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EXACT MULTIPLICITY RESULTS FOR A p-LAPLACIAN POSITONE PROBLEM WITH CONCAVE-CONVEX-CONCAVE NONLINEARITIES

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Abstract. We study the exact number of positive solutions of a two-point Dirichlet boundary-value problem involving the p-Laplacian operator. We consider the case $p = 2$ and the case $p > 1$, when the nonlinearity f satisfies $f(0) > 0$ (positone) and has three distinct simple positive zeros and such that f'' changes sign exactly *twice* on $(0, \infty)$. Note that we may allow f'' to change sign more than twice on $(0, \infty)$. We also present some interesting examples.

1. Introduction

In this paper we present *exact* multiplicity results of *positive solutions* for the nonlinear two-point Dirichlet boundary-value problem

$$\begin{gathered} -(\varphi_p(u'(x)))' = \lambda f(u(x)), \ -1 < x < 1, \\ u(-1) = u(1) = 0, \end{gathered} \tag{1.1}$$

where $p > 1$, $\varphi_p(y) = |y|^{p-2}y$ and $(\varphi_p(u'))'$ is the one-dimensional p-Laplacian, $\lambda > 0$ and f is a concave-convex-concave nonlinearity. Precise conditions are listed below.

This paper is intended as a second part of a previous paper by the present authors [3]. In fact, whereas the previous paper was a study of (1.1) with $f \in C^2[0, \infty)$ satisfying $f(0) = 0$ and has *two* distinct simple positive zeros $b < c$ and such that f'' changes sign exactly *twice* on $(0, \infty)$, here we wish to complete the picture by studying the same sort of (1.1) but with the nonlinearity $f \in C^2[0, \infty)$ satisfying instead, $f(0) > 0$ (positone) and has *three* distinct simple positive zeros $a < b < c$ and such that f'' changes sign exactly *twice* on $(0, \infty)$.

Note that besides being complementary to our previous paper [3], the present article contains an important originality which deserves to be mentioned in this introduction. A familiar feature related to positive solutions, say u, of a one-dimensional Dirichlet boundary-value problem with the p-Laplacian differential operator, is that we think that the interior zero set of the derivative u' is a *connected set.* (That is, if $u \in C^1[-1, 1]$ is a positive solution and $Z(u) = \{x \in [-1, 1] : u'(x) = 0\}$ then $Z(u) \cap (-1, 1)$ is either a single point or a closed interval.) For the particular case

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$p = 2$, it is easy to prove that $Z(u) \cap (-1, 1)$ is indeed a connected set, but what about the more general case $p > 1$? None known results in the literature prove or disprove this feature. This paper provide an example which disproves this fact. Indeed, for some $p > 1$, $p \neq 2$, *we have obtained some positive solutions of* (1.1) *such that the interior zero set of their derivative is not a connected set.* Note that this situation is not known even for (1.1) when $f(0) = 0$ in our previous paper Addou and Wang [3].

For $p = 2$, $(\varphi_p(u'))' = u''$, and (1.1) reduces to

$$\begin{gathered} -u''(x) = \lambda f(u(x)), \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \end{gathered} \tag{1.2}$$

and several *exact* multiplicity results are known when f vanishes three times on $(0, \infty)$, see [6, 8, 9, 10, 11]. However, in all of them, f'' changes sign exactly once on $(0, \infty)$. In fact, first studies go back to Smoller and Wasserman [9] in which they studied exact multiplicity results of (classical) positive solutions of (1.2) for cubic-polynomial nonlinearities $f(u) = -(u - a)(u - b)(u - c)$ satisfying $0 < a < b < c$, $c > 2b - a$ ($\Leftrightarrow \int_a^c f(u)du > 0$), and a certain condition; see also Wang [10]. One can note that, here, $f''(u)$ changes sign exactly once on $(0, \infty)$. Subsequently, Wang and Kazarinoff [12] and Wang [11] studied (1.2) when f is a cubic-like nonlinearity. In particular, Wang and Kazarinoff proved the next theorem. Define $F(u) = \int_0^u f(t)dt$.

Theorem 1.1 ([12, Theorem 1 and Remark 2])**.** *Suppose $f \in C^2[0, \infty)$ and there exist $0 < a < b < c$ such that the following conditions are satisfied:*

$$f(a) = f(b) = f(c) = 0; \tag{1.3}$$

$$\begin{gathered} f(u) > 0 \quad \text{for } u \in (0, a), \\ f(u) > 0 \quad \text{for } u \in (b, c), \\ f(u) < 0 \quad \text{for } u \in (a, b) \cup (c, \infty); \end{gathered} \tag{1.4}$$

$$\int_a^c f(u)du > 0; \tag{1.5}$$

there exists a unique $\beta \in (b, c)$ defined by $\int_a^\beta f(u)du = 0$ and such that $2F(a) - \beta f(\beta) < 0$;
there exists $r \in (0, c)$ such that $f''(u) > 0$ for $0 < u < r$ and $f''(u) < 0$ for $r < u < c$.
Then, there exists $\lambda_0 > 0$ such that

(i) *for $0 < \lambda < \lambda_0$, problem* (1.2) *has exactly one positive solution u_0 satisfying $0 < \|u_0\| < a$,*
(ii) *for $\lambda = \lambda_0$, problem* (1.2) *has exactly two positive solutions $u_0 < u_1$ satisfying $0 < \|u_0\| < a < \beta < \|u_1\| < c$,*
(iii) *for $\lambda > \lambda_0$, problem* (1.2) *has exactly three positive solutions $u_0 < u_1 < u_2$ satisfying $0 < \|u_0\| < a < \beta < \|u_1\| < \|u_2\| < c$.*

Remark 1.2. If $f \in C[0, \infty)$ satisfies (1.3)-(1.5), then it can be shown that

(i) By the maximum principle, every classical positive solution u of (1.2) satisfies either $0 < \|u\|_\infty < a$, or $\beta < \|u\|_\infty < c$.

(ii) Any two distinct positive solutions of (1.2) are strictly ordered. That is, let u and $\hat{u}$ be any two distinct positive solutions of (1.2) with $0 < \|u\|_\infty < \|\hat{u}\|_\infty$, then $u < \hat{u}$, see e.g. [12, Lemma 1].

Note that a similar result to Theorem 1.1 was obtained by Korman *et al.* [6, Theorem 2.7]. For f a cubic-like nonlinearity and for (1.2) ($p = 2$), similar results when $f(0) = 0$ (resp. $f(0) > 0$) and f'' changes sign exactly *once* on $(0, \infty)$ were proved by Korman and Shi [8] (resp. Korman *et al.* [7].)

But for (1.1) with $p \neq 2$, little is known. In fact for the case where $0 = a < b < c$ we refer to Addou [2]. Problem (1.1) with $p > 1$, has been recently studied in Addou and Wang [3] for the case where $f(0) = 0$ and f'' changes sign exactly *twice* on $(0, \infty)$. We note that the case where $f(0) > 0$ and f'' changes sign exactly *twice* has not been studied yet.

The paper is organized as follows. Section 2 is devoted to the definitions of the sets which contain the solutions of (1.1) and stating the main tool used subsequently, namely, the quadrature method. Next, in Section 3, we state our main results. In Section 4, a weakened condition and two examples are given. Finally, in Section 5, we prove the main results.

2. Quadrature method

To state the main results, we first define the subsets of $C^1[-1, 1]$ which contain the solutions of (1.1). By a positive solution to (1.1) we mean a positive function $u \in C^1[-1, 1]$ with $\varphi_p(u') \in C^1[-1, 1]$ satisfying (1.1). Recall that $Z(u) = \{x \in [-1, 1] : u'(x) = 0\}$. We note that it is easy to show that, if $f \in C$ and u is a positive solution of (1.1), then $u \in C^2[-1, 1]$ if $1 < p \leq 2$ and $u \in C^2([-1, 1] - Z)$ if $p > 2$. For the proof we refer to Addou [1, Lemma 6].

Let A^+ (resp. B^+) be the subset of $C^1[-1, 1]$ consisting of the functions u satisfying

(i) $u(x) > 0$ for all $x \in (-1, 1)$, $u(-1) = u(1) = 0$ and $u'(-1) > 0$ (resp. $u'(-1) = 0$),
(ii) u is symmetrical with respect to 0 (i.e., u is even).

Note that the derivative of any function $u \in A^+$ (resp. B^+) satisfies $u'(0) = 0$. Therefore $Z^+(u)$ contains at least 0. Also, $Z^+(u)$ may be connected or is an union of many connected components. Furthermore, each connected component is either a single point or an interval $[\tilde{a}, \tilde{b}]$, $\tilde{a} < \tilde{b}$. (Note that u' is continuous). So, for each integer $k = 1, 2, \dots,$, one can consider the subsets of A^+ (resp. B^+) which are composed by functions u such that $Z^+(u)$ is an union of k connected components exactly. These sets can be designed by $A^+_{a_1 a_2 \dots a_k}$ (resp. $B^+_{b_1 b_2 \dots b_k}$) where for all $j \in \{1, 2, \dots, k\}$, $a_j = 0$ (resp. $b_j = 0$) if the j^{th} connected component is a single point and $a_j = 1$ (resp. $b_j = 1$) if it is an interval (not reduced to a single point). For example, A^+_0 (resp. B^+_0) is the subset of A^+ (resp. B^+) consisting of the functions u such that their derivative u' vanishes once and only once (at 0 necessarily). An example of a function in A^+_0 (resp. B^+_0) is given by Fig. 1(a) (resp. Fig. 2(a)). Also, A^+_1 (resp. B^+_1) is the subset of A^+ (resp. B^+) such that $u \in A^+_1$ (resp. $u \in B^+_1$) if and only if $u \in A^+$ (resp. $u \in B^+$) and there exists $x_0 \in (0, 1)$ such that for all $x \in [0, 1]$, $u'(x) = 0$ if and only if $0 \leq x \leq x_0$ (resp. $0 \leq x \leq x_0$ or $x = 1$). An example of a function in A^+_1 (resp. B^+_1) is given by Fig. 1(b) (resp. Fig. 2(b)).

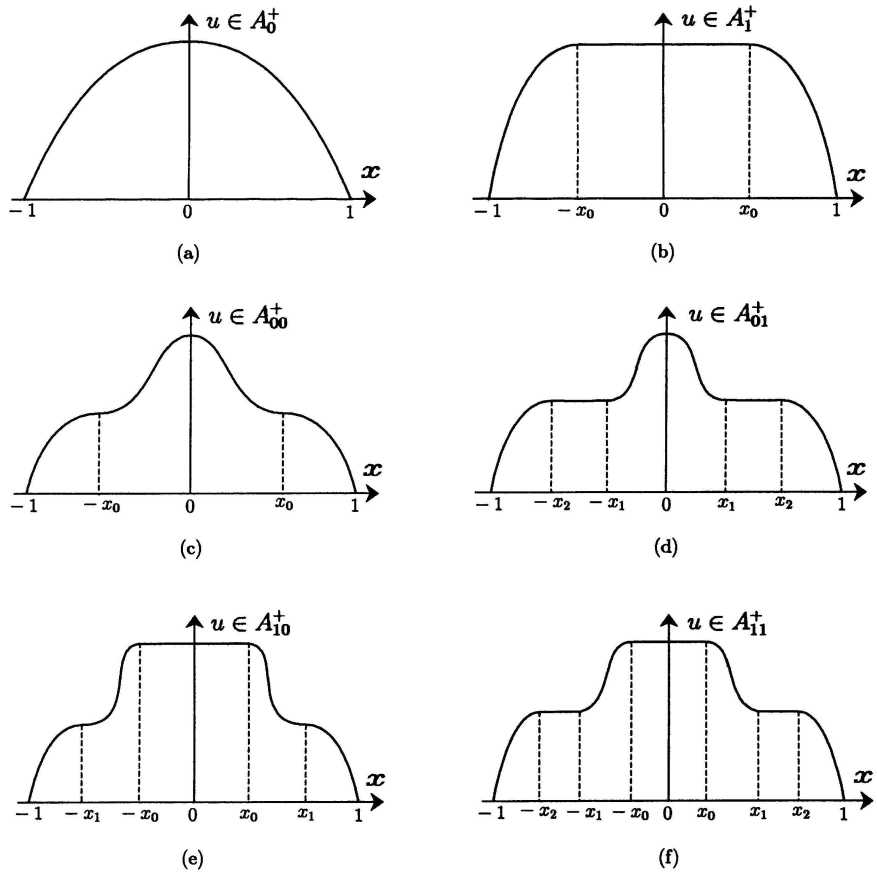


FIGURE 1. Typical graph: (a) of $u \in A_0^+$; (b) of $u \in A_1^+$; (c) of $u \in A_{00}^+$; (d) of $u \in A_{01}^+$; (e) of $u \in A_{10}^+$; (f) of $u \in A_{11}^+$.

