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# TRIPLE POSITIVE SOLUTIONS FOR THE Φ-LAPLACIAN WHEN Φ IS A SUP-MULTIPLICATIVE-LIKE FUNCTION

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Abstract. The existence of triple positive solutions for a boundary-value problem governed by the Φ-Laplacian is investigated, when Φ is a so-called sup-multiplicative-like function (in a sense introduced in [22]) and the boundary conditions include nonlinear expressions at the end points (as in [21, 28]). The Leggett-Williams fixed point theorem in a cone is used. The results improve and generalize known results given in [21].

## 1. Introduction

We call *sup-multiplicative-like function* (SML in short) an odd homeomorphism $\Phi$ of the real line $\mathbb{R}$ onto itself for which there exists a homeomorphism $\phi$ of $\mathbb{R}^+ := [0, +\infty)$ onto $\mathbb{R}^+$ such that for all $v_1, v_2 \geq 0$ it holds

$$\phi(v_1)\Phi(v_2) \leq \Phi(v_1 v_2).$$

We then say that $\phi$ *supports* $\Phi$. This is a meaning introduced in [22] and some properties of it were given therein. In Section 2 we shall present more properties by connecting the meaning of SML functions with the literature. In particular we shall see the relation of this meaning with the "uniform quietness at zero" introduced in [24].

It is clear that any sup-multiplicative function is a SML function with $\phi = \Phi$. Also any function of the form

$$\Phi(u) := \sum_{j=0}^{k} c_j |u|^{r_j} u, \quad u \in \mathbb{R}$$

is SML, provided that $0 < r_0 < r_1 < \cdots < r_k$, $c_k c_0 \neq 0$ and $0 \leq c_j, \quad j = 0, 1, \ldots, k$. Here a supporting function is defined by

$$\phi(u) := \min\{u^{r_k+1}, \quad u^{r_0+1}\}, \quad u \geq 0.$$

Let $\Phi$ be a differentiable SML function and, for each $i = 1, 2, \ldots, n$, let $g_i : [0, 1] \to [0, 1]$ be measurable functions.

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In this paper we investigate when there exist triple positive solutions of the one-dimensional differential equation with deviated arguments of the form

$$[\Phi(x')]' + p(t)f(t, x(g_1(t)), x(g_2(t)), \dots, x(g_n(t))) = 0, \quad \text{a. a.} \quad t \in I := [0,1], \tag{1.1}$$

which satisfy one of the following three pairs of conditions

$$x(0) - B_0(x'(0)) = 0, \quad x(1) + B_1(x'(1)) = 0, \tag{1.2}$$

$$x(0) - B_0(x'(0)) = 0, \quad x'(1) = 0, \tag{1.3}$$

$$x'(0) = 0, \quad x(1) + B_1(x'(1)) = 0. \tag{1.4}$$

When the leading function $\Phi$ is of the form $\Phi(u) := |u|^{m-2}u$, equation (1.1) comes from the nonautonomous $m$-Laplacian elliptic equation in the $n$-dimensional space which has radially symmetric solutions. Also equation (1.1) is generated from an equation of the form

$$x'' + q(x')f(t, x) = 0,$$

where $\inf\{q(u) : |u| \le r\} > 0$, for all $r > 0$, by setting

$$\Phi(u) := \int_0^u \frac{d\xi}{q(\xi)}.$$

The problem of existence of positive solutions for boundary value problems generated by applications in applied mathematics, physics, mechanics, chemistry, biology, etc., and described by ordinary or functional differential equations was extensively studied in the literature, see the bibliography in this article. Most of these works make use of the well known Leggett-Williams Fixed Point Theorem [17, 27], since it may also provide information for the multiplicity of the solutions. Boundary value problems with boundary conditions of the form (1.2)–(1.4) were first discussed in [18], where a problem of the form

$$x'' + f(t, x) = 0, \quad t \in (0,1)$$

is considered associated with the boundary conditions

$$ax(0) - bx'(0) = 0, \quad cx(1) + dx'(1) = 0, \tag{1.5}$$

where all the coefficients $a, b, c, d$ are positive reals. Section 6 in [18] is devoted to boundary value problems with retarded arguments associated with Dirichlet boundary conditions.

In [16] the existence of positive solutions of the equation

$$x'' + c(t)f(x) = 0, \quad t \in (0,1)$$

associated with the conditions (1.5), was investigated. Motivated by [16] Wang [28] considered the function $g(u) = |u|^{p-2}u$, $p > 1$ and he studied the boundary-value problem

$$(g(x'))' + c(t)f(x) = 0, \quad t \in (0,1) \tag{1.6}$$

associated with the boundary conditions of the form (1.2)–(1.4), where $B_0$ and $B_1$ are both nondecreasing, continuous, odd functions defined on the whole real line and at least one of them is sub-linear. The function $c$ satisfies an integral condition through the inverse function of $g$. An analog condition will also be assumed in this paper.

