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# POSITIVE SOLUTIONS FOR THE Φ-LAPLACIAN WHEN Φ IS A SUP - MULTIPLICATIVE - LIKE FUNCTION

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ABSTRACT. We provide sufficient conditions for the existence of positive solutions of a boundary-value problem for a one dimensional Φ-Laplacian ordinary differential equation with deviating arguments, where Φ is a sup-multiplicative-like function (in a sense introduced here) and the boundary conditions include nonlinear expressions at the end points. For this end, we use the Krasnoselskii fixed point theorem in a cone. The results obtained improve and generalize known results in [17] and elsewhere.

## 1. INTRODUCTION

We call *sup-multiplicative-like function* an odd homeomorphism $\Phi$ of the real line $\mathbb{R}$ onto itself for which there exists a homeomorphism $\phi$ of $\mathbb{R}_+ := [0, +\infty)$ onto $\mathbb{R}_+$ which *supports* $\Phi$ in the sense that for all $v_1, v_2 \geq 0$ it holds

$$\phi(v_1)\Phi(v_2) \leq \Phi(v_1 v_2).$$

Note that any sup-multiplicative function is sup-multiplicative-like function. Also any function of the form

$$\Phi(u) := \sum_{0}^{k} c_j |u|^j u, \quad u \in \mathbb{R}$$

is sup-multiplicative-like, provided that $c_j \geq 0$. Here a supporting function is defined by $\phi(u) := \min\{u^{k+1}, \quad u\}$, $u \geq 0$.

It is clear that a sup-multiplicative-like function $\Phi$ and any corresponding supporting function $\phi$ are increasing functions vanishing at zero and moreover their inverses $\Psi$ and $\psi$ respectively are increasing and such that

$$\Psi(w_1 w_2) \leq \psi(w_1)\Psi(w_2),$$

for all $w_1, w_2 \geq 0$. From this relation it follows easily that for all $M, u > 0$ it holds

$$M\Phi(u) \geq \Phi\Big(\frac{u}{\psi(1/M)}\Big). \tag{1.1}$$

More facts from the pathology of this meaning will be presented later in this section. For the moment we notice only that if $\Phi_1$ and $\Phi_2$ are two sup-multiplicative -like

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functions, then the functions $\Phi_1+\Phi_2$, $\Phi_1|\Phi_2|$ and $\Phi_1\circ\Phi_2$ are also sup-multiplicative-like functions, hence this class is closed with respect to the addition, multiplication and composition. Indeed to see this, assume that $\phi_1, \phi_2$ are functions which support $\Phi_1$ and $\Phi_2$ respectively. Then for all $u, v > 0$ we have

$$\big[\Phi_1 + \Phi_2\big](uv) \geq \Phi_1(u)\phi_1(v) + \Phi_2(u)\phi_2(v) \geq \big[\Phi_1 + \Phi_2\big](u)\phi(v),$$

where $\phi(v) := \min\{\phi_1(v), \phi_2(v)\}$. Also we have

$$\Phi_1(uv)\Phi_2(uv) \geq \Phi_1(u)\phi_1(v)\Phi_2(u)\phi_2(v) \geq \big(\Phi_1(u)\Phi_2(u)\big)\phi(v),$$

where $\phi(v) := \phi_1(v)\phi_2(v)$. Finally we have

$$\Phi_1\big(\Phi_2(uv)\big) \geq \Phi_1\big(\Phi_2(u)\phi_2(v)\big) \geq \big(\Phi_1\big(\Phi_2(u)\big)\phi(v),$$

where $\phi(v) := \phi_1\big(\phi_2(v)\big)$.

Let $\Phi$ be a differentiable sup-multiplicative-like function and, for each $j = 1, 2, \dots, n$, let $g_j : [0, 1] \to [0, 1]$ be measurable functions.

In this paper we investigate the case when positive solutions of the one dimensional differential equation (with deviated arguments) of the form

$$[\Phi(x')]' + c(t)f(t, x(g_1(t)), x(g_2(t)), \dots, x(g_n(t))) = 0, \quad \text{a.a.} \quad t \in I \tag{1.2}$$

exist which satisfy one of the following three pairs of conditions

$$x(0) - B_0(x'(0)) = 0, \quad x(1) + B_1(x'(1)) = 0, \tag{1.3}$$

$$x(0) - B_0(x'(0)) = 0, \quad x'(1) = 0, \tag{1.4}$$

$$x'(0) = 0, \quad x(1) + B_1(x'(1)) = 0. \tag{1.5}$$

Here we extend and in some cases improve the results given in [17] and elsewhere. For instance, we show existence of positive solutions when at least one of $B_0$ or $B_1$ is sub-linear only near zero, thus they might be exponential. The existence of multiple positive solutions of this problem will be given in a forthcoming paper.

