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PERMANENCE OF PREDATOR-PREY SYSTEM WITH INFINITE DELAY

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Abstract. In this paper we consider a predator-prey system with periodic coefficients and infinite delay, in which the prey has a history that takes them through two stages, immature and mature. We provide a sufficient and necessary condition to guarantee the permanence of the system. The system has a positive periodic solution under this condition. Some known results are extended to the delay case.

1. Introduction

In this paper, we consider the following periodic predator-prey system with infinite delay and stage structure

$$
\begin{gathered}
\dot{x_1} = a(t)x_2 - b(t)x_1 - d(t)x_1^2 - p(t)x_1 \int_{-\infty}^{0} k_{12}(s)y(t+s)\,ds, \\
\dot{x_2} = c(t)x_1 - f(t)x_2^2, \\
\dot{y} = y\big[-g(t) + h(t)\int_{-\infty}^{0} k_{21}(s)x_1(t+s)\,ds - q(t)\int_{-\infty}^{0} k_{22}(s)y(t+s)\,ds\big],
\end{gathered}
\tag{1.1}
$$

where x_1 and x_2 denote the density of immature and mature population A(prey) respectively, and y is the density of the predator B that prey on x_1. The coefficients in (1.1) are all ω-periodic and continuous for $t \geq 0$, $a(t), b(t), c(t), d(t)$ and $f(t)$ are all positive, $p(t), h(t)$ and $q(t)$ are nonnegative, and $\int_0^\omega q(t)dt > 0$, $\int_0^\omega g(t)dt \geq 0$. The functions $k_{ij}(s) (i,j = 1,2)$ defined on $\mathbb{R}_- = (-\infty, 0]$ are nonnegative and integrable, $\int_{-\infty}^0 k_{ij}(s) = 1$. The biological background for (1.1) can be found in [4, 14].

Predator-prey systems have been studied in many articles; see for example [5, 6, 8, 9, 10]. However, for the predator-prey systems with stage structure and infinite time delay, we have not obtained necessary and sufficient conditions for its permanence. In the natural world, however, there are many species whose individual members have a life history that take them through two stages, immature and mature. In particularly, we have in mind mammalian populations and some

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amphibious animals, which exhibit these two stages. Recently, autonomous systems with stage structure of species have been considered in [1, 2, 13, 14], in particularly, the effect of dispersal on the permanence of a single species with stage structure was discussed in [11]. And two species predator-prey Lotka-Volterra type dispersal systems with periodic coefficients and infinite delays have been studied in [11].

In this paper, we consider system (1.1) with periodic coefficients. Our purpose is to establish sufficient and necessary conditions of integrable form for the permanence of system (1.1) .

2. Main Results

When $f(t)$ is a continuous ω-periodic function defined on $[0,+\infty)$, we set

$$A_\omega(f)=\omega^{-1}\int_0^\omega f(t)dt,\quad f^M=\max_{t\geq 0}f(t),\quad f^L=\min_{t\geq 0}f(t).$$

Let set C_+ $=\{\phi=(\phi_1,\phi_2,\phi_3):\phi_i(t)$ is continuous and nonnegative on $\mathbb{R}_-$ and $\phi_i(0)>0,\ i=1,2,3\}$

Definition The system $\dot{x}=F(t,x)$, $x\in\mathbb{R}^n$ is said to be permanent if there are constants $M\geq m>0$ such that every positive solution $x(t)=(x_1(t),\dots,x_n(t))\in R^n_+=\{(x_1,\dots,x_n):x_i>0,\ i=1,\dots,n\}$ of this system, satisfies

$$m\leq\liminf_{t\to\infty}x_i(t)\leq\limsup_{t\to\infty}x_i(t)\leq M\,.$$

Lemma 2.1 ([4]). *The system*

$$\begin{gathered}\dot{x_1}=a(t)x_2-b(t)x_1-d(t)x_1^2,\\ \dot{x_2}=c(t)x_1-f(t)x_2^2\end{gathered}\tag{2.1}$$

has a positive ω-periodic solution $(x_1^*(t),x_2^*(t))$ which is globally asymptotically stable with respect to $R^2_{+0}=\{(x_1,x_2):x_1>0,x_2>0\}$.

Lemma 2.2 ([12]). *For the well-known periodic logistic equation*

$$\dot{u}=u[a(t)-b(t)u]\tag{2.2}$$

where $a(t)$ and $b(t)$ are $\omega-$periodic continuous functions, $b^l\geq 0$ and $A_\omega(b)>0$, there is a constant $M>0$ such that every positive solution $u(t)$ of (2.2) *satisfies* $\limsup_{t\to\infty}u(t)\leq M$.