In [15], where an equation of the form (1.1) was discussed (but without deviating arguments and with simple Dirichlet conditions), the leading factor depends on an odd homeomorphism $\Phi$, which, in order to guarantee the nonexistence of solutions, it satisfies the condition

$$\limsup_{u\to+\infty} \frac{\Phi(uv)}{\Phi(u)} < +\infty,$$

for all $v > 0$. But it is clear that such a condition is not enough. Instead, one should adopt the condition

$$\sup_{u>0} \frac{\Phi(uv)}{\Phi(u)} < +\infty. \tag{1.7}$$

In Section 2 we shall see how this condition is related to the meaning of SML functions and quietness at zero, see, [23, 24].

Our results extend and improve the results given in [21]. Indeed, we show existence of solutions under rather mild conditions on the functions $B_0$ and $B_1$ (they are not necessarily sublinear), where the Leggett-Williams Fixed Point Theorem on (topologically) closed cones in Banach spaces is applied. In [29] the same conditions were imposed to a one dimensional $p$-Laplacian differential equation, where the derivative affects the response function.

Our paper is organized as follows: In Section 2 we present some properties of the sup-multiplicative-like functions. In Section 3 we give some auxiliary facts needed in the sequel. The main results are stated in Section 4. The article closes with a specific case and an application in Section 5, where a retarded boundary-value problem is given with no sublinear functions $B_0, B_1$. It is proved that there exist constants $a, b, c$, with $0 < a < b < c$ and three positive solutions $x_1, x_2, x_3$ such that $\|x_j\| < c$, $\quad j = 1, 2, 3$ and $\|x_1\| \le a < \|x_2\|$ and for a subinterval $J$ of $I$ it holds $\inf_{s\in J} x_2(s) < b < \inf_{s\in J} x_3(s)$. In particular we notice that the constants are chosen uniformly with respect to the retardation.

## 2. On Sup-Multiplicative-Like functions

In this section we shall present some properties of the SML functions. And although most of them are exhibited in [22], we shall repeat them here and shall present new ones. In particular we shall see how these functions are connected with the uniformly quiet at zero functions introduced in [24]. See, also [23].

¿From the definition it follows that a SML function $\Phi$ and any corresponding supporting function $\phi$ are increasing unbounded functions vanishing at zero and moreover their inverses $\Psi$ and $\psi$ respectively are increasing unbounded and such that

$$\Psi(w_1w_2) \le \psi(w_1)\Psi(w_2),$$

for all $w_1, w_2 \ge 0$. From this relation it follows easily that, for all $M > 0$ and $u \ge 0$, it holds

$$M\Phi(u) \ge \Phi\Big(\frac{u}{\psi\big(\frac{1}{M}\big)}\Big). \tag{2.1}$$

**Proposition 2.1.** *If $\Phi_1$ and $\Phi_2$ are two SML functions, then so do the functions $\Phi$ defined by*

(i) $\Phi := \Phi_1 + \Phi_2$
(ii) $\Phi := \Phi_1|\Phi_2|$
(iii) $\Phi := \Phi_1 \circ \Phi_2$,
(iv) $\Phi(u) := \begin{cases} \Phi_1(u)|[\Phi_2(u^{-1})]^{-1}|, & \text{if } u \neq 0 \\ 0, & \text{if } u = 0\,. \end{cases}$

*Proof.* Indeed, let $\phi_1, \phi_2$ be functions which support the SML functions $\Phi_1$ and $\Phi_2$, respectively. Then, for all $u, v \geq 0$, we have

$$\big[\Phi_1 + \Phi_2\big](uv) \geq \Phi_1(u)\phi_1(v) + \Phi_2(u)\phi_2(v) \geq \big[\Phi_1 + \Phi_2\big](u)\phi(v),$$

where $\phi(v) := \min\{\phi_1(v), \phi_2(v)\}$. This proves (i). Also

$$\Phi_1(uv)\Phi_2(uv) \geq \Phi_1(u)\phi_1(v)\Phi_2(u)\phi_2(v) \geq \big(\Phi_1(u)\Phi_2(u)\big)\phi(v),$$

where $\phi(v) := \phi_1(v)\phi_2(v)$. This proves (ii). Moreover we have

$$\Phi_1\big(\Phi_2(uv)\big) \geq \Phi_1\big(\Phi_2(u)\phi_2(v)\big) \geq \big(\Phi_1\big(\Phi_2(u)\big)\phi(v),$$