It is clear that in case $\Phi$ is a function of the form $\Phi(u) := |u|^{m-2}u$, Equation (1.2) comes from the nonautonomous $m$-Laplacian elliptic equation in the $n-$dimensional space which has radially symmetric solutions. Also (1.2) is generated from an equation of the form

$$x'' + p(x')f(t, x) = 0,$$

where $\inf\{p(u) : |u| \leq r\} > 0$, for all $r > 0$, by setting

$$\Phi(u) := \int_0^u \frac{d\xi}{p(\xi)}.$$

Boundary-value problems with boundary conditions of the form (1.3)–(1.5) were discussed first by Gustafson and Schmitt [11] who considered a problem of the form

$$x'' + f(t, x) = 0, \quad t \in (0, 1)$$

with the boundary conditions

$$ax(0) - bx'(0) = 0, \quad cx(1) + dx'(1) = 0, \tag{1.6}$$

where the coefficients $a, b, c, d$ are positive reals. Notice that [11, Section 6] is devoted to boundary value problems with retarded arguments associated with Dirichlet boundary conditions. Boundary value problems with delays were investigated by many authors, because of their importance in variational problems, in control theory, mechanics, physics and a variety of areas in applied mathematics, see, e. g., [3, 4, 5, 9, 10, 13, 18, 20] and the references therein. Related topics can be found in [2, 7].

In [8] the existence of positive solutions of the equation

$$x'' + c(t)f(x) = 0, \quad t \in (0, 1)$$

associated with the conditions (1.6), was investigated. The same subject, but with a nonlocal boundary condition was, also, treated in [14].

Motivated by [8], Wang [17] considered the function $g(u) = |u|^{p-2}u$, $p > 1$ and he studied the boundary -value problem

$$(g(x'))' + c(t)f(x) = 0, \quad t \in (0, 1)$$

associated with the boundary conditions of the form (1.3)–(1.5) where $B_0$ and $B_1$ are both nondecreasing, continuous, odd functions defined on the whole real line and at least one of them is sub-linear. The function $c$ satisfies an integral condition through the inverse function of $g$. An analog condition we shall also assume in this paper. Conditions for the existence of multiple positive solutions of the same problem were recently given in [12, 16]. A case with simpler boundary conditions is discussed in [1]. In [19] the same conditions were imposed to a one dimensional $p$-Laplacian differential equation, where the derivative affects the response function.

In [6], where an equation of the form (1.2) was discussed (but without deviating arguments and with simple Dirichlet conditions), the leading factor depends on an odd homeomorphism $\Phi$, which, in order to guarantee the nonexistence of solutions, actually, satisfies the condition

$$\sup_{u>0} \frac{\Phi(uv)}{\Phi(u)} < +\infty,$$

for all $v > 0$. It is clear that the condition $\limsup_{u\to+\infty} \Phi(uv)/\Phi(u) < +\infty$, as it is placed in [6], is not sufficient; see [6, Lemma 4].

It is well known that in order to seek for positive solutions of operator equations Krasnoselskii presented in [15] a fixed point theorem, which is stated below and which has been proved as a powerful tool in investigating the existence of positive solutions of boundary value problems, see, e. g., most of the papers cited above.

**Theorem 1.1** (Krasnoselskii [15]). *Let* $\mathbf{B}$ *be a Banach space and let* $\mathbf{K}$ *be a cone in* $\mathbf{B}$. *Assume that* $\Omega_1$, $\Omega_2$ *are open subsets of* $E$, *with* $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, *and let*

$$A : \mathbf{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathbf{K}$$

*be a completely continuous operator such that either*

$$\|Au\| \le \|u\|, \quad u \in \mathbf{K} \cap \partial\Omega_1 \quad \text{and} \quad \|Au\| \ge \|u\|, \quad u \in \mathbf{K} \cap \partial\Omega_2,$$

*or*

$$\|Au\| \ge \|u\|, \quad u \in \mathbf{K} \cap \partial\Omega_1 \quad \text{and} \quad \|Au\| \le \|u\|, \quad u \in \mathbf{K} \cap \partial\Omega_2.$$

*Then* $A$ *has a fixed point in* $\mathbf{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

This article is organized as follows: Section 2 is devoted to the conditions of the problem and to some facts needed in the sequel. The main results are exhibited in Section 3. The paper closes with some illustrative examples in Section 4.