Theorem 2.3. *System* (1.1) *is permanent if and only if*

$$\int_0^\omega[-g(t)+h(t)\int_{-\infty}^0 k_{21}(s)x_1^*(t+s)\,ds]dt>0,\tag{2.3}$$

where $(x_1^*(t),x_2^*(t))$ is the positive ω-periodic solution of (2.1).

To prove this theorem, we need several Lemmas. In this paper we always assume that solutions of system (1.1) satisfy the initial conditions:

$$x_i(s)=\varphi_i(s),y(s)=\psi(s),\quad(i=1,2),\ (\varphi_1,\varphi_2,\psi)\in C_+,\ s\in(-\infty,0].$$

Lemma 2.4. *There exist positive constants M_x and M_y such that*

$$\limsup_{t\to\infty}x_i(t)\leq M_x,\quad\limsup_{t\to\infty}y(t)\leq M_y.$$

Proof. Obviously, R^3_+ is a positively invariant set of system (1.1). Given any positive solution $(x_1(t), x_2(t), y(t))$, we have

$$\dot{x_1} \le a(t)x_2 - b(t)x_1 - d(t)x_1^2,$$
$$\dot{x_2} = c(t)x_1 - f(t)x_2^2.$$

By the vector comparison theorem [7], we obtain

$$x_i(t) \le \bar{x}_i(t) \quad (i = 1, 2)$$

for all $t \ge 0$, where $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t))$ is the solution of (2.1) with $\bar{x}(0) = x(0)$. By the global asymptotic stability of $x^*(t)$, there is a $T_0 > 0$ such that for all $t \ge T_0$,

$$\bar{x}_i(t) \le M_x \quad (i = 1, 2);$$

hence, for all $t \ge T_0$,

$$x_i(t) \le M_x \quad (i = 1, 2). \tag{2.4}$$

Consequently, $\limsup_{t\to\infty} x_i(t) \le M_x$, $i = 1, 2$. Let $\bar{e}(t) = -g(t) + 2M_x h(t)$ and $H_0 = \sup\{x_1(t + s) + x_2(t + s) \mid t \ge 0, s \le 0\}$ and let the constant $\tau > 0$ be such that

$$H_0 \int_{-\infty}^{-\tau} k_{21}(s)\, ds < M_x, \tag{2.5}$$

$$\int_{-\tau}^{0} k_{22}(s) \exp(\bar{e}^m s)\, ds > 0. \tag{2.6}$$

From (2.4), for any $t \ge T_0 + \tau$ we have

$$\begin{aligned}
\dot{y} &\le y[-g(t) + h(t) \int_{-\infty}^{0} k_{21}(s)x_1(t + s)\, ds] \\
&\le y\big[- g(t) + h(t) \int_{-\infty}^{-\tau} k_{21}(s)H_0\, ds + h(t) \int_{-\tau}^{0} k_{21}(s)M_x\, ds\big] \\
&\le y[-g(t) + 2M_x h(t)] = y\bar{e}(t).
\end{aligned}$$

Hence, for any $t \ge t + s \ge T_0 + \tau$ we obtain

$$y(t + s) \ge y(t) \exp \int_{t}^{t+s} \bar{e}(u)du \ge y(t) \exp(\bar{e}^m s).$$

From this, for any $t \ge T_0 + 2\tau$, we have

$$\begin{aligned}
\dot{y} &\le y[\bar{e}(t) - q(t) \int_{-\infty}^{0} k_{22}(s)y(t + s)\, ds] \\
&\le y\big[\bar{e}(t) - q(t) \int_{-\tau}^{0} k_{22}(s)y(t + s)\, ds\big] \\
&\le y\big[\bar{e}(t) - q(t) \int_{-\tau}^{0} k_{22}(s) \exp(\bar{e}^m s)\, ds\; y(t)\big].
\end{aligned}$$

Let $u(t)$ be the solution of the auxiliary equation

$$\dot{u} = u[\bar{e}(t) - q(t) \int_{-\tau}^{0} k_{22}(s) \exp(\bar{e}^m s)\, dsu]$$

with the initial condition $u(T_0 + 2\tau) = y(T_0 + 2\tau)$. Then we obtain

$$y(t) \le u(t) \tag{2.7}$$

for all $t \geq T_0 + 2\tau$. From Lemma 2.2 , (2.6), $q(t) \geq 0$ and $A_\omega(q) > 0$, we know that there is a constant $M_y > 0$ such that

$$\limsup_{t\to\infty} u(t) \leq M_y.$$

Consequently, by (2.7) we have

$$\limsup_{t\to\infty} y(t) \leq M_y. \tag{2.8}$$

□

Lemma 2.5. *There is a positive constant δ_x ($\delta_x < M_x$ such that*

$$\liminf_{t\to\infty} x_i(t) \geq \delta_x.$$

Proof. There exists a constant $\sigma > 0$ such that

$$H_1 \int_{-\infty}^{-\sigma} k_{12}(s)\, ds < M_y,$$

where $H_1 = \sup\{y(t+s) \mid t \geq 0, s \leq 0\}$. From (2.8), there is a constant $T_1 > 0$ such that for all $t \geq T_1$ $y(t) \leq M_y$. For every $t \geq T_1 + \sigma$, we have