An example of a function in A_{00}^+ (resp. B_{00}^+) is given by Fig. 1(c) (resp. Fig. 2(c)). That is, there exists $x_0 \in (0,1)$ such that, for all $0 \le x \le 1$,

$$u'(x) = 0 \text{ if and only if } x \in \{0, x_0\} (\text{resp. } x \in \{0, x_0, 1\}).$$

An example of a function in A_{01}^+ (resp. B_{01}^+) is given by Fig. 1(d) (resp. Fig. 2(d)). That is, there exist $0 < x_1 < x_2 < 1$ such that, for all $0 \le x \le 1$,

$$u'(x) = 0 \text{ if and only if } x \in \{0\} \cup [x_1, x_2] \ (\text{resp. } x \in \{0\} \cup [x_1, x_2] \cup \{1\}).$$

An example of a function in A_{10}^+ (resp. B_{10}^+) is given by Fig. 1(e) (resp. Fig. 2(e)). That is, there exist $0 < x_0 < x_1 < 1$ such that for all $0 \le x \le 1$,

$$u'(x) = 0 \text{ if and only if } x \in [0, x_0] \cup \{x_1\} \ (\text{resp. } x \in [0, x_0] \cup \{x_1, 1\}).$$

An example of a function in A_{11}^+ (resp. B_{11}^+) is given by Fig. 1(f) (resp. Fig. 2(f)). That is, there exist $0 < x_0 < x_1 < x_2 < 1$ such that, for all $0 \le x \le 1$,

$$u'(x) = 0 \text{ if and only if } x \in [0, x_0] \cup [x_1, x_2] \ (\text{resp. } x \in [0, x_0] \cup [x_1, x_2] \cup \{1\}\ ).$$

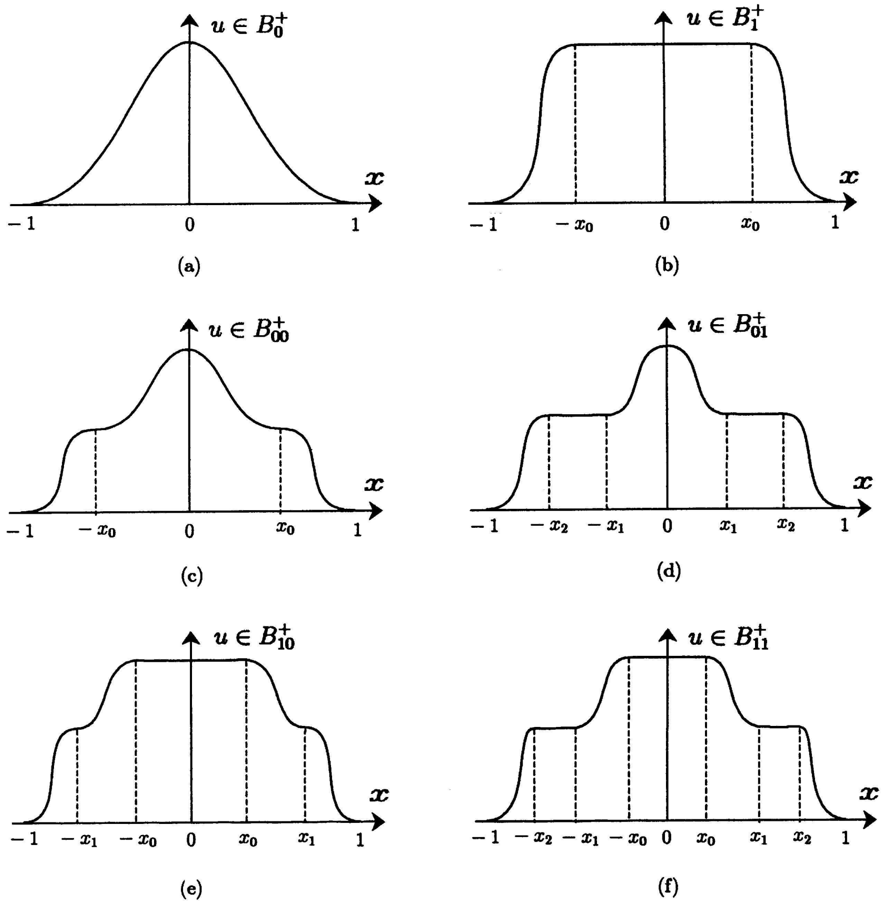


FIGURE 2. Typical graph: (a) of $u \in B_0^+$; (b) of $u \in B_1^+$; (c) of $u \in B_{00}^+$; (d) of $u \in B_{01}^+$; (e) of $u \in B_{10}^+$; (f) of $u \in B_{11}^+$.

Note that if a solution $u \in A_1^+ \cup B_1^+$, then it is usually called a dead core solution of (1.1). In this paper we extend this terminology to the case where a solution $u \in A_{a_1a_2}^+ \cup B_{a_1a_2}^+$ for some $k = 2$ and $a_j = 1$ for some $j \in \{1, 2\}$, and call it a *dead core* solution too.

First it is easy to derive an *energy relation* of solutions u of (1.1); see e.g. [4, p. 421] and [1, Lemma 7]. Denote by $p' = p/(p-1)$ the conjugate exponent of p.

Lemma 2.1 (Energy relation). *Let $p > 1$ and assume that u is a positive solution of* (1.1)*, then $(|u'(x)|^p + p'\lambda F(u(x)))' = 0$ for all $x \in [-1, 1]$.*

Lemma 2.2. *Suppose f satisfies conditions* (1.3)–(1.5) *and u is a positive solution of problem* (1.1)*. Then $u \in A_0^+ \cup A_1^+ \cup A_{00}^+ \cup A_{01}^+$.*

Proof. Suppose f satisfies conditions (1.3)–(1.5), $f(0) > 0$ and f changes sign exactly twice on $(0, \infty)$. Suppose u is a positive solution of (1.1), then u is symmetrical with respect to 0. It can be easily proved that either $0 < \|u\|_\infty \le a$ or

$\beta \le \|u\|_\infty \le c$ by applying Lemma 2.1; cf. Remark 1.2(i). Thus

$$u \in A_0^+ \cup A_1^+ \cup A_{00}^+ \cup A_{01}^+ \cup A_{10}^+ \cup A_{11}^+ \cup B_0^+ \cup B_1^+ \cup B_{00}^+ \cup B_{01}^+ \cup B_{10}^+ \cup B_{11}^+.$$

The proof is easy but tedious, so we omit it. More precisely,

(i) Since $f(0) > 0$, if u is a positive solution u of (1.1) satisfying $\|u\|_\infty = u(0) = \eta \in (0, a] \cup [\beta, c]$, then $u'(-1) = (p'\lambda F(\eta))^{1/p} > 0$ by applying Lemma 2.1. Hence $u \notin B_0^+ \cup B_1^+ \cup B_{00}^+ \cup B_{01}^+ \cup B_{10}^+ \cup B_{11}^+$.

(ii) We then show that $u \notin A_{10}^+ \cup A_{11}^+$. Suppose that $u \in A_{10}^+ \cup A_{11}^+$. Then either $\|u\|_\infty = c$ or $\|u\|_\infty = a$, and there exists $x_1 \in (0, 1)$ such that $u'(x_1) = 0$. If $\|u\|_\infty = c$, then by applying Lemma 2.1, $p'\lambda F(c) = p'\lambda F(x_1)$, which contradicts the fact that $F(c) > F(x_1)$. So $\|u\|_\infty \ne c$. Similarly, $\|u\|_\infty \ne a$. We conclude that $u \notin A_{10}^+ \cup A_{11}^+$.

By above (i) and (ii), we obtain that $u \in A_0^+ \cup A_1^+ \cup A_{00}^+ \cup A_{01}^+$. □

To study (1.1), we make use of the quadrature method. Suppose $f \in C[0, \infty)$ satisfies conditions (1.3)–(1.5). For any $E \ge 0$ and $s > 0$, let $G(E, s) := E^p - p'\lambda F(s)$. It can be shown that, the function $G(E, \cdot)$ has at most four zeros in $(0, \infty)$. For any $E \ge 0$, define

$$X_1(E) = \{s > 0 : s \in \operatorname{dom} G(E, \cdot) \text{ and } G(E, u) > 0 \text{ for all } u \in (0, s)\}$$

and

$$r_1(E) = \begin{cases} 0 & \text{if } X_1(E) = \emptyset, \\ \sup(X_1(E)) & \text{otherwise.} \end{cases}$$

Next for any $E \ge 0$, define

$$X_2(E) = \{s > r_1(E) : s \in \operatorname{dom} G(E, \cdot) \text{ and } G(E, u) > 0 \text{ for all } u \in (r_1(E), s)\}$$

and

$$r_2(E) = \begin{cases} \infty & \text{if } X_2(E) = \emptyset, \\ \sup(X_2(E)) & \text{otherwise.} \end{cases}$$

Note that $X_2(E)$ and $r_2(E)$ are well defined even if $r_1(E) = \infty$. In fact, in this case, $X_2(E) = \emptyset$ and $r_2(E) = \infty$. Let

$$\begin{aligned} \tilde{D}_1 = \big\{ &E \ge 0 : r_1(E) \in \operatorname{dom} G(E, \cdot),\ G(E, r_1(E)) = 0, \\ &\text{and } \int_0^{r_1(E)} (E^p - p'\lambda F(t))^{-1/p} dt < \infty \big\}, \end{aligned}$$

$$\begin{aligned} \tilde{D}_2 = \big\{ &E \ge 0 : r_2(E) \in \operatorname{dom} G(E, \cdot),\ G(E, r_2(E)) = 0, \\ &\text{and } \int_0^{r_2(E)} (E^p - p'\lambda F(t))^{-1/p} dt < \infty \big\}. \end{aligned}$$

Define the time maps

$$\begin{aligned} T_1(E) &= \int_0^{r_1(E)} (E^p - p'\lambda F(t))^{-1/p} dt,\ E \in \tilde{D}_1, \\ T_2(E) &= \int_0^{r_2(E)} (E^p - p'\lambda F(t))^{-1/p} dt,\ E \in \tilde{D}_2, \end{aligned}$$

whenever $\tilde{D}_1 \ne \emptyset$ (resp. $\tilde{D}_2 \ne \emptyset$).

By Lemma 2.1 and arguments in [4], we have the following theorem. Note that in this paper, by Lemma 2.2, we restrict ourself on positive solutions $u \in A_0^+ \cup A_1^+ \cup A_{00}^+ \cup A_{01}^+$.