where $\phi(v) := \phi_1\big(\phi_2(v)\big)$, which proves (iii). Finally, we have

$$\frac{\Phi(uv)}{\Phi(u)} = \frac{\Phi_1(uv)}{\Phi_2(u^{-1}v^{-1})}.\frac{\Phi_2(u^{-1})}{\Phi_1(u)} = \frac{\Phi_1(uv)}{\Phi_1(u)}.\frac{\Phi_2(u^{-1})}{\Phi_2(u^{-1}v^{-1})} \geq \phi_1(v)\phi_2(v),$$

which completes the proof. □

To proceed we shall repeat the meaning of the so called *uniformly quiet at zero* functions introduced in [24]. But first we start with a definition from [23].

**Definition 2.2.** A continuous function $f : [0, +\infty) \to \mathbb{R}^+$, with $f(x) > 0$, when $x > 0$, is said to be *quiet at zero*, if for any pair of sequences $(x_n)$, $(y_n)$ with $0 \leq x_n \leq y_n$, $n = 1, 2, \dots$, which converge to zero, it holds

$$f(x_n) = O(f(y_n)).$$

An equivalent definition is the following (see [23]): A continuous function $f :[0, +\infty) \to \mathbb{R}^+$, with $f(x) > 0$, when $x > 0$ is quiet at zero, if and only if for each $T > 0$ there is a $\mu \geq 1$ such that for all $\tau \in (0, T)$ it holds

$$\sup\{f(x) : x \in [0, \tau]\} \leq \mu \inf\{f(x) : x \in [\tau, T]\}.$$

Now, a continuous function $f : [0, +\infty) \to \mathbb{R}^+$, with $f(x) > 0$, when $x > 0$, is *uniformly quiet at zero*, if it is quiet at zero and the constant $\mu$ works uniformly with respect to all $T > 0$. See [24]. We shall show the following result.

**Theorem 2.3.** *Let $\Phi$ be a differentiable odd homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, whose the derivative $\Phi'$ is uniformly quiet at zero. Then $\Phi$ is a SML function, if and only if it satisfies relation* (1.7).

*Proof.* Observe that (1.7) is equivalent to

$$\inf_{u>0} \frac{\Phi(uv)}{\Phi(u)} > 0, \quad v > 0.$$

Indeed, if for each $v > 0$ we have

$$\sup_{u>0} \frac{\Phi(uv)}{\Phi(u)} =: H(v) < +\infty,$$

then

$$\inf_{u>0} \frac{\Phi(uv)}{\Phi(u)} = \frac{1}{H(1/v)} =: h(v) > 0.$$

Clearly, the function $h$ is increasing. We claim that $h$ is unbounded. Indeed, take any $M > 1$ and let $\mu$ be the constant in the definition of the uniform quieteness at zero of the derivative $\Phi'$. We let $v := 1 + (M-1)\mu$, which is greater than 1. Then, for each $u > 0$, we have $uv > u$ and

$$\begin{aligned}(M-1)\Phi(u) &= (M-1)\int_0^u \Phi'(s)ds \le (M-1)u \sup_{s\in[0,u]} \Phi'(s)\\ &\le (M-1)\mu u \inf_{s\in[u,uv]} \Phi'(s) = u(v-1) \inf_{s\in[u,uv]} \Phi'(s)\\ &\le \int_u^{uv} \Phi'(s)ds = \Phi(uv) - \Phi(u),\end{aligned}$$

from which we get $M\Phi(u) \le \Phi(uv)$. Thus we have $M \le h(v)$, which proves the claim.

Now the continuous function $\phi$ defined by

$$\phi(0) = 0, \quad \phi(v) := \frac{1}{v}\int_0^v h(s)ds, \quad v > 0$$

is (strictly) increasing and unbounded. Indeed, the first is obvious. If it is bounded, then, for some $K > 0$ and all $0 < r < v$, it holds

$$Kv \ge \int_0^v h(s)ds \ge \int_r^v h(s)ds \ge (v-r)h(r)$$

and so

$$h(r) \le \frac{Kv}{v-r} \to K, \quad \text{as} \quad v \to +\infty.$$

This implies that $h$ is bounded, a contradiction.

Since $\phi(v) \le h(v)$, for all $v \ge 0$, the function $\phi$ supports $\Phi$, thus the latter is a sup-multiplicative-like function. The *only if* part is evident. □

## 3. Preliminaries and some basic lemmas

To formulate the problem, we let $C(I, \mathbb{R})$ be the set of all continuous functions $x : I \to \mathbb{R}$ endowed with the sup-norm

$$\|x\| := \sup_{t\in I} |x(t)|.$$

Let, also, $L_\infty(I, \mathbb{R})$ be the B-space of all (Lebesgue) measurable functions $z : I \to \mathbb{R}$ such that

$$|z|_\infty := \operatorname*{ess\,sup}_{t\in I} |z(t)| < +\infty.$$

(Here $\operatorname{ess\,sup} |z(t)|$ stands for the essential supremum of $|z|$, namely the infimum of all $N > 0$ such that the set of all $t \in I$ satisfying $|z(t)| > N$ has measure zero.)