$$\begin{aligned}
\dot{x_1} &= a(t)x_2 - b(t)x_1 - d(t)x_1^2 - p(t)x_1 \int_{-\infty}^{-\sigma} k_{12}(s) y(t+s)\, ds \\
&\quad - p(t)x_1 \int_{-\sigma}^{0} k_{12}(s) y(t+s)\, ds \\
&\geq a(t)x_2 - b(t)x_1 - d(t)x_1^2 - 2M_y p(t) x_1, \\
\dot{x_2} &= c(t)x_1 - f(t)x_2^2\,.
\end{aligned}$$

Consider the auxiliary system

$$\begin{gathered}
\dot{u_1} = a(t)u_2 - [b(t) + 2M_y p(t)]u_1 - d(t)u_1^2, \\
\dot{u_2} = c(t)u_1 - f(t)u_2^2.
\end{gathered} \tag{2.9}$$

Let $u(t) = (u_1(t), u_2(t))$ is the solution of system (2.9) with the initial condition $u(T_1 + \sigma) = x(T_1 + \sigma)$, then for all $t \geq T_1 + \sigma$,

$$x_i(t) \geq u_i(t).$$

By Lemma 2.1, (2.9) has a positive ω-periodic solution $u^*(t) = (u_1^*(t), u_2^*(t))$, which is globally asymptotically stable. By the global asymptotic stability of $u^*(t)$, there exist constants $\delta_x > 0$ and $T_2 > T_1 + \sigma$ such that for all $t \geq T_2$,

$$u_i(t) \geq \delta_x.$$

Hence, for all $t \geq T_2$, $x_i(t) \geq \delta_x$. So we have

$$\liminf_{t\to\infty} x_i(t) \geq \delta_x.$$

□

Lemma 2.6. *There is a positive constant β ($\beta < M_y$) such that*

$$\limsup_{t\to\infty} y(t) > \beta. \tag{2.10}$$

Proof. By (2.3), we can choose constant ε_0, $0 < \varepsilon_0 < 1$, such that

$$\int_0^{\omega} [-g(t) + h(t) \int_{-\infty}^{0} k_{21}(s) x_1^*(t+s)\, ds - 2h(t)\varepsilon_0 - 2q(t)\varepsilon_0] dt > \varepsilon_0. \tag{2.11}$$

Consider the following equations with a positive parameter α,

$$\begin{gathered} \dot{x_1} = a(t)x_2 - [b(t) + 2\alpha p(t)]x_1 - d(t)x_1^2, \\ \dot{x_2} = c(t)x_1 - f(t)x_2^2. \end{gathered} \tag{2.12}$$

By Lemma 2.1, (2.12) has a positive ω-periodic solution $x_\alpha^*(t) = (x_{1\alpha}^*(t), x_{2\alpha}^*(t))$, which is globally asymptotically stable. Let $x_\alpha(t) = (x_{1\alpha}(t), x_{2\alpha}(t))$ be the solution of (2.12) with initial condition $x_\alpha(0) = x^*(0)$, where $x^*(t) = (x_1^*(t), x_2^*(t))$is the positive periodic solution of (2.1). Then for the above ε_0, there exists $T_2 > 0$ such that for all $t \geq T_2$,

$$|x_{i\alpha}^*(t) - x_{i\alpha}(t)| < \varepsilon_0/4, \quad i = 1, 2.$$

By the continuity of the solution in the parameter, we have $x_\alpha(t) \to x^*(t)$ uniformly in $[T_2, T_2 + \omega]$ as $\alpha \to 0$. Hence for $\varepsilon_0 > 0$, there exists $\alpha_0 = \alpha_0(\varepsilon_0)$ $(0 < \alpha_0 < \varepsilon_0)$ such that

$$|x_{i\alpha}(t) - x_i^*(t)| < \varepsilon_0/4, \quad i = 1, 2,\ 0 \leq \alpha \leq \alpha_0,\ t \in [T_2, T_2 + \omega].$$

So we have

$$|x_{i\alpha}^*(t) - x_i^*(t)| < \varepsilon_0/2, \quad i = 1, 2,\ 0 \leq \alpha \leq \alpha_0,\ t \in [T_2, T_2 + \omega].$$

Since $x_\alpha^*(t)$ and $x^*(t)$ are all ω-periodic, we have

$$|x_{i\alpha}^*(t) - x_i^*(t)| < \varepsilon_0/2, \quad i = 1, 2,\ 0 \leq \alpha \leq \alpha_0,\ t \geq 0. \tag{2.13}$$