Theorem 2.3 (Quadrature method). *Consider* (1.1). *Suppose* $f \in C[0,\infty)$ *satisfies conditions* (1.3)–(1.5). *Let* $E \geq 0$. *Then* T_1 *(resp.* T_2*) is a continuous function of* $E \in \tilde{D}_1$. *(resp.* $E \in \tilde{D}_2$*). Moreover,*

(i) *Problem* (1.1) *has a solution* $u \in A_0^+$ *satisfying* $u'(-1) = E > 0$ *if and only if* $E \in \tilde{D}_1 - \{0\}$, $f(r_1(E)) \geq 0$ *and* $T_1(E) = 1$, *and in this case the solution is unique.*

(ii) *Problem* (1.1) *has a solution* $u \in A_1^+$ *satisfying* $u'(-1) = E > 0$ *if and only if* $E \in \tilde{D}_1 - \{0\}$, $f(r_1(E)) = 0$ *and* $T_1(E) < 1$, *and in this case the solution is unique.*

(iii) *Problem* (1.1) *has a solution* $u \in A_{00}^+$ *satisfying* $u'(-1) = E > 0$ *if and only if* $E \in \tilde{D}_2 - \{0\}$, $f(r_1(E)) \geq 0$, $f(r_2(E)) \geq 0$, *and* $T_2(E) = 1$, *and in this case the solution is unique.*

(iv) *Problem* (1.1) *has a solution* $u \in A_{01}^+$ *satisfying* $u'(-1) = E > 0$ *if and only if* $E \in \tilde{D}_2 - \{0\}$, $f(r_1(E)) = 0$, $f(r_2(E)) \geq 0$, *and* $T_2(E) < 1$, *and in this case the solution is unique.*

Remark 2.4. In practice, we first study the variations of the real-valued function $G(E,\cdot)$, then compute $X_1(E)$ and deduce $r_1(E)$ (resp. compute $X_2(E)$ and deduce $r_2(E)$). Next, we compute $\tilde{D}_1$ (resp. $\tilde{D}_2$). For this, we first compute the set

$$D_1 = \{E > 0 : r_1(E) \in \operatorname{dom} G(E,\cdot),\ G(E, r_1(E)) = 0,\ f(r_1(E)) > 0\},$$

(resp.

$$D_2 = \{E > 0 : r_2(E) \in \operatorname{dom} G(E,\cdot),\ G(E, r_2(E)) = 0,\ f(r_2(E)) > 0\}),$$

and then we deduce $\tilde{D}_1$ (resp. $\tilde{D}_2$) by observing that $D_1 \subset \tilde{D}_1 - \{0\} \subset \overline{D}_1$ (resp. $D_2 \subset \tilde{D}_2 - \{0\} \subset \overline{D}_2$) ; we omit the proof. (Note that $\overline{D}_1$ is the closure of D_1 (resp. $\overline{D}_2$ is the closure of D_2).) After that, we define the time map T_1 on $\tilde{D}_1$ and then compute its limits at the boundary points of $\tilde{D}_1$. We next study the variations of T_1 on $\tilde{D}_1$. For T_2, we shall show that its definition domain $\tilde{D}_2$ is restricted to a single point; there is no variation to study for T_2. We achieve our study by discussing the number of solutions to

(i) Equation $T_1(E) = 1$ and $f(r_1(E)) \geq 0$ for $E \in \tilde{D}_1 - \{0\}$ in case of looking for solutions u in A_0^+.

(ii) Inequality $T_1(E) < 1$ and $f(r_1(E)) = 0$ for $E \in \tilde{D}_1 - \{0\}$ in case of looking for solutions u in A_1^+.

(iii) Equation $T_2(E) = 1$ and $f(r_1(E)) \geq 0$, $f(r_2(E)) \geq 0$, for $E \in \tilde{D}_2 - \{0\}$ in case of looking for solutions u in A_{00}^+.

(iv) Inequality $T_2(E) < 1$ and $f(r_1(E)) = 0$, $f(r_2(E)) \geq 0$, for $E \in \tilde{D}_2 - \{0\}$ in case of looking for solutions u in A_{01}^+.

3. Main results

We determine the exact multiplicity of positive solutions of (1.1) for $\lambda > 0$ under hypotheses (H1)-(H5) stated below. In particular, we assume that f satisfies the "*convexity*" condition (H4) which implies for the particular case $p = 2$, that f'' changes sign exactly *twice* on $(0,\infty)$, i.e., f is concave-convex-concave on $(0,\infty)$.

Note that if f satisfies (H1)-(H3) then it satisfies (1.3)–(1.5). Also note that we may allow that f'' changes sign more than twice, i.e., we may allow that f is concave-convex-concave-convex; see Section 4.

For f, recalling that $F(u) = \int_0^u f(t)dt$, we let

$$\theta_p(u) := pF(u) - uf(u),$$
$$\Psi_p(u) := u\theta_p'(u) - \theta_p(u) = puf(u) - u^2 f'(u) - pF(u),$$
$$\nu_p := \Big\{\int_0^c (F(c) - F(u))^{-1/p} du\Big\}^p/p' \in (0, \infty],$$
$$\alpha_p := \Big\{\int_0^a (F(a) - F(u))^{-1/p} du\Big\}^p/p' \in (0, \infty], \tag{3.1}$$
$$\lambda_p := \Big\{\int_0^\beta (F(\beta) - F(u))^{-1/p}\Big\}^p/p' \in (0, \infty], \tag{3.2}$$
$$\mu_p := \inf_{\beta \le \xi \le c} \Big\{\int_0^\xi (F(\xi) - F(u))^{-1/p} du\Big\}^p/p',$$

where $0 < a < \beta < c$ are defined below. We shall show that $0 < \mu_p < \infty$ for $p > 1$. For all $\lambda > 0$, we denote S_λ the positive solution set of (1.1).

For fixed $p > 1$, suppose $f \in C^2[0, \infty)$ and there exist $0 < a < b < c$ such that the following conditions are satisfied:

- (H1) $f(0) > 0$
- (H2) $f(u) > 0$ for $0 < u < a$, $f(u) < 0$ for $a < u < b$, $f(u) > 0$ for $b < u < c$, $f(u) < 0$ for $u > c$
- (H3) $\int_a^c f(u)du > 0$, and there exists $\beta^* \in (0, \beta]$ such that $\theta_p(\beta^*) = pF(\beta^*) - \beta^* f(\beta^*) < 0$, where $\beta \in (b, c)$ is defined by $\int_a^\beta f(u)du = 0$,
- (H4) There exist $0 < r_p < s_p < c$ such that
$$(p-2)f'(u) - uf''(u) > 0 \quad \text{for } 0 < u < r_p,$$
$$(p-2)f'(u) - uf''(u) < 0 \quad \text{for } r_p < u < s_p,$$
$$(p-2)f'(u) - uf''(u) > 0 \quad \text{for } s_p < u < \infty,$$
- (H5) There exists a unique $\sigma_p \in (s_p, c)$ satisfying $(p-1)f(\sigma_p) - \sigma_p f'(\sigma_p) = 0$ and such that $\Psi_p(\sigma_p) \ge \Psi_p(r_p)$.

Remark 3.1 (Cf. Remark 1.2). If $f \in C[0, \infty)$ satisfies (H1)-(H3), then by applying Lemma 2.1, it can be shown that

- (i) Every positive solution u of (1.1) satisfies $0 < \|u\|_\infty \le a$ or $\beta \le \|u\|_\infty \le c$.
- (ii) Any two distinct positive solutions of (1.1) are strictly ordered. That is, let u and $\hat{u}$ be any two distinct positive solutions of (1.1) with $0 < \|u\|_\infty < \|\hat{u}\|_\infty$, then $u < \hat{u}$.

Case $1 < p \le 2$. The next theorem gives a complete description of the set S_λ for $1 < p \le 2$.

Theorem 3.2 (S_λ for $1 < p \le 2$, see Fig. 3). *Assume that $1 < p \le 2$ and $f \in C^2[0, \infty)$ satisfies (H1)-(H5). Then $0 < \mu_p < \infty$. Moreover:*

- (i) *For $0 < \lambda < \mu_p$, there exists $u_\lambda \in A_0^+$ such that $S_\lambda = \{u_\lambda\}$. Also $0 < \|u_\lambda\|_\infty < a$.*

(ii) *For* $\lambda = \mu_p$, *there exist* u_λ, $v_\lambda \in A_0^+$ *such that* $u_\lambda < v_\lambda$ *and* $S_\lambda = \{u_\lambda, v_\lambda\}$. *Moreover,* $0 < \|u_\lambda\|_\infty < a < \beta < \|v_\lambda\|_\infty < c$.

(iii) *For* $\lambda > \mu_p$, *there exist* u_λ, v_λ *and* $w_\lambda \in A_0^+$ *such that* $u_\lambda < v_\lambda < w_\lambda$ *and* $S_\lambda = \{u_\lambda, v_\lambda, w_\lambda\}$. *Moreover,* $0 < \|u_\lambda\|_\infty < a < \beta < \|v_\lambda\|_\infty < \|w_\lambda\|_\infty < c$.

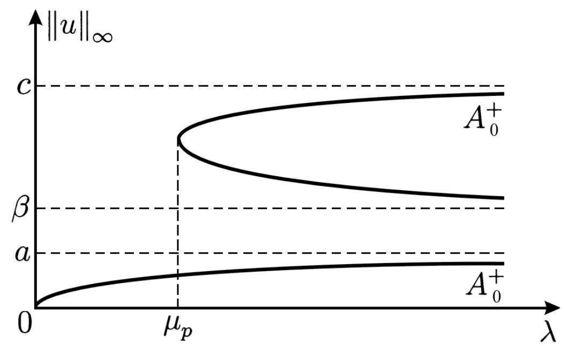


FIGURE 3. Bifurcation diagram of problem (1.1) with $f(0) > 0$ and $1 < p \le 2$.

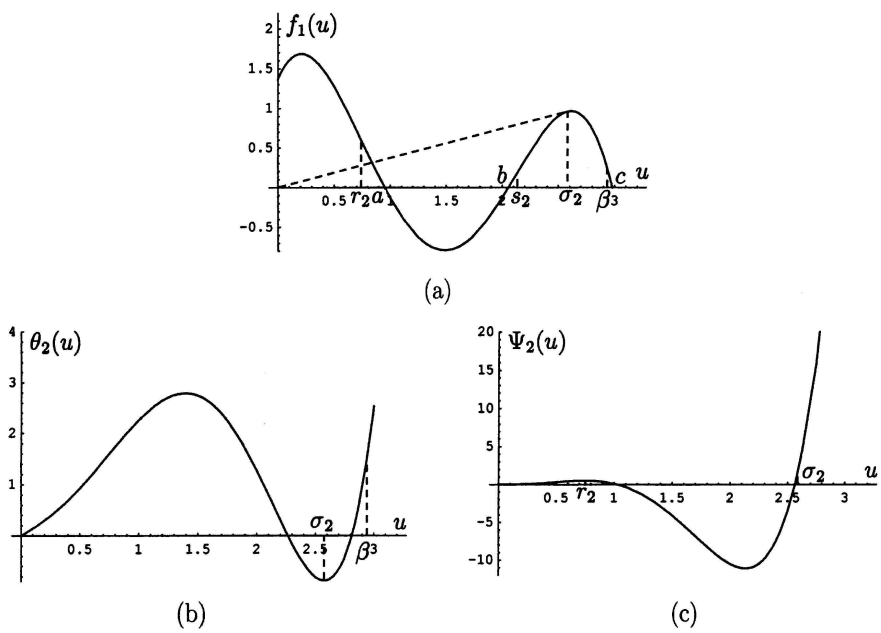


FIGURE 4. (a) Graph of $f_1(u) = -(u + 1/4)(u - 1)(u - 2)(u - 3) - 0.13$. $a \approx 0.9497$, $b \approx 2.0565$, $c \approx 2.9792$, $\beta \approx 2.9378$, $r_2 \approx 0.7425$, $s_2 \approx 2.1325$, $\sigma_2 \approx 2.5787$. (b) Graph of $\theta_2(u)$. $1.4751 \approx \theta_2(\beta) > \theta_2(\sigma_2) \approx -0.8789$. (c) Graph of $\Psi_2(u)$. $0.8792 \approx \Psi_2(\sigma_2) > \Psi_2(r_2) \approx 0.5127$.

