t\in[a,c]$. Define

$$m := d\Big(\left(\gamma+\beta(c-b)\right)\int_a^c (s-a+\delta)h(s)\nabla s\Big)^{-1}. \tag{3.5}$$

Then

$$\begin{aligned}
\theta(Au) &= \max_{t\in[b,c]} Au(t)\\
&\leq \max_{t\in[b,c]}\frac{1}{d}\left(\gamma+\beta(c-t)\right)\Big[\int_a^c (s-a+\delta)h(s)f(s,u(s))\nabla s\\
&\quad -\alpha\int_b^c (s-b)h(s)f(s,u(s))\nabla s\Big]\\
&\leq \frac{1}{d}\left(\gamma+\beta(c-b)\right)\int_a^c (s-a+\delta)h(s)f(s,u(s))\nabla s\\
&< \frac{qm}{d}\left(\gamma+\beta(c-b)\right)\int_a^c (s-a+\delta)h(s)\nabla s\\
&= q
\end{aligned}$$

using (2.4), hypothesis (ii), and (3.5).

Claim 3: If $u\in\partial P(\phi,r)$, then $\phi(Au)>r$: Since $u\in\partial P(\phi,r)$, from (3.3) we have that $\min_{t\in[a,b]} u(t)=r$ and $r\leq\|u\|\leq\frac{r}{k}$. Since $\delta\geq 0$, as in the proof of Claim 1 we have

$$\begin{aligned}
Au(b) &= \frac{\gamma+\beta(c-b)}{d}\int_a^b (s-a+\delta)h(s)f(s,u(s))\nabla s\\
&\quad + \frac{b-a+\delta}{d}\int_b^c (\gamma+\beta(c-s))h(s)f(s,u(s))\nabla s.
\end{aligned}$$

In a similar fashion from the proof of Lemma 2.2 we have

$$\begin{aligned}
Au(a) = \alpha\Big[&\frac{\gamma+\beta(c-b)}{d}\int_a^b (s-a)h(s)f(s,u(s))\nabla s \\
&+\frac{b-a}{d}\int_b^c (\gamma+\beta(c-s))h(s)f(s,u(s))\nabla s\Big] \\
&+\frac{\delta}{d}\int_a^c (\gamma+\beta(c-s))h(s)f(s,u(s))\nabla s.
\end{aligned}$$

By concavity,

$$\min_{t\in[a,b]} Au(t) = \min\{Au(a), Au(b)\}.$$

If $\alpha \geq 1$, then $Au(a) \geq Au(b)$. If $0<\alpha<1$, then $Au(a) \geq \alpha Au(b)$. It follows that

$$\begin{aligned}
\phi(Au) &= \min_{t\in[a,b]} Au(t) \\
&\geq \min\{\alpha,1\}Au(b) \\
&\geq \min\{\alpha,1\}\big(\frac{\gamma+\beta(c-b)}{d}\big)\int_a^b (s-a+\delta)h(s)f(s,u(s))\nabla s \\
&> \min\{\alpha,1\}rM\big(\frac{\gamma+\beta(c-b)}{d}\big)\int_a^b (s-a+\delta)h(s)\nabla s \\
&\geq r
\end{aligned}$$

by hypothesis (iii) and (3.4). Therefore the hypotheses of Theorem 3.1 are satisfied and there exist at least two positive fixed points u_1 and u_2 of A in $\overline{P(\phi,r)}$. Thus, the second-order triple-point boundary-value problem (1.1), (1.2), has at least two positive solutions u_1 and u_2 such that

$$\begin{aligned}
p<\eta(u_1) &\quad \text{with } \theta(u_1)<q, \\
q<\theta(u_2) &\quad \text{with } \phi(u_2)<r.
\end{aligned}$$

□

Corollary 3.3. *Let $d>0$. If there exists $q>0$ such that the function f satisfies the following conditions:*

