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LIOUVILLE'S THEOREM AND THE RESTRICTED MEAN PROPERTY FOR BIHARMONIC FUNCTIONS

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Abstract. We prove that under certain conditions, a bounded Lebesgue measurable function satisfying the restricted mean value for biharmonic functions is constant, in $\mathbb{R}^n$ with $n \geq 3$.

1. Introduction

It is well known that a biharmonic function f in $\mathbb{R}^n$, i.e. a solution of the classical biharmonic equation $\Delta^2 u = 0$, satisfies the biharmonic mean formula on every open ball $B = B(x,r)$ of center x and radius $r > 0$ in $\mathbb{R}^n$:

$$f(x) = \frac{1}{|B|}\int_B f d\lambda - \frac{r^2}{2(n+2)}\Delta f(x),$$

where $|B|$ denotes the volume of the ball B and λ is the Lebesgue measure on $\mathbb{R}^n$. This formula is due to Pizzetti and can be found in [6].

When the biharmonic function f is bounded, say

$$\sup_{x\in\mathbb{R}^n} |f(x)| = M < +\infty,$$

one has

$$\sup_{x\in\mathbb{R}^n} |\Delta f(x)| \leq \frac{4(n+2)}{r^2} M.$$

Letting $r \to \infty$, this yields $\Delta f \equiv 0$, so that f is a bounded harmonic function on $\mathbb{R}^n$, hence f is constant by the Liouville's classical Theorem. This is the Liouville's property for biharmonic functions.

Let us recall that a function f on a domain Ω of $\mathbb{R}^n$, locally integrable and whose Laplacian in the distribution sense is a function, satisfies the restricted biharmonic mean property if there exists a function $r : \Omega \to \mathbb{R}_+$ such that $0 < r(x) \leq d(x, C\Omega)$ and

$$f(x) = \frac{1}{|B(x)|}\int_{B(x)} f(y)dy - \frac{r(x)^2}{2(n+2)}\Delta f(x) \tag{1.1}$$

for every $x \in \Omega$, where $B(x)$ is the open ball of center x and radius $r(x)$.

In [1] and [2], we proved that, under certain conditions on the functions f and r, if f satisfies the biharmonic mean property on the balls $B(x)$ of center x and radius

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$r(x)$, then f is biharmonic. This result extends a result by Hansen and Nadirashvili [3, 4] on functions possessing the restricted (harmonic) mean property to functions satisfying the restricted biharmonic property.

Let $r : \mathbb{R}^n \to \mathbb{R}_+$ be such that $0 < r(x) \leq \|x\| + M_0$, $n \geq 2$, where M_0 is a constant, and let f be a Lebesgue measurable bounded function on $\mathbb{R}^n$ which satisfies

$$f(x) = \frac{1}{|B(x)|} \int_{B(x)} f(y)dy$$

for every $x \in \mathbb{R}^n$. Hansen and Nadirashvili [5] proved that if f is continuous or if r is bounded from below on every compact set of $\mathbb{R}^n$ by some positive constant, then f is constant. Our main goal in this work is to extend this result to biharmonic functions, to establish a Liouville's theorem for functions having the restricted biharmonic mean property. For doing this, we use representation measures for harmonic functions used in [1], and derived from the biharmonic mean formula in [5]. We will verify that the conditions in [5] are satisfied and then use their results.

By a function we always mean a function with values in $\overline{\mathbb{R}}$, unless otherwise stated.

The results of this paper extend easily to polyharmonic functions of order greater than 2. We have treated only the biharmonic case for reasons of simplicity.

2. Liouville's theorem and the restricted biharmonic mean property

Let us recall some results of [4] and [5] that will be used in the sequel. Let $n \geq 3$ and let G denote the Green kernel in $\mathbb{R}^n$ normalized in such a way that for every $y \in \mathbb{R}^n$, $\Delta G(.,y) = -\epsilon_y$ in the distribution sense. Recall that

$$G(x,y) = \frac{1}{\sigma_n(n-2)} \frac{1}{\|x-y\|^{n-2}}, \quad \forall (x,y) \in (\mathbb{R}^n)^2,$$

where σ_n is the area of the unit sphere of $\mathbb{R}^n$.

Let $\mathcal{B}(\mathbb{R}^n)$ be the Borel field. A kernel on $\mathbb{R}^n$ is a function $N : \mathbb{R}^n \times \mathcal{B}(\mathbb{R}^n) \to \overline{R}_+$ such that

(1) For every $A \in \mathcal{B}(\mathbb{R}^n)$, $x \mapsto N(x,A)$ is Borel measurable,
(2) For every $x \in \mathbb{R}^n$, $A \mapsto N(x,A)$ is a measure μ_x on $\mathcal{B}(\Omega)$.