Next, we give two interesting examples of quartic polynomials of Theorem 3.2 with $p=2$, of which one satisfies $\theta_2(\beta)>0$ and the other satisfies $\theta_2(\beta)<0$.

Two examples of Theorem 3.2. (i) (See Fig. 4, $\theta_2(\beta)>0$) Let $p=2$. The function $f=f_1(u)=-(u+1/4)(u-1)(u-2)(u-3)-0.13$ satisfies all conditions (H1)-(H5) in Theorem 3.2 with $a\approx 0.9497$, $b\approx 2.0565$, $c\approx 2.9792$, $\int_a^c f_1(s)ds\approx 0.004695>0$, $r_2\approx 0.7425$, $s_2\approx 2.1325$, $2.5787\approx\sigma_2<\beta\approx 2.9387$, $1.4751\approx\theta_2(\beta)>\theta_2(\sigma)\approx -0.8789$. Note that, in (H4)-(H5), $0.8792\approx\Psi_2(\sigma_2)>\Psi_2(r_2)\approx 0.5127$.
(ii) ($\theta_2(\beta)<0$) Let $p=2$. Let $f=f_2(u)=-(u-d)(u-a)(u-b)(u-c)$ with

$$d=-\frac{1}{6}<0<a=1<b=2<c.$$

Thus f_2 satisfies (H1), (H2) and (H4) with

$$f_2''(u)=-12u^2+(17+6c)u-3-\frac{17}{3}c$$

and $f_2''(0)=-3-\frac{17}{3}c<0$. Let $r_2<s_2$ be two positive zeros of $f_2''(u)$ on $(0,c)$. So

$$\begin{gathered}r_2=\frac{1}{24}(6c+17-\sqrt{36c^2-68c+145}),\\ s_2=\frac{1}{24}(6c+17+\sqrt{36c^2-68c+145}).\end{gathered}$$

There exists $c_1\approx 2.8380$, the biggest positive zero of $18c^3-49c^2-26c+25$, such that

$$c>c_1\Leftrightarrow\int_a^c f_2(u)du=\frac{1}{360}(c-1)^2(18c^3-49c^2-26c+25)>0 \text{ on } (2,\infty).$$

So for $c>c_1\approx 2.8380$, there exists a unique $\beta=\beta(c)\in(b,c)=(2,c)$ satisfying

$$\int_a^\beta f_2(u)du=0.$$

or,

$$\frac{1}{360}(\beta-1)^2[-72\beta^3+(90c+111)\beta^2+(-160c+114)\beta-140c+57]=0.$$

Note that $\beta(c)$ can be expressed explicitly by Cartan's formulas; see e.g. [5]. We compute that

$$\theta_2(u)=\frac{3}{5}u^5-\frac{6c+17}{12}u^4+\frac{17c+9}{18}u^3+\frac{c}{3}u$$

and

$$\theta_2(\beta(c))<0 \text{ for } c>c_2\approx 2.9056,$$

where c_2 is the unique zero of $\theta_2(\beta(c))$ on (c_1,∞). Thus f_2 satisfies (H3) for $c>c_2\approx 2.9056$.

Finally, we check (H5) for $c>c_2$. Let $\sigma_2=\sigma_2(c)$ be the unique zero of

$$\theta_2'(u)=3u^4-\frac{6c+17}{3}u^3+\frac{17c+9}{6}u^2+\frac{c}{3}$$

in (s_2,c). Note that $\sigma_2(c)$ can be expressed explicitly; see e.g. [5]. We compute that

$$\begin{gathered}\Psi_2(u)=\frac{1}{180}u^3[432u^2-(270c+765)u+340c+180]\,,\\ \Psi_2(\sigma_2)>\Psi_2(s_2) \text{ for } c>c_2\approx 2.9056.\end{gathered}$$

We summarize above results and conclude that f_2 satisfies (H1)-(H5) for $c > c_2 \approx 2.9056$.

3.1. **The case** $p > 2$. By Lemma 2.2, $S_\lambda \subset A_0^+ \cup A_1^+ \cup A_{00}^+ \cup A_{01}^+$. Hence

$$S_\lambda = (S_\lambda \cap A_0^+) \cup (S_\lambda \cap A_1^+) \cup (S_\lambda \cap A_{00}^+) \cup (S_\lambda \cap A_{01}^+);$$

Theorem 3.4 (resp. 3.5, 3.6, 3.7) gives complete description of the set $S_\lambda \cap A_0^+$ (resp. $S_\lambda \cap A_1^+$, $S_\lambda \cap A_{00}^+$, $S_\lambda \cap A_{01}^+$); see Fig. 5.

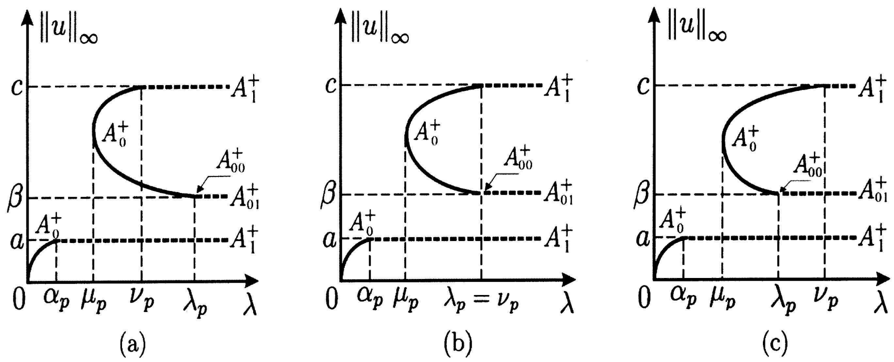


FIGURE 5. Bifurcation diagrams for Eq. (1.1) with $p > 2$ and $0 < \alpha_p < \mu_p$. (a) $\mu_p < \nu_p < \lambda_p$; (b) $\mu_p < \nu_p = \lambda_p$; (c) $\mu_p < \lambda_p < \nu_p$.

Remark 3.3. Fig. 5 describes the solution set of (1.1) when $p > 2$ and $0 < \alpha_p < \mu_p$. There are two connected branches. The lower branch bifurcates at the origin and represents solutions in A_0^+ until $\lambda = \alpha_p$; for all $\lambda > \alpha_p$, the branch is horizontal and represents solutions in A_1^+. The upper branch is formed by two parts: The upper horizontal curve represents solutions in A_1^+ with norm equal to c, and the $\subset$-shaped curve represents solutions in A_0^+ until $\lambda = \lambda_p$ where there is a solution in A_{00}^+ with norm equal to β and on its right the lower horizontal curve which represents solutions in A_{01}^+ with norm equal to β.

We shall show that, for $p > 2$, $0 < \mu_p < \lambda_p < \infty$ and $0 < \mu_p < \nu_p < \infty$; see Lemmas 5.3-5.4. We also show that, for $p > 2$, $0 < \alpha_p < \lambda_p < \infty$; see Lemma 5.7. For $p > 1$, let

$$B(0,a) := \{u \in C^1[-1,1] : \|u\|_\infty < a\},$$
$$B(0,\beta) := \{u \in C^1[-1,1] : \|u\|_\infty < \beta\},$$
$$B(0,c) := \{u \in C^1[-1,1] : \|u\|_\infty < c\}.$$

Theorem 3.4 ($S_\lambda \cap A_0^+$ for $p > 2$, see Fig. 5)**.** *Assume that $p > 2$ and $f \in C^2[0,\infty)$ satisfies (H1)-(H5). Then $0 < \mu_p < \lambda_p < \infty$, $0 < \mu_p < \nu_p < \infty$, and $0 < \alpha_p < \lambda_p < \infty$. $S_\lambda \cap A_0^+ \cap (B(0,\beta) - \overline{B(0,a)}) = \emptyset$. Also,*

(i) *For $0 < \lambda \le \alpha_p$, there exists $u_\lambda \in A_0^+ \cap \overline{B(0,a)}$ such that $S_\lambda \cap A_0^+ \cap \overline{B(0,a)} = \{u_\lambda\}$. Moreover, $0 < \|u_\lambda\|_\infty \le a$, and $\|u_\lambda\|_\infty = a$ if and only if $\lambda = \alpha_p$.*
(ii) *For $\lambda > \alpha_p$, $S_\lambda \cap A_0^+ \cap \overline{B(0,a)} = \emptyset$.*

Moreover, ***(a) If*** $\mu_p < \nu_p < \lambda_p$***, then:***

(iii) *For* $0 < \lambda < \mu_p$, $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \emptyset$.
(iv) *For* $\lambda = \mu_p$, *there exists* $u_\lambda \in A_0^+ \cap (\overline{B(0,c)} - B(0,\beta))$ *such that* $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \{u_\lambda\}$ *and* $\beta < \|u_\lambda\|_\infty < c$.
(v) *For* $\mu_p < \lambda \le \nu_p$, *there exist* u_λ, $v_\lambda \in A_0^+ \cap (\overline{B(0,c)} - B(0,\beta))$ *such that* $u_\lambda < v_\lambda$ *and* $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \{u_\lambda, v_\lambda\}$. *Moreover,* $\beta < \|u_\lambda\|_\infty < \|v_\lambda\|_\infty \le c$, *and* $\|v_\lambda\|_\infty = c$ *if and only if* $\lambda = \nu_p$.
(vi) *For* $\nu_p < \lambda < \lambda_p$, *there exists* $u_\lambda \in A_0^+ \cap (\overline{B(0,c)} - B(0,\beta))$ *such that* $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \{u_\lambda\}$ *and* $\beta < \|u_\lambda\|_\infty < c$.
(vii) *For* $\lambda \ge \lambda_p$, $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \emptyset$.

(b) If $\mu_p < \nu_p = \lambda_p$***, then:***

(viii) *For* $0 < \lambda < \mu_p$, $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \emptyset$.
(ix) *For* $\lambda = \mu_p$, *there exists* $u_\lambda \in A_0^+ \cap (\overline{B(0,c)} - B(0,\beta))$ *such that* $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \{u_\lambda\}$ *and* $\beta < \|u_\lambda\|_\infty < c$.
(x) *For* $\mu_p < \lambda < \nu_p = \lambda_p$, *there exist* u_λ, $v_\lambda \in A_0^+ \cap (\overline{B(0,c)} - B(0,\beta))$ *such that* $u_\lambda < v_\lambda$ *and* $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \{u_\lambda, v_\lambda\}$. *Moreover,* $\beta < \|u_\lambda\|_\infty < \|v_\lambda\|_\infty < c$.
(xi) *For* $\lambda = \nu_p = \lambda_p$, *there exists* $u_\lambda \in A_0^+ \cap (\overline{B(0,c)} - B(0,\beta))$ *such that* $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \{u_\lambda\}$. *Moreover,* $\|u_\lambda\|_\infty = c$.
(xii) *For* $\lambda > \lambda_p = \nu_p$, $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \emptyset$.