## 2. The conditions and some auxiliary facts

In this section we present the basic conditions used throughout this paper and give some auxiliary results. We shall denote by $\langle \cdot, \cdot \rangle$ the inner product in the $n$-dimensional space $\mathbb{R}^n$. Also, for each vector $a := (\alpha_1, \alpha_2, \dots \alpha_n)$ we shall denote by $|a|$ its *sum-norm*, namely

$$|a| := |\alpha_1| + |\alpha_2| + \dots + |\alpha_n|.$$

Also we shall denote by $W_a$ the set

$$W_a := \{u \in (\mathbb{R}_+)^n : \langle a, u \rangle \neq 0\}.$$

It is clear that, if the vector $a$ has nonnegative coordinates, then for each $u \in W_a$ the inner product $\langle a, u \rangle$ is positive.

(H1) Assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is a differentiable sup-multiplicative-like function.

Let $\phi$ be a corresponding supporting function. In the sequel we shall assume that $\Psi$ and $\psi$ are the inverses of $\Phi$ and $\phi$, respectively. Moreover we notice that both functions are defined on the whole real line.

(H2) $f : I \times \mathbb{R}^n \to \mathbf{R}$ is a continuous function such that $f(t, u) \geq 0$, for all $u \in {\mathbb{R}_+}^n$ and $t \in I$.

(H3) $c : I \to \mathbb{R}_+$ is a (Lebesgue) integrable function such that for some nontrivial subinterval $J := [\alpha, \beta]$ of $I$ it holds $c(t) > 0$ almost everywhere on $J$. We set

$$\|c\|_1 := \int_0^1 c(t)dt.$$

(H4) The functions $g_j : I \to I$, $j = 1, 2, \dots, n$ are measurable and such that

$$\gamma := \inf_{t \in J} \min\{g_j(t), \quad 1 - g_j(t) : \quad j = 1, 2, \dots, n\} > 0,$$

where $J$ is the interval defined in (H3).

(H5) There exist vectors $a := (\alpha_1, \alpha_2, \dots \alpha_n)$ and $b := (\beta_1, \beta_2, \dots, \beta_n)$ with nonnegative coordinates such that

$$\lim_{\{u \in W_a,\ u \to 0\}} \sup_{t \in I} \frac{f(t, u)}{\Phi(\langle a, u \rangle)} = 0$$

and

$$\lim_{\{u \in W_b, \quad |u| \to \infty\}} \inf_{t \in I} \frac{f(t, u)}{\Phi(\langle b, u \rangle)} = +\infty.$$

$(\text{H6})_1$ There exists $a := (\alpha_1, \alpha_2, \dots \alpha_n)$ in ${\mathbb{R}_+}^n$ such that

$$\lim_{\{u \in W_a,\ u \to 0\}} \inf_{t \in I} \frac{f(t, u)}{\Phi(\langle a, u \rangle)} = +\infty.$$

$(\text{H6})_2$ There exist $a := (\alpha_1, \alpha_2, \dots \alpha_n)$ in ${\mathbb{R}_+}^n$ and $j_0 \in \{1, 2, \dots, n\}$ such that

$$\lim_{\{u \in W_a,\ u_{j_0} \to +\infty\}} \sup_{t \in I} \frac{f(t, u_1, u_2, \dots, u_{j_0}, \dots, u_n)}{\Phi(\langle a, u \rangle)} = 0,$$

uniformly with respect to the variables $u_i$ for all $i \neq j_0$.

(H7) For each $i = 0, 1$ the function $B_i$ is continuous nondecreasing and such that $\alpha B_i(\alpha) \geq 0$.

(H8) $\limsup_{\alpha \to 0+} \frac{B_0(\alpha)}{\alpha} < +\infty$.

(H9) $\limsup_{\alpha \to 0+} \frac{-B_1(-\alpha)}{\alpha} < +\infty$.

(H10) $\sup_{\alpha>0} \frac{B_0(\alpha)}{\alpha} < +\infty$.
(H11) $\sup_{\alpha>0} \frac{-B_1(-\alpha)}{\alpha} < +\infty$.

In this paper we shall work in the Banach space $C := C(I, \mathbb{R}_+)$ of all continuous functions $x : I \to \mathbb{R}$ furnished with the usual sup-norm $\|\cdot\|$. To apply Theorem 1.1 we need the set

$$\mathbf{K} := \{x \in C : x \text{ is concave}\},$$

which is a cone in $C$. We start with the following lemma which follows by the concavity.

**Lemma 2.1.** *For each $x$ in $\mathbf{K}$ and $t \in I$ it holds $x(t) \geq \min\{t,\ 1-t\}\|x\|$.*

To give an integral equivalent formula of the problem we need the following auxiliary results.