A kernel N on $\mathbb{R}^n$ is said to be Markovian if $N(x,\mathbb{R}^n) = 1$ for $x \in \mathbb{R}^n$.

Fix a Markovian kernel $(x,A) \mapsto \mu_x(A)$ on $\mathbb{R}^n$ such that, for some constants $M_0 \geq 1$, $0 < \eta < 1$, $\gamma > 0$ and $a > 0$, the following conditions are satisfied:

(i) $\mu_x(s) \leq s$ for every non-negative superharmonic function s on $\mathbb{R}^n$.
(ii) There exist a function r_0 on $\mathbb{R}^n$, $0 < r_0(x) \leq \|x\| + M_0 =: \rho(x)$ and a constant $\gamma > 0$ such that $\mu_x \geq \gamma \lambda_{B(x,r_0(x))}$ and $(G^{\epsilon_x} - G^{\mu_x}) 1_{CB(x,\eta r_0(x))} \lambda \leq a\rho^2 \mu_x$.

Remarks (See [5]) 1. The condition (i) is satisfied if and only if $G^{\mu_x} \leq G^{\epsilon_x}$.

2. Let r be a Borel measurable function on $\mathbb{R}^n$ such that $0 < r \leq \|\cdot\| + M_0$ for some constant $M_0 \geq 1$. Then the kernel $(x,E) \mapsto \lambda_{B(x,r(x))}(E)$ satisfies the conditions (i) and (ii), where λ_A denotes the probability measure $\frac{1_A}{|A|}\lambda$ if $|A| \neq 0$ (for Condition (ii), it is sufficient to take $r_0(x) = r(x)/2$).

Theorem 2.1 ([5, Cor. 2.4]). *Let f be a Lebesgue measurable bounded function on $\mathbb{R}^n$ such that*

$$f(x) = \int f d\mu_x$$

for every $x \in \mathbb{R}^n$. Assume that f is continuous or that r_0 is locally bounded from below by a constant > 0. Then f is constant.

3. Liouville's Theorem and the restricted biharmonic mean property

For a ball $B = B(x, r)$ of $\mathbb{R}^n$, put

$$w_B(x, z) = G(x, z) - \frac{1}{|B|} \int_B G(y, z) d\lambda(y).$$

It is not difficult to see that the function w_B satisfies the following properties:

(1) $w_B(x, y) = 0$ if $y \notin B$.
(2) The function $w_B(x, .)$ is invariant under rotations around x.
(3) One has

$$\frac{2(n+2)}{r^2} \int w_B(x, y) d\lambda(y) = 1.$$

This equality is an immediate consequence of the harmonic mean formula on B applied to the function $\int_{B'} G(., y)dy$, where B' is any open ball of center x such that $\overline{B} \subset B'$.

It follows that for $x \in \mathbb{R}$, the measure μ_x^r on $\mathbb{R}^n$ of density $\frac{2(n+2)}{r^2} w_{B(x)}(x, .)$ with respect to the Lebesgue measure is a probability measure supported by $\overline{B(x, r)}$ and invariant under rotations around x. In particular, one has,

$$\int s(y) d\mu_x^r(y) \le s(x)$$

for any non-negative superharmonic function s on $\mathbb{R}^n$. The map $(x, A) \mapsto \mu_x^r(A)$, where A is a Borel set of $\mathbb{R}^n$, is a Markovian kernel on $\mathbb{R}^n$.

Let $r > 0$ be a function on $\mathbb{R}^n$ such that there exists a constant $M_0 \ge 0$ satisfying $r(x) \le \|x\| + M_0 = \rho(x)$ for any $x \in \mathbb{R}^n$. We denote by μ_x the measure $\mu_x^{r(x)}$. When r is Borel measurable, the Markovian kernel $(x, A) \mapsto \mu_x^{r(x)}(A)$ satisfies the condition (i) of the section above.

Lemma 3.1. *Let f be a bounded function of class $\mathcal{C}^2$ in $\mathbb{R}^n$ such that Δf is constant, then f is constant.*

Proof. By replacing f by $-f$ if necessary one can assume that $\Delta f = c$ with $c \le 0$, so that f is a superharmonic function which we assume to be ≥ 0 by adding to it a constant if needed. When $n = 2$ the result is then immediate. If $n \ge 3$, f is the Green potential of a measure $c\lambda$, hence $c = 0$. We then deduce that f is a bounded harmonic function, hence constant by the Liouville's classical theorem. □

Theorem 3.2. *Let r be a real function on $\mathbb{R}^n$, $n \ge 3$, such that $0 < r(x) \le \|x\| + M_0$, and let f be a bounded Lebesgue measurable function whose Laplacian in the distribution sense is a bounded function and such that*

$$f(x) = \frac{1}{|B(x)|} \int_{B(x)} f d\lambda - \frac{r(x)^2}{2(n+2)} \Delta f(x), \tag{3.1}$$

for every $x \in \mathbb{R}^n$. If f is continuous or if r is locally bounded from below by a positive constant, then f is constant.