(c) If $\mu_p < \lambda_p < \nu_p$***, then:***

(xiii) *For* $0 < \lambda < \mu_p$, $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \emptyset$.
(xiv) *For* $\lambda = \mu_p$, *there exists* $u_\lambda \in A_0^+ \cap (\overline{B(0,c)} - B(0,\beta))$ *such that* $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \{u_\lambda\}$. *Moreover,* $\beta < \|u_\lambda\|_\infty < c$.
(xv) *For* $\mu_p < \lambda < \lambda_p$, *there exist* u_λ, $v_\lambda \in A_0^+ \cap (\overline{B(0,c)} - B(0,\beta))$ *such that* $u_\lambda < v_\lambda$ *and* $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \{u_\lambda, v_\lambda\}$. *Moreover,* $\beta < \|u_\lambda\|_\infty < \|v_\lambda\|_\infty < c$.
(xvi) *For* $\lambda_p \le \lambda \le \nu_p$, *there exists* $u_\lambda \in A_0^+ \cap (\overline{B(0,c)} - B(0,\beta))$ *such that* $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \{u_\lambda\}$ *and* $\beta < \|u_\lambda\|_\infty \le c$, *and* $\|u_\lambda\|_\infty = c$ *if and only if* $\lambda = \nu_p$.
(xvii) *For* $\lambda > \nu_p$, $S_\lambda \cap A_0^+ \cap (\overline{B(0,c)} - B(0,\beta)) = \emptyset$.

Theorem 3.5 ($S_\lambda \cap A_1^+$ for $p > 2$, see Fig. 5)**.** *Assume that* $p > 2$ *and* $f \in C^2[0,\infty)$ *satisfies (H1)-(H5). Then each solution* u_λ *of* (1.1) *in* A_1^+ *satisfies* $\|u_\lambda\|_\infty = a$ *or* $\|u_\lambda\|_\infty = c$. *Moreover:*

(i) *For* $0 < \lambda \le \alpha_p$, $S_\lambda \cap A_1^+ \cap \partial\overline{B(0,a)} = \emptyset$.
(ii) *For* $\lambda > \alpha_p$, *there exists* $u_\lambda \in A_1^+ \cap \partial\overline{B(0,a)}$ *such that* $S_\lambda \cap A_1^+ \cap \partial\overline{B(0,a)} = \{u_\lambda\}$ *and* $\|u_\lambda\|_\infty = a$.
(iii) *For* $0 < \lambda \le \nu_p$, $S_\lambda \cap A_1^+ \cap \partial\overline{B(0,c)} = \emptyset$.
(iv) *For* $\lambda > \nu_p$, *there exists* $u_\lambda \in A_1^+ \cap \partial\overline{B(0,c)}$ *such that* $S_\lambda \cap A_1^+ \cap \partial\overline{B(0,c)} = \{u_\lambda\}$ *and* $\|u_\lambda\|_\infty = c$.

Theorem 3.6 ($S_\lambda \cap A_{00}^+$ for $p > 2$, see Fig. 5)**.** *Assume that* $p > 2$ *and* $f \in C^2[0,\infty)$ *satisfies (H1)-(H5). Then each solution* u_λ *of* (1.1) *in* A_{00}^+ *satisfies* $\|u_\lambda\|_\infty = \beta$. *Moreover,*

(i) *For* $\lambda \neq \lambda_p$ *and* $\lambda > 0$, $S_\lambda \cap A_{00}^+ = \emptyset$.
(ii) *For* $\lambda = \lambda_p$, *there exists* $u_\lambda \in A_{00}^+$ *such that* $S_\lambda \cap A_{00}^+ = \{u_\lambda\}$ *and* $\|u_\lambda\|_\infty = \beta$.

Theorem 3.7 ($S_\lambda \cap A_{01}^+$ for $p > 2$, see Fig. 5)**.** *Assume that* $p > 2$ *and* $f \in C^2[0,\infty)$ *satisfies (H1)-(H5). Then each solution* u_λ *of* (1.1) *in* A_{01}^+ *satisfies* $\|u_\lambda\|_\infty = \beta$. *Moreover,*
(i) *For* $0 < \lambda \leq \lambda_p$, $S_\lambda \cap A_{01}^+ = \emptyset$.
(ii) *For* $\lambda > \lambda_p$, *there exists* $u_\lambda \in A_{01}^+$ *such that* $S_\lambda \cap A_{01}^+ = \{u_\lambda\}$ *and* $\|u_\lambda\|_\infty = \beta$.

4. A weakened condition and two examples

We point out that the *convexity* condition of $\theta_p(u) = pF(u) - uf(u)$ on $(0,c)$ in (H4) in Theorems 3.2-3.7 can actually be weakened; cf. Remark 9 in Addou and Wang [3], which holds true in the case $f(0) = 0$ as well as in the positone case $f(0) > 0$. More precisely, condition (H4) can be weakened as

(H4$'$) There exist $0 \leq r_p < s_p < c$ such that
$$(p-2)f'(u) - uf''(u) > 0 \text{ for } 0 < u < r_p, \text{(It is not necessary if } r_p = 0.)$$
$$(p-2)f'(u) - uf''(u) < 0 \text{ for } r_p < u < s_p,$$
$$(p-2)f'(u) - uf''(u) > 0 \text{ for } s_p < u < c, \text{ (It can be weakened below.)}$$

We note that in (H4$'$) if $r_p = 0$ then condition (H5) is automatically satisfied since it can be easily shown that, for $p > 1$, $\Psi_p(\sigma_p) > 0 = \Psi_p(r_p)$.

We also note that in (H4$'$) the condition
$$(p-2)f'(u) - uf''(u) > 0 \quad \text{for } s_p < u < c$$
can actually be weakened as
$$\theta_p'(u) = (p-1)f(u) - uf'(u) \begin{cases} \leq 0 & \text{for } s_p < u < \sigma_p, \\ \geq 0 & \text{for } d \leq u \leq c, \end{cases}$$
$$\theta_p''(u) = (p-2)f'(u) - uf''(u) \geq 0 \text{ for } \sigma_p \leq u < d,$$
where $d \in (\sigma_p, c]$ is defined by
$$d := \begin{cases} c & \text{if } \theta_p(c) \leq \theta_p(t_p), \\ \inf\{u \in (\sigma_p, c] : \theta_p(\xi) > \theta_p(t_p) \text{ for all } \xi \in (u,c]\} & \text{otherwise,} \end{cases} \tag{4.1}$$
where t_p is the unique zero of $\theta_p'(u)$ on (r_p, s_p).

Thus we summarize that (H4$'$) can be weakened as follows:

(H4$''$) There exist $0 \leq r_p < s_p < \sigma_p < d < c$ such that
$$(p-2)f'(u) - uf''(u) > 0 \text{ for } 0 < u < r_p, \text{ (It is not necessary if } r_p = 0.)$$
$$(p-2)f'(u) - uf''(u) < 0 \text{ for } r_p < u < s_p,$$
$$\theta_p'(u) = (p-1)f(u) - uf'(u) \begin{cases} \leq 0 & \text{for } s_p < u < \sigma_p, \\ \geq 0 & \text{for } d \leq u \leq c, \end{cases}$$
$$\theta_p''(u) = (p-2)f'(u) - uf''(u) \geq 0 \text{ for } \sigma_p \leq u < d,$$
where d is defined in (4.1).

For example, Theorem 3.2 can be generalized as

Theorem 4.1 (S_λ for $1 < p \le 2$, see Fig. 3)**.** *Assume that $1 < p \le 2$ and $f \in C^2[0,\infty)$ satisfies (H1)-(H3), ((H4′) or (H4″)) and (H5). Then the results in Theorem 3.2 hold.*

Therefore, in the case that $f(0) > 0$, $p = 2$ and $r_2 = 0$, Theorem 4.1 generalizes [12, Theorem 1]. We give two examples of classes of nonlinearities of Theorem 4.1.

Proposition 4.2. *Let $p = 2$. $f = f_3(u) = -(u-a)(u-b)(u-c)$ with $a = 1$, $b = 3$, and*

$$c > 2b - a = 5 \ (\Leftrightarrow \int_a^c f_3(u)du > 0).$$

Then f_3 satisfies all conditions (H1)-(H3), (H4′) and (H5) in Theorem 4.1.

Proof. It is easy to see that $f_3(u) = -(u-1)(u-3)(u-c)$ satisfies (H1), (H2), (H5) and (H4′) with $r_2 = 0$ for $c > 5$. Finally, we check (H3) for $c > 5$. We compute that

$$\theta_2(u) = \frac{1}{2}u^4 - \frac{1}{3}(c+4)u^3 + 3cu,$$
$$\beta = \beta(c) = \frac{1}{3}(2c + 5 - 2\sqrt{c^2 - 7c + 10}) \in (3, c),$$

and $\theta_2(\beta) < 0$ for $c > \tilde{c} \approx 5.1193$, where $\tilde{c}$ is the unique zero of $\theta_2(\beta)$ on $(5,\infty)$. Although for $5 < c \le \tilde{c} \approx 5.1193$, $\theta_2(\beta) \ge 0$, and thus Theorem 1.1 does not apply. We compute that

$$\theta_2'(u) = 2u^4 - (c+4)u^2 + 3c$$

and find that

$$\sigma_2 = \sigma_2(c) = \frac{1}{6}\Big\{c + 4 + \frac{(c+4)^2}{(c^3 + 12c^2 + 114c + 64 + 18\sqrt{-c^4 - 12c^3 + 33c - 64c})^{1/3}} + \big(c^3 + 12c^2 - 114c + 64 + 18\sqrt{-c^4 - 12c^3 + 33c - 64c}\big)^{1/3}\Big\}.$$

satisfies $0 < \sigma_2 < \beta$ and $\theta_2(\sigma_2) < 0$ for $5 < c \le \tilde{c}$; we omit the detailed numerical simulations here. So f_3 satisfies (H3) for $5 < c \le \tilde{c}$.

We conclude that $f_3(u) = -(u-1)(u-3)(u-c)$ satisfies all conditions (H1)-(H3), (H4′) and (H5) in Theorem 4.1 for $c > 5$. □

Proposition 4.3 (See Fig. 6 for $\varepsilon = 0.2$)**.** *Let $p = 2$. For $0 < \varepsilon < 1$, let*

$$0 < a = \sin^{-1}\varepsilon < b = \pi - \sin^{-1}\varepsilon < c = 2\pi + \sin^{-1}\varepsilon$$

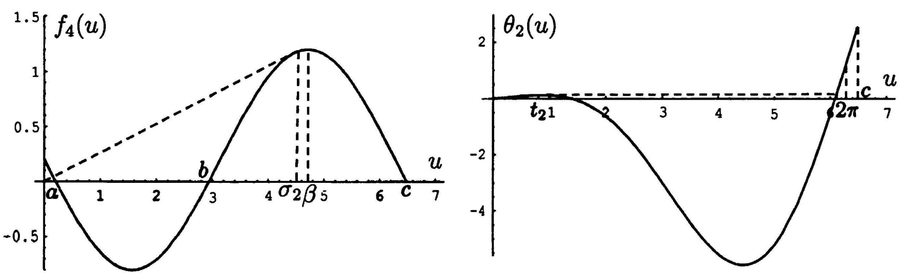
and $f = f_4(u)$ satisfy

$$f_4(u) = \begin{cases} -\sin u + \varepsilon & \text{for } 0 < u < c, \\ < 0 & \text{for } u > c. \end{cases}$$

Then f_4 satisfies all conditions (H1)-(H3), (H4″) and (H5) in Theorem 4.1 for $\varepsilon > 0$ small enough.

Proof. For $f = f_4(u)$, we find that

$$\theta_2(u) = u\sin u + 2\cos u + \varepsilon u - 2,$$
$$\theta_2'(u) = u\cos u - \sin u + \varepsilon.$$



(a) (b)

FIGURE 6. (a) Graph of $f_4(u) = -\sin u + \varepsilon$ on $(0, c)$ for $\varepsilon = 0.2$, $a \approx 0.2014$, $b \approx 2.9402$, $c \approx 6.4845$, $\beta \approx 4.7770$, $r_2 = 0$, $s_2 = \pi \approx 3.1416$, $\sigma_2 \approx 4.4473$. (b) Graph of $\theta_2(u)$ on $(0, c)$. $t_2 \approx 0.8650$, $d \approx 6.1084$.