In the sequel we assume that

(H1) $\Phi : \mathbb{R} \to \mathbb{R}$ is a differentiable SML function.

Let $\phi$ be a corresponding supporting function of $\Phi$. Then we let $\Psi$ and $\psi$ be the inverses of $\Phi$ and $\phi$, respectively. Note that both these functions are defined on the whole real line.

For the other statements of the problem we assume the following:

(H2) $f(t, u)$, $(t, u) \in I \times \mathbb{R}^n$ is a real valued function measurable in the first variable and continuous in the second one. Moreover assume that for all $u \in \mathbb{R}^n$ with nonnegative coordinates it holds $f(\cdot, u) \in L_\infty(I, \mathbb{R})$ and $f(t, u) > 0$ for a.a. $t \in I$.

(H3) $p : I \to \mathbb{R}^+$ is a (Lebesgue) integrable function such that for some nontrivial subinterval $J := [\alpha, \beta]$ of $I$ it holds $p(t) > 0$ a.e. on $J$. We set
$$\|p\|_1 := \int_0^1 p(t)dt.$$
(H4) The functions $g_j : I \to I$, $j = 1, 2, \dots, n$ are measurable and such that
$$\gamma := \inf_{t\in J} \min\{t, \quad 1-t, \quad \min_{j=1,2,\dots,n}\{g_j(t), \quad 1-g_j(t)\}\} > 0,$$
where $J$ is the interval defined in (H3).

(H5) For each $i = 0, 1$ the function $B_i$ is continuous nondecreasing and such that $uB_i(u) \ge 0$, $u \in \mathbb{R}$.

As we stated above, to prove our main results we will use the Leggett-Williams Fixed Point Theorem, which we state here. First we give some notation.

**Definition 3.1.** A cone $\mathbb{P}$ in a real Banach space $\mathbb{E}$ is a nonempty closed subset of $\mathbb{E}$ such that

(i) $\kappa\mathbb{P} + \lambda\mathbb{P} \subset \mathbb{P}$, for all $\kappa$, $\lambda \ge 0$

(ii) $\mathbb{P} \cap (-\mathbb{P}) = \{0\}$.

Let $\mathbb{P}$ be a cone in $\mathbb{E}$. For any $r > 0$ and any cone $\mathbb{P}$ let
$$\mathbb{P}_r := \{y \in \mathbb{P} : \|y\| < r\}.$$
Then, $\overline{\mathbb{P}}_r$ is the closure of $\mathbb{P}_r$, i.e. the set $\{y \in \mathbb{P} : \|y\| \le r\}$.

Also, for all $\rho, r > 0$ and any real valued function $h$ defined on the cone $\mathbb{P}$, consider the set
$$\mathbb{P}(h; r, \rho) := \{y \in \overline{\mathbb{P}}_\rho : \quad r \le h(y)\}.$$

**Theorem 3.2** (Leggett-Williams [17, 27])**.** *Let $T : \overline{\mathbb{P}}_c \to \overline{\mathbb{P}}_c$ be a completely continuous operator and $h : \mathbb{P} \to \mathbb{R}$ a nonnegative continuous concave function on $\mathbb{P}$ such that $h(y) \le \|y\|$ for all $y \in \overline{\mathbb{P}}_c$. Suppose that there exist numbers $a, b, c, d$, with $0 < a < b < d \le c$ and such that*

(i) *$\{x \in \mathbb{P}(h; b, d) : h(x) > b\} \ne \emptyset$ and $h(T(x)) > b$, for all $x \in \mathbb{P}(h; b, d)$,*

(ii) *$\|T(x)\| < a$, for all $x \in \overline{\mathbb{P}}_a$,*

(iii) *$h(Ty) > b$, for $y \in \mathbb{P}(h; b, c)$ with $\|Ty\| > d$.*

*Then $T$ has at least three fixed points $y_1$, $y_2$, and $y_3$ in $\overline{\mathbb{P}}_c$ such that*
$$\|y_1\| \le a < \|y_2\| \quad \text{and} \quad h(y_2) < b < h(y_3).$$

The following result follows easily from the concavity.