**Lemma 2.2.** *Suppose the functions $B_0, B_1$ satisfy condition (H7). Then for each $(\Theta, y)$ with $0 \leq y(t) \leq \Theta$, $\quad t \in I$, there exists a unique real number $U(\Theta, y)$ which depends continuously on $\Theta, y$ and it satisfies*

$$\begin{gathered} 0 \leq U(\Theta, y) \leq \Psi(\Theta), \\ \Omega\Big(U(\Theta, y)\Big) = 0, \end{gathered} \tag{2.1}$$

*where*

$$\Omega(w) := B_0(w) + B_1\big[\Psi\big(\Phi(w) - \Theta\big)\big] + \int_0^1 \Psi\big[\Phi(w) - y(s)\big]ds, \quad w \geq 0.$$

*Proof.* We observe that

$$\begin{gathered} \Omega(0) = B_1\big[\Psi\big(-\Theta\big)\big] + \int_0^1 \Psi\big(-y(s)\big)ds \leq 0, \\ \Omega\big[\Psi(\Theta)\big] = B_0\big[\Psi(\Theta)\big] + \int_0^1 \Psi\big(\Theta - y(s)\big)ds \geq 0. \end{gathered}$$

Thus the existence follows. The uniqueness is trivial since $\Omega$ is (strictly) increasing.

To show the continuity of $U$ we let $0 \leq y_n \leq \Theta_n$, $y_n \to y$, (uniformly,) $\Theta_n \to \Theta$, but $U(\Theta_n, y_n) \to w$, where $w \neq U(\Theta, y)$. The latter is impossible by the continuity and monotonicity of $\Psi$. □

## 3. Main results

Suppose that $x(t)$, $t \in I$ solves the boundary -value problem (1.2)–(1.3). Integrate twice both sides of (1.2) from 0 to $t$ and take into account Lemma 2.2. Then we obtain

$$x(t) = (Ax)(t), \tag{3.1}$$

where $A$ is the operator defined on the set $C$ by the formula

$$(Ay)(t) = B_0\big[U\big(E_y(1), E_y\big)\big] + \int_0^t \Psi\big[\Phi\big(U(E_y(1), E_y)\big) - E_y(r)\big]dr \tag{3.2}$$

and by the formula

$$\begin{aligned} (Ay)(t) = &-B_1\big(-\Psi\big[E_y(1) - \Phi\big(U(E_y(1), E_y)\big)\big]\big) \\ &+ \int_t^1 \Psi\big[E_y(r) - \Phi\big(U(E_y(1), E_y)\big)\big]dr, \end{aligned} \tag{3.3}$$

where, for simplicity, we have set

$$E_y(t) := \int_0^t z_y(s)ds,$$
$$z_y(s) := c(s)f\big(s, y(g_1(s)), y(g_2(s)), \dots, y(g_n(s))\big).$$

It is clear that a function $x$ is a solution of the operator equation (3.1) if and only if it is a solution of the boundary value problem (1.2), (1.3). Thus what we have to (and shall) do is to provide sufficient conditions for the existence of solutions of the integral equation (3.1).

Let $x \in \mathbf{K}$. We can see that for all $t$ it holds

$$(Ax)'(t) = \Psi\Big(\Phi\big(U(E_x(1), E_x)\big) - E_x(t)\Big)$$

and moreover

$$(Ax)(1) = -B_1\big(\Psi\big[\Phi\big(U(E_x(1), E_x)\big) - E_x(1)\big]\big) \geq 0,$$
$$(Ax)(0) = B_0\big(U\big(E_x(1), E_x\big)\big) \geq 0.$$

These facts ensure that the function $Ax$ is nonnegative and concave; thus $Ax \in \mathbf{K}$.

Let $\sigma$ be the smallest point in $I$ satisfying

$$\Phi\big(U(E_x(1), E_x)\big) = E_x(\sigma).$$

It is clear that such a point exists because of Lemma 2.2. Then the maximum of $Ax$ is achieved at $\sigma$ and therefore from (3.2) we have

$$\|Ax\| = B_0\big[U\big(E_x(1), E_x\big)\big] + \int_0^\sigma \Psi\Big[\int_r^\sigma z_x(s)ds\Big]dr. \tag{3.4}$$

while from (3.3)

$$\|Ax\| = -B_1\big( - \Psi\big[E_x(1) - \Phi\big(U\big(E_x(1), E_x\big)\big)\big]\big) + \int_\sigma^1 \Psi\Big[\int_\sigma^r z_x(s)ds\big)\Big]dr. \tag{3.5}$$