Let $\sigma_2 = \sigma_2(\varepsilon)$ be the unique zero of $\theta_2'(u)$ on $(\pi, 2\pi) \subset (b, c)$ and $\beta = \beta(\varepsilon)$ be the unique zero of

$$\int_a^u f_4(s)ds = \varepsilon u + \cos u - \varepsilon \sin^{-1} \varepsilon - \sqrt{1-\varepsilon^2}$$

on (b, c). It can be checked easily that

(i) f_4 satisfies (H1) and (H2) for $0 < \varepsilon < 1$.

(ii) f_4 satisfies the condition $\int_a^c f_4(u)du > 0$ in (H3) for $0 < \varepsilon < 1$. Also, for $\varepsilon > 0$ small enough, by continuity, $0 < \sigma_2(\varepsilon) < \beta(\varepsilon)$ since $\sigma_2(0) \approx 4.4934 < \beta(0) = 2\pi$, and

$$\theta_2(\sigma_2(\varepsilon)) < 0 \tag{4.2}$$

since $\theta_2(\sigma_2(0)) \approx -6.8206 < 0$. So f_4 satisfies (H3) for $\varepsilon > 0$ small enough.

(iii) We check that f_4 satisfies (H4$''$). First

$$\theta_2'(0) = f_4(0) = \varepsilon > 0, \tag{4.3}$$

$$\theta_2''(u) = -uf_4''(u) = -u \sin u \begin{cases} < 0 & \text{for } 0 = r_2 < u < s_2 = \pi, \\ > 0 & \text{for } s_2 = \pi < u < 2\pi, \\ < 0 & \text{for } 2\pi < u < c = 2\pi + \sin^{-1} \varepsilon. \end{cases} \tag{4.4}$$

Also, for $\varepsilon > 0$ small enough, let $t_2 = t_2(\varepsilon)$ be the unique zero of $\theta_2'(u) = u \cos u - \sin u + \varepsilon$ on $(0, \pi)$. It can be proved that $\lim_{\varepsilon \to 0^+} t_2(\varepsilon) = 0$. More precisely, we compute that

$$t_2(\varepsilon) \sim (2\varepsilon)^{1/3} \text{ as } \varepsilon \to 0^+$$

and hence

$$\theta_2(t_2(\varepsilon)) \sim 2^{1/3}\varepsilon^{4/3} \text{ as } \varepsilon \to 0^+.$$

Thus

$$\theta_2(2\pi) = 2\varepsilon\pi > \theta_2(t_2(\varepsilon)) \text{ for } \varepsilon > 0 \text{ small enough.} \tag{4.5}$$

We also find that

$$\theta_2'(2\pi) = 2\pi + \varepsilon > 0, \tag{4.6}$$

$$\theta_2'(c) = c\cos c = (2\pi + \sin^{-1}\varepsilon)\sqrt{1-\varepsilon^2} > 0. \tag{4.7}$$

So by (4.2)-(4.7), it can be proved that there exists $d \in (\sigma_2, 2\pi]$ such that f_4 satisfies

$$\theta_2''(u) = -uf_4''(u) = -u\sin u < 0 \quad \text{for } 0 = r_2 < u < s_2 = \pi,$$

$$\theta_2'(u) = f_4(u) - uf_4'(u) = u\cos u - \sin u + \varepsilon \begin{cases} < 0 & \text{for } s_2 < u < \sigma_2, \\ > 0 & \text{for } d \le u \le c = 2\pi + \sin^{-1}\varepsilon, \end{cases}$$

$$\theta_2''(u) = -uf_4''(u) = -u\sin u > 0 \quad \text{for } \sigma_2 \le u < d;$$

see Fig. 6(b). We omit the detailed proofs here. So f_4 satisfies (H4$''$).

(iv) f_4 satisfies (H5) automatically for $0 < \varepsilon < 1$ since $r_2 = 0$.

We conclude that f_4 satisfies all conditions (H1)-(H3), (H4$''$) and (H5) in Theorem 4.1 for $\varepsilon > 0$ small enough. □

5. Proofs of main results

First, we have the next lemma which holds for nonlinearities $f \in C[0,\infty)$ satisfying (H1), (H2) and the condition $\int_a^c f(s)ds > 0$ in (H3). We omit the proof.

Lemma 5.1. *Assume that $f \in C[0,\infty)$ satisfies (H1), (H2) and the condition $\int_a^c f(s)ds > 0$ in (H3). Consider the function defined by*

$$s \longmapsto G(\lambda, E, s) := E^p - p'\lambda F(s),$$

where $p > 1$, $E \ge 0$ and $\lambda > 0$ are real parameters. Then

(i) *If $E > E_c := (p'\lambda F(c))^{1/p} > 0$, then the function $G(\lambda, E, \cdot)$ is strictly positive on $(0,\infty)$.*

(ii) *If $E = E_c$, then the function $G(\lambda, E, \cdot)$ is strictly positive on $(0,c)$ and vanishes at c.*

(iii) *If $E_a := (p'\lambda F(a))^{1/p} < E < E_c$, then the function $G(\lambda, E, \cdot)$ has a unique zero $s_1(\lambda, E)$ on (β, c) and is strictly positive on $(0, s_1(\lambda, E))$. Moreover,*

 (a) *The function $E \mapsto s_1(\lambda, E)$ is C^1 on (E_a, E_c) and*

$$\frac{\partial s_1}{\partial E}(\lambda, E) = \frac{(p-1)E^{p-1}}{\lambda f(s_1(\lambda, E))} > 0 \text{ for all } E \in (E_a, E_c).$$

 (b) $\lim_{E\to E_a^+} s_1(\lambda, E) = \beta$ *and* $\lim_{E\to E_c^-} s_1(\lambda, E) = c$.

(iv) *If $E = E_a$, then*

$$G(\lambda, E) \begin{cases} > 0 & \text{for } 0 < s < a, \\ = 0 & \text{for } s = a, \\ > 0 & \text{for } a < s < \beta, \\ = 0 & \text{for } s = \beta, \\ < 0 & \text{for } \beta < s < c. \end{cases}$$

(v) *If $0 < E < E_a$, then the function $G(\lambda, E, \cdot)$ has a unique zero $s_2(\lambda, E)$ on $(0, a)$ and is strictly positive on $(0, s_2(\lambda, E))$. Moreover,*

 (a) *The function $E \mapsto s_2(\lambda, E)$ is C^1 on $(0, E_a)$ and*

$$\frac{\partial s_2}{\partial E}(\lambda, E) = \frac{(p-1)E^{p-1}}{\lambda f(s_2(\lambda, E))} > 0 \text{ for all } E \in (0, E_a).$$

 (b) $\lim_{E\to 0^+} s_2(\lambda, E) = 0$ *and* $\lim_{E\to E_a^-} s_2(\lambda, E) = a$.

(vi) *If* $E = 0$, *then* $G(\lambda, 0, s) \begin{cases} < 0 & \textit{for } 0 < s \le a, \\ < 0 & \textit{for } \beta \le s \le c. \end{cases}$

Now, for $p > 1$, $\lambda > 0$ and $E \ge 0$, we let

$$X_1(\lambda, E) := \{s \in \operatorname{dom} G(\lambda, E, \cdot) = (0, \infty) : G(\lambda, E, u) > 0 \quad \text{for all } u \in (0, s)\}.$$

In view of Lemma 5.1, it follows that

$$X_1(\lambda, E) = \begin{cases} (0, \infty) & \text{if } E > E_c, \\ (0, c] & \text{if } E = E_c, \\ (0, s_1(\lambda, E)] & \text{if } E_a < E < E_c, \\ (0, a] & \text{if } E = E_a, \\ (0, s_2(\lambda, E)] & \text{if } 0 < E < E_a, \\ \emptyset & \text{if } E = 0. \end{cases}$$

Therefore, $r_1(\lambda, 0) := 0$, and

$$r_1(\lambda, E) := \sup X_1(\lambda, E) = \begin{cases} \infty & \text{if } E > E_c, \\ c & \text{if } E = E_c, \\ s_1(\lambda, E) & \text{if } E_a < E < E_c, \\ a & \text{if } E = E_a, \\ s_2(\lambda, E) & \text{if } 0 < E < E_a, \end{cases}$$

Also, we let

$$\begin{aligned} X_2(\lambda, E) := \big\{ & s > r_1(\lambda, E) : s \in \operatorname{dom} G(\lambda, E, \cdot) = (0, \infty), \\ & G(\lambda, E, u) > 0 \text{ for all } u \in (r_1(\lambda, E), s) \big\} \end{aligned}$$

In view of Lemma 5.1,

$$X_2(\lambda, E) = \begin{cases} \emptyset & \text{if } E > E_c, \\ (c, \infty) & \text{if } E = E_c, \\ \emptyset & \text{if } E_a < E < E_c, \\ (a, \beta) & \text{if } E = E_a, \\ \emptyset & \text{if } 0 < E < E_a, \\ \emptyset & \text{if } E = 0. \end{cases}$$

Therefore, $r_2(\lambda, E) := \begin{cases} \beta & \text{if } E = E_a, \\ \infty & \text{otherwise.} \end{cases}$ Let

$$\begin{aligned} D_1(p, \lambda) := \big\{ & E > 0 : r_1(\lambda, E) \in \operatorname{dom} G(\lambda, E, \cdot) = (0, \infty), \\ & G(\lambda, E, r_1(\lambda, E)) = 0, \text{ and } f(r_1(\lambda, E)) > 0 \big\} \\ = {} & (0, E_a) \cup (E_a, E_c) \end{aligned}$$

and

$$\begin{aligned} D_2(p, \lambda) := \big\{ & E > 0 : r_2(\lambda, E) \in \operatorname{dom} G(\lambda, E, \cdot) = (0, \infty), \\ & G(\lambda, E, r_2(\lambda, E)) = 0, \text{ and } f(r_2(\lambda, E)) > 0 \\ = {} & \{E_a\}. \end{aligned}$$

Note that the definition domains of the time maps T_1 and T_2 are

$$\begin{aligned}\tilde{D}_1(p,\lambda) := \Big\{ & E \geq 0 : r_1(\lambda, E) \in \operatorname{dom} G(\lambda, E, \cdot) = (0,\infty), G(\lambda, E, r_1(\lambda,E)) = 0, \\ & \text{and } \int_0^{r_1(\lambda,E)} (G(\lambda,E,u))^{-1/p} du < \infty \Big\},\end{aligned}$$

$$\begin{aligned}\tilde{D}_2(p,\lambda) := \Big\{ & E \geq 0 : r_2(\lambda, E) \in \operatorname{dom} G(\lambda, E, \cdot) = (0,\infty), G(\lambda, E, r_2(\lambda,E)) = 0, \\ & \text{and } \int_0^{r_2(\lambda,E)} (G(\lambda,E,u))^{-1/p} du < \infty \Big\}\end{aligned}$$

In the present case, $(0, E_a) \cup (E_a, E_c) \subset \tilde{D}_1(p,\lambda) \subset [0, E_a] \cup [E_a, E_c]$, and $\tilde{D}_2(p,\lambda) \subset \{E_a\}$. We define, for $E \in \tilde{D}_1(p,\lambda)$, the time map

$$\begin{aligned}T_1(\lambda, E) &:= \int_0^{r_1(\lambda,E)} (G(\lambda,E,u))^{-1/p} du \\ &= \int_0^{r_1(\lambda,E)} (E^p - p'\lambda F(u))^{-1/p} du \\ &= (p'\lambda)^{-1/p} \int_0^{r_1(\lambda,E)} (F(r_1(\lambda,E)) - F(u))^{-1/p} du,\end{aligned}$$

since $G(\lambda, E, r_1(\lambda,E)) = E^p - p'\lambda F(r_1(\lambda,E)) = 0$. For all $\lambda > 0$, $r_1(\lambda,\cdot)$ is an increasing C^1-diffeomorphism from $(0, E_a]$ onto $(0, a]$ and from $(E_a, E_c]$ onto $(\beta, c]$. Thus T_1 may be written as

$$T_1(p,\lambda,E) = (p'\lambda)^{-1/p} S(p, r_1(\lambda,E)) \text{ for } E \in \overline{D}_1(p,\lambda),$$

where for all $p > 1$, $S(p,\cdot)$ is defined by

$$S(p,\alpha) := \int_0^{\alpha} (F(\alpha) - F(u))^{-1/p} du \text{ for all } \alpha \in (0,a] \cup (\beta, c]. \tag{5.1}$$