- (i) $\liminf_{u\to 0^+} \dfrac{f(s,u)}{u} > M/k$ *for* $s\in[a,b]$,
- (ii) $f(s,u) < qm$ *for* $s\in[a,c]$ *and* $u\in[0,q/k]$,
- (iii) $\liminf_{u\to\infty} \dfrac{f(s,u)}{u} > M$ *for* $s\in[a,b]$,

then the second-order boundary-value problem (1.1), (1.2), *has at least two positive solutions u_1 and u_2 such that*

$$\begin{aligned}
p<\max_{t\in[a,c]} u_1(t) &\quad \textit{with } \max_{t\in[b,c]} u_1(t)<q, \\
q<\max_{t\in[b,c]} u_2(t) &\quad \textit{with } \min_{t\in[a,b]} u_2(t)<r.
\end{aligned}$$

Proof. From assumption (i) of the corollary we know there exists $p\in(0,q)$ such that

$$\frac{f(s,u)}{u} > \frac{M}{k}, \quad u\in(0,p],\ s\in[a,b].$$

In particular,

$$f(s,u) > uM/k \geq pk(M/k) = pM, \quad u \in [kp,p],\ s \in [a,b],$$

and (i) of Theorem 3.2 holds. Now let

$$f_\infty := \liminf_{u\to\infty}\Big(\min_{s\in[a,b]}\frac{f(s,u)}{u}\Big),$$

and $\eta \in (M, f_\infty)$. Then there exists $r' > q$ such that $\min_{s\in[a,b]} f(s,u) \geq \eta u$, $u \in [r',\infty)$. Set

$$\varpi := \min\big\{\min_{s\in[a,b]} f(s,u) : u \in [0,r']\big\}$$

and take

$$r > \max\big\{r', \frac{\varpi}{\eta - M}\big\}.$$

Then

$$\min_{s\in[a,b]} f(s,u) \geq \eta u - \varpi \geq \eta r - \varpi > rM, \quad u \in [r,\infty),$$

so that (iii) of Theorem 3.2 holds. The conclusion follows. $\square$

Similar to Corollary 3.3, one can prove the following statement.

Corollary 3.4. *Let $d > 0$. If there exists $q > 0$ such that the function f satisfies the following conditions:*

- (i) $\limsup_{u\to 0^+}\dfrac{f(s,u)}{u} < m$ *for* $s \in [a,c]$,
- (ii) $f(s,u) > qM$ *for* $s \in [a,b]$ *and* $u \in [kq,q]$,
- (iii) $\limsup_{u\to\infty}\dfrac{f(s,u)}{u} < mk$ *for* $s \in [a,c]$,

then the second-order boundary-value problem (1.1), (1.2), *has at least two positive solutions.*

4. Existence of at Least Three Positive Solutions

To prove the existence of at least three positive solutions to (1.1), (1.2) we will use the Leggett-Williams fixed point theorem [13, 18]:

Theorem 4.1. *Let P be a cone in the real Banach space $\mathcal{S}$, $A : \overline{P_r} \to \overline{P_r}$ be completely continuous and ϕ be a nonnegative continuous concave functional on P with $\phi(u) \leq \|u\|$ for all $u \in \overline{P_r}$. Suppose there exists $0 < p < q < \ell \leq r$ such that the following conditions hold:*

- (i) $\{u \in P(\phi,q,\ell) : \phi(u) > q\} \neq \emptyset$ *and* $\phi(Au) > q$ *for all* $u \in P(\phi,q,\ell)$;
- (ii) $\|Au\| < p$ *for* $\|u\| \leq p$;
- (iii) $\phi(Au) > q$ *for* $u \in P(\phi,q,r)$ *with* $\|Au\| > \ell$.