**Lemma 3.3.** *Any concave continuous function $x : I \to \mathbb{R}^+$ satisfies the inequality*
$$x(t) \ge \min\{t, \quad 1-t\}\|x\|.$$

The following result (which is proved in [22]) plays an important role in our approach.

**Lemma 3.4.** *Suppose the functions $B_0, B_1$ satisfy condition (H5). Then for each $\Theta \ge 0$ and $y \in C(I, \mathbb{R})$, with $0 \le y(t) \le \Theta$, $\quad t \in I$, there exists a unique real number $U(\Theta, y)$, which depends continuously on $\Theta, y$ and it satisfies*
$$0 \le U(\Theta, y) \le \Psi(\Theta),$$
$$\Omega\Big(U(\Theta, y)\Big) = 0,$$

*where*

$$\Omega(w) := B_0(w) + B_1\big[\Psi\big(\Phi(w) - \Theta)\big] + \int_0^1 \Psi\big[\Phi(w) - y(s)\big]ds, \quad w \geq 0.$$

## 4. Main results

Following the lines of [22] we can see that a function $x \in C(I, \mathbb{R})$ solves the boundary-value problem (1.1), (1.2), if and only if it solves the equation

$$x(t) = (Ax)(t), \tag{4.1}$$

where $A$ is the operator defined on the set $C$ by the formula

$$(Ay)(t) = B_0\big[U\big(E_y(1), E_y\big)\big] + \int_0^t \Psi\big[\Phi\big(U(E_y(1), E_y)\big) - E_y(r)\big]dr, \tag{4.2}$$

or, equivalently, by the formula

$$\begin{aligned}(Ay)(t) = -B_1\Big( - \Psi\Big[E_y(1) - \Phi\big(U(E_y(1), E_y)\big)\Big]\Big)\\ + \int_t^1 \Psi\big[E_y(r) - \Phi\big(U(E_y(1), E_y)\big)\big]dr,\end{aligned} \tag{4.3}$$

where, for simplicity, we have set

$$E_y(t) := \int_0^t z_y(s)ds, \quad z_y(s) := p(s)f\big(s, y(g_1(s)), y(g_2(s)), \dots, y(g_n(s)).$$

Next, we provide sufficient conditions for the existence of solutions of the integral equation (4.1). Consider the set

$$\mathbb{K} := \big\{x \in C(I, \mathbb{R}^+) : x \quad \text{is concave}\big\},$$

and observe that it is a cone in $C$. Let $x \in \mathbb{K}$. For all $t$ it holds

$$(Ax)'(t) = \Psi\big(\Phi\big(U(E_x(1), E_x)\big) - E_x(t)\big),$$

thus $(Ax)'$ decreases. Moreover we have

$$\begin{gathered}(Ax)(1) = -B_1\big(\Psi\big[\Phi\big(U(E_x(1), E_x)\big) - E_x(1)\big]\big) \geq 0,\\ (Ax)(0) = B_0\big(U\big(E_x(1), E_x\big)\big) \geq 0.\end{gathered}$$

These facts together with Lemma 3.4 ensure that the function $(Ax)(\cdot)$ is nonnegative and concave; thus $Ax \in \mathbb{K}$.

Let $\sigma$ be the smallest point in $I$ satisfying

$$\Phi\big(U(E_x(1), E_x)\big) = E_x(\sigma).$$

It is clear that such a point exists because of Lemma 3.4. Then the maximum of $Ax$ is achieved at $\sigma$ and therefore we have

$$\|Ax\| = (Ax)(\sigma). \tag{4.4}$$

Moreover (4.2) becomes

$$(Ay)(t) = B_0\big[E_y(\sigma)\big)\big] + \int_0^t \Psi\big[\int_r^\sigma z_y(s)ds\big]dr, \tag{4.5}$$

while (4.3) takes the form

$$(Ay)(t) = -B_1\big( - \Psi\big[\int_\sigma^1 z_x(s)ds\big]\big) + \int_t^1 \Psi\big[\int_\sigma^r z_x(s)ds\big)\big]dr. \tag{4.6}$$

Now consider the boundary-value problem (1.1), (1.3). In this case the problem is equivalent to the operator equation (4.1), where $A$ is the completely continuous operator defined by

$$(Ax)(t) = B_0\big[\Psi\big(E_x(1)\big)\big] + \int_0^t \Psi\Big(\int_s^1 z_x(s)ds\Big).$$

For each $x \in \mathbb{K}$ the image $Ax$ is a nonnegative function and the derivative $(Ax)'$ is a non-increasing function. Thus it is concave and so $A$ maps $\mathbb{K}$ into $\mathbb{K}$. Also $(Ax)(\cdot)$ is a nondecreasing function, thus we have

$$\|Ax\| = (Ax)(1) = B_0\big[\Psi\big(E_x(1)\big)\big] + \int_0^1 \Psi\Big(\int_s^1 z_x(s)ds\Big).$$