Now consider the boundary value problem (1.2)–(1.4). In this case the problem is equivalent to the operator equation (3.1), where $A$ is the completely continuous operator

$$(Ax)(t) = B_0\big[\Psi\big(E_x(1)\big)\big] + \int_0^t \Psi\Big(\int_s^1 z_x(s)ds\Big).$$

For each $x \in \mathbf{K}$ the image $Ax$ is a nonnegative function and the derivative $(Ax)'$ is a non-increasing function. Thus it is concave and so $A$ maps $\mathbf{K}$ into $\mathbf{K}$. Also $Ax$ is a nondecreasing function, thus we have

$$\|Ax\| = B_0\big[\Psi\big(E_x(1)\big)\big] + \int_0^1 \Psi\Big(\int_s^1 z_x(s)ds\Big).$$

Finally, let the boundary value problem (1.2)–(1.5) In this case the problem is equivalent to the operator equation $Ax = x$, where $A$ is the completely continuous operator defined by

$$(Au)(t) = -B_1\big[ - \Psi(E_u(1))\big] + \int_t^1 \Psi(E_u(s))ds.$$

Observe that for each $x \in \mathbf{K}$ the image $Ax$ is a nonnegative function, having its first derivative non-increasing. Thus it is a concave function and so $A$ maps $\mathbf{K}$ into $\mathbf{K}$. Also $Ax$ is a non-increasing function, thus we have

$$\|Ax\| = (Ax)(0) = -B_1\big[-\Psi(E_u(1))\big] + \int_0^1 \Psi(E_x(s))ds.$$

Next, we give the proofs of the results for the problem (1.2)–(1.3), since the other cases follow by obvious small modifications. Our first main result in this section is the following.

**Theorem 3.1.** *The boundary-value problem* (1.2), (1.3) *admits a positive solution provided that the conditions (H1)–(H5), (H7), and at least one of (H8), (H9) are satisfied.*

*Proof.* Assume that (H8) holds; thus there is a $\mu > 0$ such that

$$0 \le u \le 1 \quad \text{implies} \quad B_0(u) \le \mu u. \tag{3.6}$$

Let $\epsilon$ be a positive number such that

$$\epsilon \le \frac{1}{(\mu+1)\psi(\|c\|_1)|a|}, \tag{3.7}$$

where $a$ is the vector given in condition (H5). From the continuity at zero of the inverse function $\psi$ it follows that there is a $\delta > 0$ such that $\psi(v) \le \epsilon$, for all $v \in [0, \delta]$.

From the first condition in (H5) it follows that there is $T_1 > 0$ such that

$$T_1 \le \frac{1}{\psi(\|c\|_1)\epsilon|a|} \tag{3.8}$$

and $0 \le u_j \le T_1$, $j = 1, 2, \dots, n$ implies

$$f(t, u_1, u_2, \dots, u_n) \le \delta\Phi(\langle a, u\rangle)$$

for all $t \in I$, where $u := (u_1, u_2, \dots, u_n)$. Therefore for all $t \in I$ and $u \in W_a$ with coordinates in $(0, T_1]$ we have

$$\psi\Big(\frac{f(t, u_1, u_2, \dots, u_n)}{\Phi(\langle a, u\rangle)}\Big) \le \epsilon,$$

which implies

$$f(t, u_1, u_2, \dots, u_n) \le \phi(\epsilon)\Phi(\langle a, u\rangle) \le \Phi(\epsilon\langle a, u\rangle) \le \Phi(\epsilon|a|T_1). \tag{3.9}$$

The latter holds for all $u$ with $0 \le u_j \le T_1$.

Consider an $x \in \mathbf{K}$ with $\|x\| = T_1$. Then for each $t \in I$ we have $0 \le x(t) \le T_1$. By Lemma 2.2 and relations (3.8), (3.9) it follows that

$$\begin{aligned} U\big(E_x(1), E_x\big) &\le \Psi\big(E_x(1)\big) \le \Psi\Big(\int_0^1 z_x(s)ds\Big) \\ &\le \Psi\Big(\|c\|_1\Phi(\epsilon|a|T_1)\Big) = \Psi\Big(\phi(\psi(\|c\|_1))\Phi(\epsilon|a|T_1)\Big) \\ &\le \Psi\Big(\Phi\big(\psi(\|c\|_1)\epsilon|a|T_1\big)\Big) = \psi(\|c\|_1)\epsilon|a|T_1 \le 1. \end{aligned}$$

Hence from (3.4) and (3.6) we have

$$\begin{aligned}\|Ax\| &= B_0\big(U\big(E_x(1),E_x\big)\big)+\int_0^\sigma \Psi\Big(\int_r^\sigma z_x(s)ds\Big)dr\\ &\le (\mu+1)\Psi\Big(\int_0^1 z_x(s)ds\Big)\le (\mu+1)\psi(\|c\|_1)\epsilon|a|T_1\end{aligned}$$

and therefore by (3.7), we get

$$\|Ax\|\le T_1=\|x\|. \tag{3.10}$$

On the other hand, if condition (H9) holds, then we work with (3.5) and get (3.10).