Note that $S(p,\cdot)$ takes its values in $[0,\infty]$. We define, for $E \in \tilde{D}_2(p,\lambda)$, the time map

$$T_2(p,\lambda,E) := \int_0^{r_2(\lambda,E)} (G(\lambda,E,u))^{-1/p} du = (p'\lambda)^{-1/p} S(p, r_2(\lambda,E)).$$

Note that, if $\tilde{D}_2(p,\lambda) \neq \emptyset$ then $\tilde{D}_2(p,\lambda) = \{E_a\}$ and $r_2(\lambda,E) = \beta$. That is why we extend the definition domain of $S(p,\cdot)$ by including the eventual range of $r_2(\lambda,\cdot)$; that is, we define $S(p,\cdot)$ on $(0,a] \cup [\beta, c]$. On the other hand, continuity arguments imply that if $\tilde{D}_2(p,\lambda) = \{E_a\}$ then

$$\begin{aligned}T_2(p,\lambda,E_a) &= (p'\lambda)^{-1/p} S(p, r_2(\lambda,E)) = (p'\lambda)^{-1/p} S(p,\beta) \\ &= \lim_{\alpha \to \beta^+} (p'\lambda)^{-1/p} S(p,\alpha) = \lim_{E \to E_a^+} (p'\lambda)^{-1/p} S(p, r_1(\lambda,E)) \\ &= \lim_{E \to E_a^+} T_1(p,\lambda,E).\end{aligned}$$

So we simply study the function $\alpha \mapsto S(p,\alpha)$ for $\alpha \in (0,a] \cup [\beta, c]$, and if $S(p,\alpha) < \infty$, we intend that

$$S(p,\alpha) = \begin{cases} (p'\lambda)^{1/p} T_1(p,\lambda,E_\alpha := r_1^{-1}(\lambda,\alpha)) & \text{if } \alpha \in (0,a] \cup (\beta, c], \\ (p'\lambda)^{1/p} T_2(p,\lambda,E_a) & \text{if } \alpha = \beta. \end{cases}$$

Lemma 5.2. *$f'(a) < 0$ and $f'(c) < 0$.*

The proof of Lemma 5.2 is easy but tedious; we omit it.

Lemma 5.3. (i) *$S(p, 0) = 0$ if $p > 1$.*
(ii) *$S(p, a) = \infty$ if and only if $1 < p \le 2$.*
(iii) *$S(p, \beta) = \infty$ if and only if $1 < p \le 2$.*
(iv) *$S(p, c) = \infty$ if and only if $1 < p \le 2$.*

Proof. (i) For $p > 1$ and $0 < \alpha < a$, we write $S(p, \alpha)$ in (5.1) as

$$\begin{aligned} S(p,\alpha) &= \int_0^\alpha (F(\alpha) - F(u))^{-1/p} du \\ &= \alpha (F(\alpha))^{-1/p} \int_0^1 \big(1 - \frac{F(\alpha t)}{F(\alpha)}\big)^{-1/p} dt \quad (\text{let } u = \alpha t). \end{aligned}$$

Applying l'Hopital's rule, it is easy to see that $\lim_{\alpha\to 0^+} \alpha(F(\alpha))^{-1/p} = 0$ and $\lim_{\alpha\to 0^+} \frac{F(\alpha t)}{F(\alpha)} = t$. Therefore, $S(p, 0) = \lim_{\alpha\to 0^+} S(p, \alpha) = 0 \cdot \int_0^1 (1 - t)^{-1/p} dt = 0 \cdot p' = 0$. Hence the result follows.

(ii) Recall that for $p > 1$, $S(p, a) = \int_0^a (F(a) - F(u))^{-1/p} du$. Note that $F(a) - F(u) = -\frac{1}{2} f'(a)(a - u)^2 + o((u - a)^2)$ near a^- and by Lemma 5.2, $f'(a) < 0$. Therefore,

$$(F(a) - F(u))^{-1/p} \approx (-f'(a)/2)^{-1/p} (a - u)^{-2/p} \quad \text{near } a^-.$$

Then easy computation shows that $S(p, a) = \infty$ if and only if $1 < p \le 2$.

(iii) We write

$$S(p,\beta) = \int_0^\beta (F(\beta) - F(u))^{-1/p} du = \big(\int_0^a + \int_a^\beta\big)\big(F(\beta) - F(u)\big)^{-1/p} du.$$

(**Eventual singularity at** β^-) Note that $F(\beta) - F(u) = f(\beta)(\beta - u) + o(\beta - u)$ near β^-. Since $f(\beta) > 0$, $(F(\beta) - F(u))^{-1/p} \approx (f(\beta))^{-1/p}(\beta - u)^{-1/p}$ near β^-. Then easy computation shows that $\int_{\beta-\varepsilon}^\beta (F(\beta) - F(u))^{-1/p} du < \infty$ for $p > 1$ and $\varepsilon > 0$ sufficiently small.

(**Eventual singularity at** a^-) Since $F(\beta) = F(a)$ by (H3), the same arguments as those used in the proof of part (ii) above imply that $\int_0^a (F(\beta) - F(u))^{-1/p} du = \infty$ if and only if $1 < p \le 2$.

(**Eventual singularity at** a^+) Since $F(\beta) = F(a)$, the same arguments as those used in the proof of part (ii) above imply that $\int_a^{a+\varepsilon} (F(\beta) - F(u))^{-1/p} du = \infty$ if and only if $1 < p \le 2$ for $\varepsilon > 0$ sufficiently small.

In above analysis, $S(p, \beta) = \infty$ if and only if $1 < p \le 2$.

(iv) Recall that for $p > 1$, $S(p, c) = \int_0^c (F(c) - F(u))^{-1/p} du$. Note that $F(c) - F(u) = -\frac{1}{2} f'(c)(c - u)^2 + o((u - c)^2)$ near c^- and by Lemma 5.2, $f'(c) < 0$. Therefore,

$$(F(c) - F(u))^{-1/p} \approx (-f'(c)/2)^{-1/p} (c - u)^{-2/p} \quad \text{near } c^-.$$

Then an easy computation shows that $S(p, c) = \infty$ if and only if $1 < p \le 2$. □

Next, we study the variations of $S(p, \alpha)$ for $\alpha \in (0, a) \cup (\beta, c)$. For $p > 1$, $S'(p, \alpha)$ is given by

$$S'(p,\alpha) = \frac{1}{p\alpha} \int_0^\alpha \frac{\theta_p(\alpha) - \theta_p(u)}{(F(\alpha) - F(u))^{1/p}} du \quad \text{for } \alpha \in (0, a) \cup (\beta, c), \tag{5.2}$$

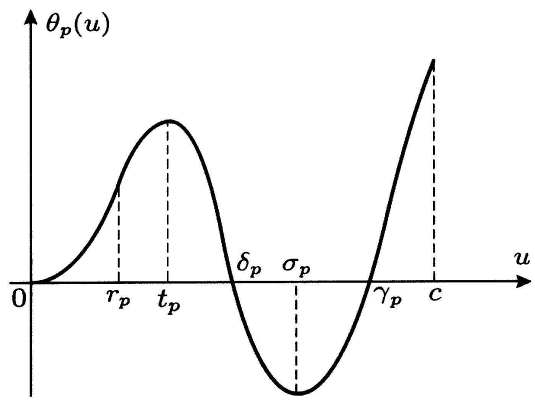


FIGURE 7. Graph of $\theta_p(u)$

where $\theta_p(u) = pF(u) - uf(u)$. This implies

$$\theta_p'(u) = (p-1)f(u) - uf'(u),$$
$$\theta_p''(u) = (p-2)f'(u) - uf''(u).$$

Thus by (H1) and (H4),

$$\begin{gathered}\theta_p(0) = 0,\\ \theta_p'(0) = (p-1)f(0) > 0,\\ \theta_p''(u)\begin{cases} > 0 & \text{for } 0 < u < r_p,\\ < 0 & \text{for } r_p < u < s_p,\\ > 0 & \text{for } s_p < u < c.\end{cases}\end{gathered} \tag{5.3}$$

In addition, by (H3), (H2) and Lemma 5.2,

$$\begin{gathered}\theta_p(\beta^*) < 0,\\ \theta_p(c) = pF(c) - cf(c) = pF(c) > 0,\\ \theta_p'(c) = (p-1)f(c) - cf'(c) = -cf'(c) > 0.\end{gathered}$$

Hence there exist $t_p \in (r_p, s_p)$ and $\sigma_p \in (s_p, c)$ such that

$$\theta_p \quad \text{is strictly increasing on } (0, t_p), \tag{5.4}$$
$$\theta_p \quad \text{is strictly decreasing on } (t_p, \sigma_p), \tag{5.5}$$
$$\theta_p \quad \text{is strictly increasing on } (\sigma_p, c). \tag{5.6}$$

In addition, there exist $\delta_p \in (t_p, \sigma_p)$ and $\gamma_p \in (\sigma_p, c)$ such that

$$\theta_p(\delta_p) = \theta_p(\gamma_p) = 0. \tag{5.7}$$

The typical graph of $\theta_p(u)$ on $[0, c]$ is depicted in Fig. 7.

Lemma 5.4. *For $p > 1$, the following statements hold*

(i) *$S(p, \alpha)$ is strictly increasing on $(0, a)$.*
(ii) *$S(p, \alpha)$ has exactly one critical point, a minimum, on (β, c). More precisely, there exists a unique $m_p \in (\beta, c)$ such that $S(p, \alpha)$ is strictly decreasing on (β, m_p) and is strictly increasing on (m_p, c).*

Proof. Part (i). By (5.2)-(5.4), it suffices to show that
$$0 < a < t_p \quad \text{for } p > 1. \tag{5.8}$$
Note that
$$\theta_p(a) = pF(a) - af(a) = pF(a) > 0,$$
and by Lemma 5.2,
$$\theta_p'(a) = (p-1)f(a) - af'(a) = -af'(a) > 0,$$
then $a \in (0, t_p) \cup (\gamma_p, c)$ by (5.3)–(5.7). If $a \in (\gamma_p, c)$ $(\subset (\sigma_p, c))$, then (5.6) implies that θ_p is strictly increasing on (a, c). Hence
$$\theta_p'(u) > 0 \quad \text{for } u \in (a, c). \tag{5.9}$$
However, (H2) implies that there exists $\eta_p \in (a, b)$ $(\subset (a, c))$ such that $f(\eta_p) < 0$ and $f'(\eta_p) = 0$. Therefore,
$$\theta_p'(\eta_p) = (p-1)f(\eta_p) - \eta_p f'(\eta_p) = (p-1)f(\eta_p) < 0 \quad \text{for } p > 1,$$
which leads to a contradiction with (5.9). Therefore, (5.8) holds and hence part (i) follows. Part (ii) follows by exactly the same arguments used to prove [3, Lemma 4.7]. To this end, it suffices to prove the following two lemmas. □

Lemma 5.5. *For $p > 1$, $S''(p, \alpha) + (p/\alpha)S'(p, \alpha) > 0$ for all $\alpha \in (\max\{\sigma_p, \beta\}, c)$.*

The proof of Lemma 5.5 is the same as that of [3, Lemma 4.6]; we omit it.