Proof. As in [1], we remark that if f satisfies (3.1) for every $x \in \mathbb{R}^n$, then $\Delta f(x) = \int \Delta f d\mu_x$ for every $x \in \mathbb{R}^n$. Following [7, Sec. 2.1], we may assume that r is Borel measurable. We have already seen that r satisfies condition (i) of the previous section, we shall prove that it satisfies also condition (ii). Then the theorem follows then from Theorem 2.1 and Lemma 3.1. Let $r_0(x) = r(x)/2$ for $x \in \Omega$. Then it is easy to verify that the condition $\mu_x \geq \gamma\lambda_{B(x,r_0(x))}$ is satisfied for some constant γ (which does not depend on x).

For simplicity, let $x = 0$ and assume that $B(x) = B$, the unit ball of $\mathbb{R}^n$, let $w = w_B(0, .)$. Then it is not difficult to see that $w = \kappa_n(|\cdot|^{-n} + (\frac{n}{2} - 1)|\cdot|^2 - \frac{n}{2})$. The function w is invariant under rotations around 0 and that $w \approx (1 - |\cdot|)^2$ on $\{\eta < |\cdot| < 1\}$. Let us take $\psi :]0, 1[\to \mathbb{R}_+$ such $\psi(|z|) = w(z)$ and let σ_t be the normalized area measure on $\{|\cdot| = 1\}$. Then

$$G_\Omega^{\epsilon_0} - G_\Omega^{\mu_0} = G_B^{\epsilon_0} - G_B^{\mu_0} = \text{const} \int_0^1 (G_B^{\epsilon_0} - G_B^{\sigma_t})\psi(t)t^{n-1}dt,$$

where for $\eta < |z| < 1$,

$$(G_B^{\epsilon_0} - G_B^{\sigma_t})(z) = \kappa_n(|z|^{2-n} - t^{2-n})^+ \leq \text{const}(t - |z|)^+,$$

and then

$$\begin{aligned}(G_\Omega^{\epsilon_0} - G_\Omega^{\mu_0})(z) &\leq C \int_{|z|}^1 (t - |z|)\psi(t)dt \\ &= C\psi(|z|) \int_0^1 (t - |z|)dt \\ &= \frac{C}{2} w(z)(1 - |z|)^2.\end{aligned}$$

Then condition (ii) is satisfied for $r_0(x) = \frac{r(x)}{2}$ and $\eta = 1$. Hence, it follows from Theorem 2.1 that the function Δf is constant, therefore f is constant by Lemma 3.1. □

We say that a function f is bounded by another function $g \geq 0$ on an open set U of $\mathbb{R}^n$ if $|f| \leq g$ on U. By combining the above theorem with [1, Theorems 9 and 10] ([2, Theorems 2.1 and 2.2]), we obtain the following result.

Corollary 3.3. *Let f be a locally integrable function in a domain U of $\mathbb{R}^n$, $n \geq 3$, whose Laplacian, in the distribution sense, is bounded by an harmonic function. Let r be a real function > 0 on U such that for any $x \in U$, $B(x, r(x)) \subset U$ and (1.1) takes place. If $U = \mathbb{R}^n$ assume that f is bounded and that $r(x) \leq \|x\| + M_0$, for some constant $M_0 \in \mathbb{R}_+$. Suppose moreover that Δf is a continuous or that r is locally bounded from below by a constant > 0. Then f is harmonic.*

We will study to the cases $n = 1$ and $n = 2$ in a forthcoming work.

Remarks. 1. We did not consider if the hypothesis that Δf is bounded in the above theorem can be dropped.

2. If the function r is bounded from below by a positive constant, then the condition that f is bounded and the restricted biharmonic mean property in the above theorem implies that that the function Δf is bounded. In fact, we have

$$\sup_{x\in\mathbb{R}^n} |\Delta f(x)| \leq \frac{4(n + 2)}{\inf_{x\in\mathbb{R}^n} r(x)} \sup_{x\in\mathbb{R}^n} |f(x)| < +\infty.$$

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