Now, conditions (H3) and (H4) imply that the function

$$Q(t):=\int_\alpha^t \frac{dr}{\psi\Big(\big(\int_r^t c(s)ds\big)^{-1}\Big)}+\int_t^\beta \frac{dr}{\psi\Big(\big(\int_t^r c(s)ds\big)^{-1}\Big)},$$

is well defined on the interval $J$ and it takes a positive minimum on it say $2q$. (Also an upper bound of $Q$ is $(\beta-\alpha)/\psi(1/\|c\|_1)$. Keep in mind the monotonicity of the function $\psi$.) Note that

$$\begin{aligned}Q(\alpha)&=\int_\alpha^\beta \frac{dr}{\psi\Big(\big(\int_\alpha^r c(s)ds\big)^{-1}\Big)}(\ge 2q),\\ Q(\beta)&=\int_\alpha^\beta \frac{dr}{\psi\Big(\big(\int_r^\beta c(s)ds\big)^{-1}\Big)}(\ge 2q).\end{aligned}$$

Let $b:=(\beta_1,\dots,\beta_n)$ be the vector in the second condition in (H5) and consider any $M$ such that

$$M\ge \frac{1}{\phi(q|b|\gamma)}. \tag{3.11}$$

From the second condition in (H5) there is a $R>0$ such that

$$u\in\mathbb{R}_+\text{and}\quad u_j\ge R,\quad j=1,2,\dots,n$$

implies that

$$f(t,u)\ge M\Phi(\langle b,u\rangle)\ge M\Phi(R|b|)\ge \Phi\Big(\frac{R|b|}{\psi(1/M)}\Big), \tag{3.12}$$

because of (1.1).

Define $T_2:=R/\gamma$ and take any $x\in\mathbf{K}$ with $\|x\|=T_2$. Then by Lemma 2.1,

$$x(g_j(s))\ge \min\{g_j(s),\quad 1-g_j(s)\}T_2\ge \gamma T_2=R,\quad j=1,2,\dots,n$$

for all $s\in J$ and so

$$f(s,x(g_1(s)),x(g_2(s)),\dots,x(g_n(s)))\ge \Phi\Big(\frac{R|b|}{\psi(1/M)}\Big),\quad s\in J.$$

Next we distinguish three cases:

Case (i) $\sigma < \alpha$. From (3.5), (1.1), and (3.12), we obtain

$$\begin{aligned}
\|Ax\| &\geq \int_\sigma^1 \Psi\Big(\int_\sigma^r z_x(s)ds\Big)dr \geq \int_\alpha^\beta \Psi\Big(\Phi\Big(\frac{R|b|}{\psi(1/M)}\Big)\int_\alpha^r c(s)ds\Big)dr \\
&\geq \int_\alpha^\beta \Psi\Big(\Phi\Big(\frac{R|b|}{\psi\big(\frac{1}{M}\big)\psi\Big(\big(\int_\alpha^r c(s)ds\big)^{-1}\Big)}\Big)\Big)dr \\
&= \frac{R|b|}{\psi(1/M)}Q(\alpha) \geq \frac{R|b|}{\psi(1/M)}q
\end{aligned}$$

and so from the choice of $M$ we obtain

$$\|Ax\| \geq T_2 = \|x\|. \tag{3.13}$$

Case (ii) $\alpha \leq \sigma \leq \beta$. Adding relations (3.4) and (3.5), we obtain

$$\begin{aligned}
2\|Ax\| &\geq \int_0^\sigma \Psi\Big(\int_r^\sigma z_x(s)ds\Big)dr + \int_\sigma^1 \Psi\Big(\int_\sigma^r z_x(s)ds\Big)dr \\
&\geq \int_\alpha^\sigma \Psi\Big(\Phi\Big(\frac{R|b|}{\psi(1/M)}\Big)\int_r^\sigma c(s)ds\Big)dr + \int_\sigma^\beta \Psi\Big(\Phi\Big(\frac{R|b|}{\psi(1/M)}\Big)\int_\sigma^r c(s)ds\Big)dr \\
&\geq \int_\alpha^\sigma \Psi\Big(\Phi\Big(\frac{R|b|}{\psi(1/M)\psi\Big(\big(\int_r^\sigma c(s)ds\big)^{-1}\Big)}\Big)\Big)dr \\
&\quad + \int_\sigma^\beta \Psi\Big(\Phi\Big(\frac{R|b|}{\psi(1/M)\psi\Big(\big(\int_\sigma^r c(s)ds\big)^{-1}\Big)}\Big)\Big)dr \\
&= \frac{R|b|}{\psi(1/M)}Q(\sigma) \geq \frac{R|b|}{\psi(1/M)}2q
\end{aligned}$$

and by the choice of $M$ (see (3.11)) we, again, obtain (3.13)
Case (iii) $\sigma > \beta$. In this case we use relation (3.4) as exactly in case (i) and, finally, obtain (3.13).