Finally, let the boundary value problem (1.1), (1.4). In this case the problem is equivalent to the operator equation $Ax = x$, where $A$ is the completely continuous operator defined by

$$(Ax)(t) = -B_1\big[-\Psi(E_x(1))\big] + \int_t^1 \Psi(E_x(s))ds.$$

For each $x \in \mathbb{K}$ the image $(Ax)(\cdot)$ is a nonnegative function, having its first derivative non-increasing. Thus it is a concave function and so $A$ maps $\mathbb{K}$ into $\mathbb{K}$. Also $(Ax)(\cdot)$ is a non-increasing function, thus we have

$$\|Ax\| = (Ax)(0) = -B_1\big[-\Psi(E_u(1))\big] + \int_0^1 \Psi(E_x(s))ds.$$

Next we shall discuss only the problem (1.1)–(1.2), since the other extreme cases follow with obvious small modifications.

Condition (H3) implies that the function

$$Q(t) := \int_\alpha^t \frac{dr}{\psi\big(\big(\int_r^t p(s)ds\big)^{-1}\big)} + \int_t^\beta \frac{dr}{\psi\big(\big(\int_t^r p(s)ds\big)^{-1}\big)},$$

is well defined on the interval $J$ and it takes a positive minimum on it, say, $2m$. (An upper bound of $Q$ is $(\beta-\alpha)/\psi(1/\|p\|_1)$. Keep in mind the monotonicity of the function $\psi$.) Note that

$$Q(\alpha) = \int_\alpha^\beta \frac{dr}{\psi\Big(\big(\int_\alpha^r p(s)ds\big)^{-1}\Big)}(\geq 2m),$$

$$Q(\beta) = \int_\alpha^\beta \frac{dr}{\psi\Big(\big(\int_r^\beta p(s)ds\big)^{-1}\Big)}(\geq 2m).$$

Let $M(u)$, $u \geq 0$ be the function

$$M(u) := \sup\{|f(\cdot, v_1, v_2, \dots, v_n)|_\infty : \quad 0 \leq v_j \leq u, \quad j = 1, 2, \dots, n\}, \quad u \geq 0.$$

It is not hard to see that $M$ is a nondecreasing. Our first main result in this section is the following:

**Theorem 4.1.** *Suppose that conditions (H1)–(H4) are satisfied. Moreover assume that there are real numbers $a, b, c$ such that $0 < a < b < \frac{b}{\gamma} < c$ and satisfying the following conditions:*

(H6) *Let $G$ stand for either the function $B_0$ or $B_1$. Then*

$$G\big(\Psi\big(\|p\|_1 M(w)\big)\big) + \Psi\big(\|p\|_1 M(w)\big) < w,$$

*for $w = a$ and $w = c$.*

(H7) *For all $t \in J$ and $v_j \in [\gamma, 1/\gamma]$ it holds*

$$m\gamma\Psi\big(f(t, bv_1, bv_2, \dots, bv_n)\big) > b, \quad \textit{for} \quad a.a. t \in I.$$

*Then the boundary-value problem (1.1)–(1.2) has at least three positive solutions $x_1$, $x_2$, $x_3$ with $\|x_j\| \le c$, for $j = 1, 2, 3$ and such that*

$$\|x_1\| \le a < \|x_2\| \quad \textit{and} \quad \inf_{t\in J} x_2(t) < b < \inf_{t\in J} x_3(t).$$

*Proof.* We shall show that all conditions of Theorem 3.2 are satisfied. Let $x \in \overline{\mathbb{K}}_w$, where $w \in \{a, c\}$. Then we have

$$f(s, x(g_1(s), x(g_2(s)), \dots, x(g_n(s))) \le M(w), \quad \textit{for} \quad a.a. s \in I.$$

Hence

$$E_x(1) = \int_0^1 z_x(s)ds \le \|p\|_1 M(w)$$

and, because of Lemma 3.4, we get

$$\|Ax\| \le G\Big(\Psi\big(\|p\|_1 M(w)\big)\Big) + \Psi\big(\|p\|_1 M(w)\big) < w.$$

This and the concavity of $Ax$ imply that $A$ maps $\overline{\mathbb{K}}_c$ into $\overline{\mathbb{K}}_c$ and $\overline{\mathbb{K}}_a$ into $\mathbb{K}_a$, hence condition $(ii)$ of Theorem 3.2 is satisfied.