Lemma 5.6. *Assume that $p > 1$.*

(i) *If $p > 2$ then $S'(p, c) = \infty$.*
(ii) *If $\beta < \sigma_p$ then $S'(p, \alpha) < 0$ for all $\alpha \in (\beta, \sigma_p]$.*
(iii) *If $\beta = \sigma_p$ then $S(p, \beta) < \infty$ then $-\infty \le S'(p, \beta) < 0$.*
(iv) *If $\beta > \sigma_p$ and $S(p, \beta) < \infty$ then $S'(p, \beta) = -\infty$.*

Proof. The proofs of parts (i) and (iii) follow exactly as those of parts (i) and (iii) of [3, Lemma 4.5]; we omit them. For part (ii) we point out that by (H3) $(\theta_p(\beta^*) < 0$ where $\beta^* \le \beta)$ it follows that $\delta_p < \beta$. Then the argument used to prove Lemma 4.5(ii) of [3] can apply to prove $S'(p, \alpha) < 0$ for all $\alpha \in (\beta, \sigma_p]$. So part (ii) holds. Proof of part (iv). Since $F(\beta) = F(a) > 0$, it follows that the integral representing $S'(p, \beta)$, has two singularities; one at a and the other at β. So we write
$$S'(p, \beta) = (p\beta)^{-1}(I_{a^-} + I_{a^+} + I_\beta),$$
where
$$I_{a^-} := \int_0^a \frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{(p+1)/p}} du,$$
$$I_{a^+} := \int_a^{(a+\beta)/2} \frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{(p+1)/p}} du,$$
$$I_\beta := \int_{(a+\beta)/2}^{\beta} \frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{(p+1)/p}} du.$$
We next show $I_\beta < \infty$ and $I_{a^\pm} = -\infty$. First we show $I_\beta < \infty$. Since $(c >)\ \beta > \sigma_p$, it follows that $\theta_p'(\beta) > 0$. Therefore,
$$\frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{(p+1)/p}} \approx \frac{\theta_p'(\beta)}{(f(\beta))^{(p+1)/p}} \frac{1}{(\beta - u)^{1/p}} \quad \text{near } \beta^-.$$

Since $1/p < 1$ and $\theta_p'(\beta)(f(\beta))^{-(p+1)/p} > 0$, easy computations show that $I_\beta < \infty$. We then show $I_{a^\pm} = -\infty$. Note that

$$\frac{\theta_p(\beta) - \theta_p(u)}{(F(\beta) - F(u))^{(p+1)/p}} \approx \frac{\theta_p(\beta) - \theta_p(a)}{(-f'(a)/2)^{(p+1)/p}} \frac{1}{(a-u)^{2(p+1)/p}} \quad \text{near } a.$$

Since $2(p+1)/p > 1$ and

$$\frac{\theta_p(\beta) - \theta_p(a)}{(-f'(a)/2)^{(p+1)/p}} = \frac{-\beta f(\beta)}{(-f'(a)/2)^{(p+1)/p}} < 0,$$

easy computations show that $I_{a^\pm} = -\infty$. This completes the proof of Lemma 5.6. Therefore, the proof Lemma 5.4 is also complete. □

Lemma 5.7. *For $p > 2$, $0 < \alpha_p < \lambda_p < \infty$.*

Proof. By (3.2), (5.1) and Lemma 5.3(iii), for $p > 2$,

$$\lambda_p = \Big\{ \int_0^\beta (F(\beta) - F(u))^{-1/p} du \Big\}^p / p' = \Big\{ \lim_{\alpha \to \beta^+} S(p, \alpha) \Big\}^p / p' < \infty.$$

We then find that

$$\begin{aligned}
\lambda_p &= \{ \int_0^\beta (F(\beta) - F(u))^{-1/p} du \}^p / p' \\
&= \{ \int_0^a (F(\beta) - F(u))^{-1/p} du + \int_a^\beta (F(\beta) - F(u))^{-1/p} du \}^p / p' \\
&> \{ \int_0^a (F(\beta) - F(u))^{-1/p} du \}^p / p' \\
&= \{ \int_0^a (F(a) - F(u))^{-1/p} du \}^p / p' \quad (\text{since } F(\beta) = F(a)) \\
&= \alpha_p > 0 \quad (\text{by } (3.1)).
\end{aligned}$$

This completes the proof. □

Let u be a positive solution of (1.1), then $0 < \|u\|_\infty \le a$ or $\beta \le \|u\|_\infty \le c$. In addition, $u \in A_0^+ \cup A_1^+ \cup A_{00}^+ \cup A_{01}^+$ by Lemma 2.2.

By Lemma 5.3(ii)-(iv), for $p > 2$, $S(p, a) < \infty$, $S(p, \beta) < \infty$, $S(p, c) < \infty$. In this case we have the following three statements:

(i) Suppose for $\lambda = \alpha_p = (S(p, a))^p / p'$, u_{α_p} is the corresponding solution of (1.1) satisfying $\|u_{\alpha_p}\|_\infty = u_{\alpha_p}(0) = a$. Then

$$u_{\alpha_p}'(x) = \{ p' \alpha_p [F(a) - F(u(x))] \}^{1/p} > 0 \quad \text{for } -1 \le x < 0$$

by Lemma 2.1. So $u_{\alpha_p} \in A_0^+$. Then for each $\lambda > \alpha_p$,

$$u_\lambda(x) := \begin{cases} u_{\alpha_p}\big((\frac{\lambda}{\alpha_p})^{1/p} (|x| - 1 + (\frac{\alpha_p}{\lambda})^{1/p}) \big) & \text{if } 1 - (\frac{\alpha_p}{\lambda})^{1/p} < |x| \le 1, \\ a & \text{if } |x| \le 1 - (\frac{\alpha_p}{\lambda})^{1/p} \end{cases}$$

is a C^1 dead core solution of (1.1) satisfying $\|u_\lambda\|_\infty = a$,

$$u_\lambda'(-1) = (p' \lambda F(a))^{1/p} > (p' \alpha_p F(a))^{1/p} = u_{\alpha_p}'(-1) > 0,$$

and $u_\lambda \in A_1^+$.

(ii) Suppose for $\lambda = \nu_p = (S_p(c))^p/p'$, u_{ν_p} is the corresponding solution of (1.1) satisfying $\|u_{\nu_p}\|_\infty = u_{\nu_p}(0) = c$. Then

$$u'_{\nu_p}(x) = \{p'\nu_p[F(c) - F(u(x))]\}^{1/p} > 0 \text{ for } -1 \le x < 0$$

by Lemma 2.1. So $u_{\nu_p} \in A_0^+$. Then for each $\lambda > \nu_p$,

$$u_\lambda(x) := \begin{cases} u_{\nu_p}\big((\frac{\lambda}{\nu_p})^{1/p}(|x| - 1 + (\frac{\nu_p}{\lambda})^{1/p})\big) & \text{if } 1 - (\frac{\nu_p}{\lambda})^{1/p} < |x| \le 1, \\ c & \text{if } |x| \le 1 - (\frac{\nu_p}{\lambda})^{1/p} \end{cases}$$

is a C^1 dead core solution of (1.1) satisfying $\|u_\lambda\|_\infty = c$,

$$u'_\lambda(-1) = (p'\lambda F(c))^{1/p} > (p'\nu_p F(c))^{1/p} = u'_{\nu_p}(-1) > 0,$$

and $u_\lambda \in A_1^+$.
(iii) Suppose for $\lambda = \lambda_p = (S(p,\beta))^p/p'$, u_{λ_p} is the corresponding solution of (1.1) satisfying $\|u_{\lambda_p}\|_\infty = u_{\lambda_p}(0) = \beta$. Then, by Lemma 2.1, there exists a unique negative number $-x_0 \in (-1, 0)$ such that

$$u_{\lambda_p}(-x_0) = a \text{ and } u'_{\lambda_p}(-x_0) = 0$$

and

$$u'_{\lambda_p}(x) = \{p'\lambda_p[F(\beta) - F(u(x))]\}^{1/p} > 0 \quad \text{for } x \in [-1, 0) - \{-x_0\}.$$

So $u_{\lambda_p} \in A_{00}^+$. Then for each $\lambda > \lambda_p$,

$$u_\lambda(x) := \begin{cases} u_{\lambda_p}\big((\frac{\lambda}{\lambda_p})^{1/p}|x|\big) & \text{if } |x| \le (\frac{K_{0a}}{\lambda})^{1/p}, \\ a & \text{if } (\frac{K_{0a}}{\lambda})^{1/p} \le |x| \le 1 - (\frac{K_{a\beta}}{\lambda})^{1/p}, \\ u_{\lambda_p}\Big(\frac{|x| - 1 + (\frac{K_{a\beta}}{\lambda})^{1/p} + (\frac{K_{a\beta}}{\lambda_p})^{1/p}}{(\frac{K_{a\beta}}{\lambda})^{1/p} + (\frac{K_{a\beta}}{\lambda_p})^{1/p}}\Big) & \text{if } 1 - (\frac{K_{a\beta}}{\lambda})^{1/p} \le |x| \le 1 \end{cases}$$

is a positive solution of (1.1) satisfying $\|u_\lambda\|_\infty = \beta$,

$$u'_\lambda(-1) = (p'\lambda F(\beta))^{1/p} > (p'\lambda_p F(\beta))^{1/p} = u'_{\lambda_p}(-1) > 0,$$

and $u_\lambda \in A_{01}^+$, where

$$K_{0a} := (\int_0^a (F(\beta) - F(u))^{-1/p} du)^p/p',$$
$$K_{a\beta} := (\int_a^\beta (F(\beta) - F(u))^{-1/p} du)^p/p'.$$

Hence Theorems 3.2-3.7 follow immediately by Theorem 2.3 and Lemmas 5.3-5.7.

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References

[1] I. Addou, Multiplicity of solutions for quasilinear elliptic boundary-value problems, *Electron. J. Differential Equations* **1999** (1999), No. 21, 1-27.

[2] I. Addou, Exact multiplicity results for quasilinear boundary-value problems with cubic-like nonlinearities, *Electron. J. Differential Equations* **2000** (2000), No. 01, 1-26, (Addendum, 27-29).

[3] I. Addou, and S.-H. Wang, Exact multiplicity results for some p-Laplacian nonpositone problems with concave-convex-concave nonlinearities. *Nonlinear Analysis* **53** (2003), 111-137.

[4] M. Guedda and L. Veron, Bifurcation phenomena associated to the p-Laplace operator, *Trans. Amer. Math. Soc.* **310** (1988), 419-431.

[5] N. Jacobson, *Basic Algebra*, Freeman, New York, 1989.

[6] P. Korman, Y. Li and T. Ouyang, Exact multiplicity results for boundary problems with nonlinearities generalizing cubic, *Proc. Roy. Soc. Edinburgh* **126A** (1996), 599-616.

[7] P. Korman, Y. Li and T. Ouyang, Perturbation of global solution curves for semilinear problems, *Adv. Nonlinear Stud.* **3** (2003), 289–299.

[8] P. Korman and J. Shi, Instability and exact multiplicity of solutions of semilinear equations, *Electron. J. Diff. Equ. Conf.,* **5** (2000), 311-322.

[9] J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions, *J. Differential Equations* **39** (1981), 269-290.

[10] S.-H. Wang, A correction for a paper by J. Smoller and A. Wasserman, *J. Differential Equations* **77** (1989), 199-202.

[11] S.-H. Wang, On the time map of a nonlinear two-point boundary value problem, *Differential Integral Equations* **7** (1994), 49-55.

[12] S.-H. Wang and N. D. Kazarinoff, Bifurcation of steady-state solutions of a scalar reaction-diffusion equation in one space variable, *J. Austral. Math. Soc.* (Series A) **52** (1992), 343-355.

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