Since the operator $A$ is obviously completely continuous, inequalities (3.10) and (3.13) imply that Theorem 1.1 applies with $\Omega_1$ and $\Omega_2$ being the open balls in $C$ with center the origin and radius $T_1$ and $T_2$ respectively. □

**Theorem 3.2.** *Assume that the function $f$ is (upper) bounded and the conditions (H1)–(H4), (H6)$_1$ and (H7) are satisfied. Then the boundary-value problem* (1.2), (1.3) *admits a positive solution.*

*Proof.* Assume that $R_1$ is a bound of $f$. Take any $x \in \mathbf{K}$ with

$$\|x\| = S_1 := B_0\big[\Psi(\|c\|_1 R_1)\big] + \Psi(\|c\|_1 R_1).$$

Then from (3.4) we obtain

$$\begin{aligned}
\|Ax\| &= B_0\big[U\big(E_x(1), E_x\big)\big] + \int_0^\sigma \Psi\Big[\int_r^\sigma z_x(s)ds\Big]dr \\
&\leq B_0\big[\Psi(E_x(1))\big] + \Psi\big(E_x(1)\big) \\
&\leq B_0\big[\Psi(\|c\|_1 R_1)\big] + \Psi(\|c\|_1 R_1) \\
&= S_1 = \|x\|,
\end{aligned}$$

for all $x$ with $\|x\| = S_1$.

Now we let any $M > 0$ satisfying (3.11), but $|a|$ in place of $|b|$. From condition $(\mathrm{H6})_1$ it follows that there is a $S_2 > 0$ with $S_2 < S_1$ such that each $u := (u_1, \dots, u_n)$ with $0 \le u_j \le S_2$, satisfies relation (3.12), but with $|a|$ in the place of $|b|$. Take an $x \in \mathbf{K}$ with $\|x\| = S_2$. Then proceed as in the second part of Theorem 3.1. $\square$

**Theorem 3.3.** *Assume that the conditions (H1)–(H4), $(H6)_1$ and (H7) are satisfied and the function $f(t, u_1, u_2, \dots, u_n)$ is not (upper) bounded with respect to a variable $u_{j_0}$. If the condition $(H6)_2$ is satisfied with respect to the index $j_0$ and at least one of the conditions (H10), (H11) is satisfied, then the boundary value problem* (1.2), (1.3) *admits a positive solution.*

*Proof.* Consider an $\epsilon > 0$ such that

$$\epsilon \le \frac{1}{\|c\|_1}\phi\Big(\frac{1}{|a|(\rho+1)}\Big),$$

where $a$ is the vector in condition $(\mathrm{H6})_2$. From this condition it follows that there is a $P > 0$ such that for all $t \in I$ and $u \in \mathbb{R}_+$ the inequality $u_{j_0} \ge P$ implies

$$f(t,u) \le \epsilon\Phi(\langle a, u\rangle),$$

for all $t \in I$. Since $f$ is unbounded with respect to the variable $u_{j_0}$, there is a $S_1 > P$ such that if $0 \le u_{j_0} \le S_1$ then

$$\begin{aligned} f(t, u_1, \dots, u_{j_0}, \dots, u_n) &\le \sup_{t\in I} f(t, u_1, \dots, S_1, \dots, u_n) \\ &\le \epsilon\Phi(\alpha_1 u_1 + \dots + \alpha_{j_0} u_{j_0} + \dots + \alpha_n u_n), \end{aligned}$$

for all $t \in I$ and $u_i \ge 0$, $i \ne j_0$. For all $x \in \mathbf{K}$ with $\|x\| = S_1$ it holds

$$\begin{aligned} \Psi\big(E_x(1)\big) &\le \Psi\big(\|c\|_1 \epsilon\Phi(|a|S_1)\big) \\ &= \Psi\big(\phi(\psi(\|c\|_1\epsilon))\Phi(|a|S_1)\big) \\ &\le \Psi\big(\Phi[|a|\psi(\|c\|_1\epsilon)S_1]\big) \\ &= |a|\psi(\|c\|_1\epsilon)S_1\,. \end{aligned}$$