Then A has at least three fixed points u_1, u_2, and u_3 in $\overline{P_r}$ satisfying:

$$\|u_1\| < p, \quad \phi(u_2) > q, \quad p < \|u_3\| \ \text{with}\ \phi(u_3) < q.$$

Again define the continuous concave functional $\phi : P \to [0,\infty)$ to be $\phi(u) := \min_{t\in[a,b]} u(t)$, the cone P as in (3.1), M as in (3.4), and

$$m := d\Big((\gamma + \beta(c-a))\int_a^c (s-a+\delta)h(s)\nabla s\Big)^{-1}.$$

Moreover, we take

$$P_r := \{u \in P : \|u\| < r\}, \quad P(\phi, p, q) := \{u \in P : p \le \phi(u),\ \|u\| \le q\}.$$

Theorem 4.2. *Let $d > 0$. Suppose that there exist constants $0 < p < q < q/k < r$ such that*

- (D1) $f(s,u) \le rm$ *for* $s \in [a,c]$, $u \in [0,r]$;
- (D2) $f(s,u) \ge qM$ *for* $s \in [a,b]$, $u \in [q, q/k]$;
- (D3) $f(s,u) < pm$ *for* $s \in [a,c]$, $u \in [0,p]$,

where k is given in (2.6). *Then the boundary-value problem* (1.1), (1.2) *has at least three positive solutions u_1, u_2, u_3 satisfying*

$$\|u_1\| < p, \quad q < \min_{t\in[a,b]} u_2(t), \quad \|u_3\| > p \text{ with } \min_{t\in[a,b]} u_3(t) < q.$$

Proof. Again the solutions are the fixed points of the operator A defined by

$$\begin{aligned} Au(t) = \frac{1}{d}\left(\gamma + \beta(c-t)\right)\Big[\int_a^c (s-a+\delta)h(s)f(s,u(s))\nabla s \\ - \alpha \int_b^c (s-b)h(s)f(s,u(s))\nabla s\Big] - \int_t^c (s-t)h(s)f(s,u(s))\nabla s. \end{aligned}$$

The conditions of Theorem 4.1 will now be shown to be satisfied. For all $u \in P$ we have $\phi(u) \le \|u\|$. If $u \in \overline{P_r}$, then $\|u\| \le r$ and assumption (D1) implies $f(s,u(s)) \le rm$ for $s \in [a,c]$. Consequently,

$$\begin{aligned} \|Au\| &= \max_{t\in[a,c]} Au(t) \\ &\le \max_{t\in[a,c]} \frac{1}{d}\left(\gamma + \beta(c-t)\right)\Big[\int_a^c (s-a+\delta)h(s)f(s,u(s))\nabla s \\ &\quad - \alpha \int_b^c (s-b)h(s)f(s,u(s))\nabla s\Big] \\ &\le \frac{1}{d}\left(\gamma + \beta(c-a)\right)\int_a^c (s-a+\delta)h(s)f(s,u(s))\nabla s \\ &< \frac{rm}{d}\left(\gamma + \beta(c-a)\right)\int_a^c (s-a+\delta)h(s)\nabla s \\ &= r. \end{aligned}$$

This proves that $A : \overline{P_r} \to \overline{P_r}$. Similarly, if $u \in P_p$, then assumption (D3) yields $f(s,u(s)) < pm$ for $s \in [a,c]$. Just as above, we have $A : \overline{P_p} \to P_p$. It follows that condition (ii) of Theorem 4.1 is satisfied.

We now consider condition (i) of Theorem 4.1; pick $u_P(t) \equiv q/k$ for $t \in [a,c]$, for k given in (2.6). Then $u_P \in P(\phi, q, q/k)$ and $\phi(u_P) = \phi(q/k) > q$, so that $\{u \in P(\phi, q, q/k) : \phi(u) > q\} \ne \emptyset$. Consequently, if $u \in P(\phi, q, q/k)$, then $q \le u(s) \le q/k$ when $s \in [a,b]$. From assumption (D2) we have that

$$f(s,u(s)) \ge qM$$

for $s \in [a,b]$. As in Claim 3 of the proof of Theorem 3.2,

$$\min_{t\in[a,b]} Au(t) = \min\{Au(a), Au(b)\}.$$

If $\alpha \geq 1$, then $Au(a) \geq Au(b)$; if $0 < \alpha < 1$, then $Au(a) \geq \alpha Au(b)$. It follows that