Consider the nonnegative continuous concave function $h$ defined by

$$h(x) := \inf_{t\in J} x(t), \quad x \in \mathbb{K}.$$

Since the set $\overline{\mathbb{K}}_{\frac{b}{\gamma}}$ contains the constant function $y(t) := b/\gamma$ and, moreover, it holds $h(y) = b/\gamma > b$, we have

$$\mathbb{K}(h; b, \frac{b}{\gamma}) \ne \emptyset.$$

Therefore, the first requirement of condition $(i)$ of Theorem 3.2 is satisfied.

Now, consider any $x \in \overline{\mathbb{K}}(h; b) \cap \overline{\mathbb{K}}_{\frac{b}{\gamma}}$; then we have

$$\|x\| \le b/\gamma \quad \text{and} \quad x(t) \ge b, \quad t \in J.$$

This implies that for all $s \in J$ and $j = 1, 2, \dots, n$ it holds

$$\frac{b}{\gamma} \ge x(g_j(s)) \ge \min\{g_j(s), \quad 1 - g_j(s)\}\|x\| \ge \gamma\|x\| \ge \gamma b.$$

Therefore, from condition (H7), we have

$$f(s, x(g_1(s)), x(g_2(s)), \dots, x(g_n(s))) \ge \Phi\big(\frac{b}{m\gamma}\big), \quad \text{for a. a. } s \in J \tag{4.7}$$

Now, we claim that

$$\|Ax\| > \frac{b}{\gamma}\,. \tag{4.8}$$

To prove this claim we distinguish three cases:

Case (i) $\sigma < \alpha$. From (4.4), (4.6), (4.7) and (2.1) we obtain

$$\begin{aligned}\|Ax\| &\geq \int_\sigma^1 \Psi\Big(\int_\sigma^r z_x(s)ds\Big)dr > \int_\alpha^\beta \Psi\Big(\Phi(\frac{b}{m\gamma})\int_\alpha^r p(s)ds\Big)dr\\ &\geq \int_\alpha^\beta \Psi\Big(\Phi\Big(\frac{b}{m\gamma\psi\big(\big(\int_\alpha^r p(s)ds\big)^{-1}\big)}\Big)\Big)dr\\ &= \frac{b}{m\gamma}Q(\alpha) \geq \frac{2b}{\gamma} > \frac{b}{\gamma}.\end{aligned}$$

Case (ii) $\alpha \leq \sigma \leq \beta$. From relations (4.4), (4.5), (4.6) and (4.7) we obtain

$$\begin{aligned}&2\|Ax\|\\ &\geq \int_0^\sigma \Psi\Big(\int_r^\sigma z_x(s)ds\Big)dr + \int_\sigma^1 \Psi\Big(\int_\sigma^r z_x(s)ds\Big)dr\\ &> \int_\alpha^\sigma \Psi\Big(\Phi(\frac{b}{m\gamma})\int_r^\sigma p(s)ds\Big)dr + \int_\sigma^\beta \Psi\Big(\Phi(\frac{b}{m\gamma})\int_\sigma^r p(s)ds\Big)dr\\ &\geq \int_\alpha^\sigma \Psi\Big(\Phi\Big(\frac{b}{m\gamma\psi\big(\big(\int_r^\sigma p(s)ds\big)^{-1}\big)}\Big)\Big)dr + \int_\sigma^\beta \Psi\Big(\Phi\Big(\frac{b}{m\gamma\psi\big(\big(\int_\sigma^r p(s)ds\big)^{-1}\big)}\Big)\Big)dr\\ &= \frac{b}{m\gamma}Q(\sigma) \geq \frac{2b}{\gamma}.\end{aligned}$$

Case (iii) $\sigma > \beta$. In this case we use relations (4.4) and (4.5) as in case (i) and obtain (4.8). This proves the claim.

Now by Lemma 3.3 and (4.8) we get

$$h(Ax) = \min_{t\in J}(Ax)(t) \geq \min_{t\in J}\min\{t, \quad 1-t\}\|Ax\| \geq \gamma A > b,$$

which shows that condition (i) of Theorem 3.2 is true.

Finally, we show that condition (iii) of Theorem 3.2 is satisfied. Indeed, according to Lemma 3.3, for every $x \in \overline{\mathbb{K}}(h; b) \cap \overline{\mathbb{K}}_c$, with $\|Ax\| > \frac{b}{\gamma}$, we have

$$h(Ax) = \inf_{t\in J}(Ax)(t) \geq \gamma\|Ax\| > b.$$

Consequently Theorem 3.2 is applicable, with $\mathbb{P} := \mathbb{K}$, $T := A$, $a, b$ the points as they are defined above and $d := \frac{b}{\gamma}$. The proof is complete. □

## 5. A specific case and an application

Consider a differentiable SML function $\Phi$ and the retarded differential equation of the form

$$[\Phi(x'(t))]' + f(x(kt)) = 0, \tag{5.1}$$

with $0 < k \leq 1$, associated with the boundary conditions (1.2).