Assume that (H10) holds. Then for some $\rho > 0$ from Lemma 2.2 and relation (3.4) we obtain

$$\begin{aligned} \|Ax\| &\le B_0\big[U\big(E_x(1), E_x\big)\big] + \int_0^\sigma \Psi\big[E_x(\sigma)\big]\,dr \\ &\le (\rho+1)\Psi\big[E_x(1)\big] \\ &\le |a|(\rho+1)\psi(\|c\|_1\epsilon)S_1 \le S_1, \end{aligned}$$

because of the choice of $\epsilon$. So we have $\|Ax\| \le \|x\|$ for all $x$ with $\|x\| = S_1$.

If (H11) holds, we use (3.5) and obtain the same result. The rest of the proof is similar as that of Theorem 3.2. $\square$

## 4. Examples

**Example 4.1.** Consider the boundary-value problem

$$\begin{gathered} [(\lambda_1|x'| + \lambda_2|x'|^2)x']' + c(t)[\Lambda_1 x(t/2) + \Lambda_2 x(t)]^4 = 0, \quad t \in [0,1], \\ x(0) - e^{x'(0)} + 1 = 0, \quad x(1) + [x'(1)]^{1/3} = 0, \end{gathered}$$

where $\lambda_i, \Lambda_i > 0$ for $i = 1, 2$ and the function $c$ satisfies condition (H3). It is easy to see that Theorem 3.1 is applicable, where here we have

$$\Phi(u) := (\lambda_1|u| + \lambda_2|u|^2)u, \quad \phi(u) := \min\{u^2, u^3\},$$
$$B_0(u) = e^u - 1, \quad B_1(u) = u^{1/3}$$

and $a = b$ is the vector with coordinates $(\Lambda_1, \Lambda_2)$. Note that (H8) is satisfied.

**Example 4.2.** Consider the boundary-value problem

$$[(\lambda_1|x'| + \lambda_2|x'|^2)x']' + c(t)[\Lambda_1 x(t/2) + \Lambda_2 x(t)] = 0, \quad t \in [0, 1],$$
$$x(0) - \lambda\big(x'(0) + \sin[x'(0)]\big) = 0, \quad x(1) + [x'(1)]^{1/3} = 0,$$

where $\lambda > 0$ and the other coefficients are as in Example 4.1. It is not hard to see that Theorem 3.3 applies. Here one can get $j_0 = 1$, or 2. Also (H10) holds.

**Example 4.3.** Consider the boundary-value problem

$$x'' + c(t)(1 + e^{-x'})\Big[\sin\Big(\Lambda_1 x(t/2) + \Lambda_2 x(t)\Big)\Big]^{2/3} = 0, \quad t \in [0, 1], \tag{4.1}$$

associated with any of the three pairs of conditions (1.2)–(1.5), where $\lambda, \Lambda_1, \Lambda_2 > 0$ and the function $c$ is continuous. Here (4.1) depends on the first derivative of the solution. Thus in order to investigate existence of solutions one has to follow a standard method where the space $C^1(I, \mathbb{R})$ should be taken into account. Nevertheless, we can easily see that (4.1) can be written as

$$[\Phi(x')]' + a(t)\Big[\sin\Big(\Lambda_1 x(t/2) + \Lambda_2 x(t)\Big)\Big]^{2/3} = 0, \quad t \in [0, 1], \tag{4.2}$$

where $\Phi(u) := \ln[\frac{1}{2}(1 + e^u)]$. We claim that for all $u, v > 0$ it holds

$$\frac{\Phi(uv)}{\Phi(u)} \geq v, \tag{4.3}$$

thus $\Phi$ is a sup-multiplicative-like function, with a supporting function $\phi(v) = v$. To prove the claim it is sufficient to show that the function

$$h(u) := 1 + e^{uv} - \frac{1}{2^{v-1}}(1 + e^u)^v, \quad u \geq 0$$

takes nonnegative values. Indeed, we have $h(0) = 0$ and

$$h'(u) = ve^{uv} - \frac{v}{2^{v-1}}(1 + e^u)^{v-1}.$$

Clearly is sufficient to show that $h'(u) \geq 0$, or equivalently, that

$$\eta(u) := 2^{v-1}(1 + e^u) - (1 + e^{-u})^v \geq 0.$$

The latter is true since $\eta$ is increasing and it vanishes at zero. Thus (4.3) holds. Now we can apply Theorem 3.2 to conclude the existence of solutions of the problem (4.2)–(1.3) provided that the functions $B_0$, and $B_1$ satisfy condition (H7).

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