$$\begin{aligned}
\phi(Au) &= \min_{t\in[a,b]} Au(t) \\
&\geq \min\{\alpha, 1\} Au(b) \\
&\geq \min\{\alpha, 1\}\big(\frac{\gamma+\beta(c-b)}{d}\big)\int_a^b (s-a+\delta)h(s)f(s,u(s))\nabla s \\
&> \min\{\alpha, 1\} qM\big(\frac{\gamma+\beta(c-b)}{d}\big)\int_a^b (s-a+\delta)h(s)\nabla s \\
&\geq q
\end{aligned}$$

by hypothesis (D2) and (3.4). Therefore,

$$\phi(Au) > q, \quad u \in P(\phi, q, q/k),$$

so that condition (i) of Theorem 4.1 holds. To check on Theorem 4.1 (iii), we suppose that $u \in P(\phi, q, r)$ with $\|Au\| > q/k$. Then, Lemma 2.3 and the definition of ϕ yield

$$\phi(Au) = \min_{t\in[a,b]} Au(t) \geq k\|Au\| > kq/k = q.$$

□

Using the ideas in the proof of the above theorem, we can establish the existence of an arbitrary odd number of positive solutions of (1.1), (1.2), assuming the right conditions on the nonlinearity f.

Theorem 4.3. *Let $d > 0$. Suppose that there exist constants*

$$0 < p_1 < q_1 < q_1/k < p_2 < q_2 < q_2/k < p_3 < \cdots < p_n, \quad n \in \{2, 3, 4, \cdots\},$$

such that

- (D1) $f(s, u) \geq q_i M$ *for* $s \in [a, b]$, $u \in [q_i, q_i/k]$;
- (D2) $f(s, u) < p_i m$ *for* $s \in [a, c]$, $u \in [0, p_i]$,

where k is given in (2.6). *Then the boundary-value problem* (1.1), (1.2) *has at least* $2n - 1$ *positive solutions.*

5. Example

Let $\mathbb{T} = \{1 - (1/2)^{\mathbb{N}_0}\} \cup \{1\}$. Taking $a = 0$, $b = 31/32$, $c = \beta = 1$, $\alpha = 20$, $\gamma = 1/19$, and $\delta = 3/4$, we have $d = 1/8$ and $k = 1/32$. If we let $h(s) \equiv 1$, then $m = 57/680$ and $M = 77824/71145$. Suppose

$$f(t, u) = f(u) := \frac{2000e^u}{99e^{700} + e^u}, \quad t \in [0, 1], \quad u \geq 0.$$

Clearly f is always increasing. If we take $p = 701$, $q = 705$, and $r = 24000$, then

$$0 < p < q < q/k < r.$$

We check that (D1), (D2), and (D3) of Theorem 4.2 are satisfied. Since $f(715) \approx 1999.94$ and $\lim f(u) = 2000$,

$$f(u) \leq 24000m \approx 2011.76, \quad u \in [0, r],$$

so that (D1) is met. To verify (D2), note that $f(705) \approx 1199.72$, so that

$$f(u) \geq 705M \approx 771.18, \quad u \in [q, 32q].$$

Lastly, as $f(701) \approx 53.45$,

$$f(u) < 701m \approx 58.76, \quad u \in [0,p]$$

and (D3) holds. Therefore by Theorem 4.2, the boundary-value problem

$$u^{\Delta\nabla}(t) + f(u(t)) = 0, \quad u(0) = 20u(31/32) + \frac{3}{4}u^{\Delta}(0), \quad u(1) + \frac{1}{19}u^{\Delta}(1) = 0$$

has at least three positive solutions u_1, u_2, u_3 satisfying

$$\|u_1\| < 701, \quad 705 < \min_{t\in[0,31/32]} u_2(t), \quad \|u_3\| > 701 \quad \text{with} \quad \min_{t\in[0,31/32]} u_3(t) < 705.$$

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