**Theorem 5.1.** *Assume that $f$ is a nonnegative continuous increasing function such that $f(0) = 0$. Assume, also, that for $G = B_0$, or $G = B_1$ it holds*

$$\limsup_{w\to\zeta} \frac{f\Big(G(w) + w\Big)}{\Phi(w)} < 1 \tag{5.2}$$

*for each $\zeta = 0$ and $\zeta = +\infty$ and moreover*
$$\sup_{w>0} \frac{f(w)}{\Phi(512w)} > 1. \tag{5.3}$$
*If the conditions (H1) and (H5) are satisfied, then there exist real numbers $a, b, c$, with $a < b < c$ such that for each $k \in (0,1]$ there are solutions $x_1$, $x_2$, $x_3$ of the problem (5.1)–(1.2) satisfying the conclusion of Theorem 3.2.*

*Proof.* Fix any $k \in (0,1]$ and consider (5.1) associated, for instance, with the boundary condition (1.2). We are going to apply Theorem 4.1. To do it consider the interval $J := [1/4, 3/4]$. Then we obtain
$$\gamma = \inf_{t\in J} \min\{t, \quad 1-t, \quad kt, \quad 1-kt\} = \frac{1}{4} \quad \text{and} \quad m = \frac{1}{32}.$$
By (5.3) there exists $b_0 > 0$ such that
$$f(b_0) > \Phi(512 b_0). \tag{5.4}$$
Define $b := 4b_0$. From (5.2) it follows that there are $a', c'$ with
$$0 < a' < \Psi(f(b)) < \Psi(f(4b)) < c'$$
and such that
$$f(G(w) + w) < \Phi(w), \quad w \in \{a', c'\}.$$
Let $a, c$ be the positive real numbers satisfying $f(a) = \Phi(a')$ and $f(c) = \Phi(c')$. Then we have $0 < a < b < 4b < c$ and moreover
$$G(\Psi(f(w))) + \Psi(f(w)) < w, \quad w \in \{a, c\},$$
which means that condition (H6) is satisfied, since in (5.1) we have $p(t) = 1$ and $M(u) = f(u)$. Also from (5.4) it follows that, for all $v \in [1/4, 4]$, it holds
$$f(bv) \geq f(\frac{b}{4}) = f(b_0) > \Phi(512 b_0) = \Phi(\frac{b}{m\gamma}).$$
Therefore, Theorem 4.1 applies and the result follows. $\square$

**An application.** Consider the retarded differential equation
$$[\Phi(x'(t))]' + \sum_{j=0}^{\mu} \alpha_j x^{\tau_j}(kt) = 0, \tag{5.5}$$
where $k \in (0,1]$, $0 < \tau_0 < \tau_1 < \dots < \tau_\mu$ and
$$\Phi(u) = \begin{cases} \sum_{j=0}^{\xi} \gamma_j |u|^{r_j} u, & \text{if } |u| \leq 1, \\ \big(\sum_{j=0}^{\xi} \gamma_j\big) |u|^{\rho} u, & \text{if } |u| > 1. \end{cases}$$
Here $\mu, \xi$ are positive integers, $0 < r_0 < r_1 < \dots < r_\xi$, $0 < \rho$ and all the coefficients are nonnegative real numbers with $\gamma_0 \neq 0$. Associate Eq. (5.5) with the boundary conditions
$$x(0) - \sum_{j=0}^{\nu} \beta_j |x'(0)|^{\eta_j} x'(0) = 0, \quad x(1) + e^{x'(1)} = 0,$$
where $\nu$ is a positive integer and $\beta_j \geq 0$, with $\beta_\nu > 0$ and $0 < \eta_0 < \eta_1 < \dots < \eta_\nu$.

Assume that the inequalities $r_0 + 1 < \tau_0$, $(\eta_\nu + 1)\tau_\mu < \rho + 1$ and
$$\sum_{j=0}^{\mu} \alpha_j > \Big(\sum_{j=0}^{\xi} \gamma_j\Big)(512)^{\rho+1}$$

are true. Then it is not hard to see that Theorem 5.1 is applicable. (In this case the constant $b_0$ in the proof of Theorem 5.1 is taken equal to 1.) It is noteworthy that the functions

$$B_0(u) := \sum_{j=0}^{\nu} \beta_j |u|^{\eta_j} u \quad \text{and} \quad B_1(u) := e^u$$

are not sublinear, thus the results of [21] cannot be applied, even if the leading factor $\Phi$ has the classical form $|u|^{p-2}u$ of the $p$-Laplacian.

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