Suppose that the condition (2.10) is not true, then for any positive constant $\alpha < \alpha_0$ we have

$$\limsup_{t \to \infty} y(t) < \alpha.$$

So there exists $T_3 > 0$ such that for all $t \geq T_3$ $y(t) < \alpha$. We can choose constant $\tau_0 > 0$ such that

$$H_2 \int_{-\infty}^{-\tau_0} k(s)\, ds < \alpha,$$

where $k(s) = k_{12}(s) + k_{21}(s) + k_{22}(s)$, $H_2 = H_1 + \max\{x_1^*(t)\}$. For any $t \geq T_3 + \tau_0$, we have

$$\begin{aligned} \dot{x_1} &= a(t)x_2 - b(t)x_1 - d(t)x_1^2 - p(t)x_1 \int_{-\tau_0}^{0} k_{12}(s) y(t+s)\, ds \\ &\quad - p(t)x_1 \int_{-\infty}^{-\tau_0} k_{12}(s) y(t+s)\, ds \\ &\geq a(t)x_2 - b(t)x_1 - d(t)x_1^2 - 2p(t)x_1\alpha \\ &= a(t)x_2 - [b(t) + 2\alpha p(t)]x_1 - d(t)x_1^2, \\ \dot{x_2} &= c(t)x_1 - f(t)x_2^2. \end{aligned}$$

Then by the vector comparison theorem we obtain

$$x_i(t) \geq x_{i\alpha}(t), \quad i = 1, 2,\ t \geq T_3 + \tau_0, \tag{2.14}$$

where $x_\alpha(t) = (x_{1\alpha}(t), x_{2\alpha}(t))$ is the solution of (2.12) with initial condition $x_\alpha(T_3+\tau_0) = x(T_3+\tau_0)$. By the global asymptotic stability of $x^*_\alpha(t)$, there exists $T_4 > T_3+\tau_0$ such that

$$|x_{i\alpha}(t) - x^*_{i\alpha}(t)| < \varepsilon_0/2, \quad i = 1,2,\ t \geq T_4. \tag{2.15}$$

Hence, by (2.13), (2.14) and (2.15)

$$x_i(t) \geq x^*_i(t) - \varepsilon_0, \quad i = 1,2,\ t \geq T_4.$$

Then for any $t \geq T_4 + \tau_0$, we have

$$\begin{aligned}
\dot{y} &\geq y\big[-g(t) + h(t)\int_{-\tau_0}^{0} k_{21}(s)x_1(t+s)\,ds - q(t)\int_{-\tau_0}^{0} k_{22}(s)y(t+s)\,ds \\
&\quad - q(t)\int_{-\infty}^{-\tau_0} k_{22}(s)y(t+s)\,ds\big] \\
&\geq y\big[-g(t) + h(t)\int_{-\tau_0}^{0} k_{21}(s)(x^*_1(t+s)-\varepsilon_0)\,ds - q(t)\alpha\int_{-\tau_0}^{0} k_{22}(s)\,ds - q(t)\alpha\big] \\
&\geq y\big[-g(t) + h(t)\int_{-\infty}^{0} k_{21}(s)x^*_1(t+s)\,ds - 2h(t)\varepsilon_0 - 2q(t)\varepsilon_0\big](\alpha < \varepsilon_0)
\end{aligned}$$

Integrating the above inequality from $T_4+\tau_0$ to t ($t \geq T_4+\tau_0$) yields

$$y(t) \geq y(T_4+\tau_0)\exp\int_{T_4+\tau_0}^{t} [-g(t)+h(t)\int_{-\infty}^{0} k_{21}(s)x^*_1(t+s) - 2h(t)\varepsilon_0 - 2q(t)\varepsilon_0]\,ds.$$

By (2.11) we know that $y \to \infty$ as $t \to \infty$, which is a contradiction. This completes the proof. $\square$

Lemma 2.7. *There exists a positive constant δ_y ($\delta_y < M_y$) such that*

$$\liminf_{t\to\infty} y(t) \geq \delta_y.$$

Proof. Otherwise, there must exist a sequence $\{\phi_k\} \subset C_+$ such that

$$\liminf_{t\to\infty} y(t,\phi_k) < \frac{\beta}{k+1}, \quad k = 1,2,\dots.$$

By Lemma 2.6, we have $\limsup_{t\to\infty} y(t,\varphi_k) > \beta$, $k = 1,2,\dots$. Hence, for each k there are time sequences $\{s^{(k)}_q\} and \{t^{(k)}_q\}$, satisfying $0 < s^{(k)}_1 < t^{(k)}_1 < s^{(k)}_2 < t^{(k)}_2 < \dots < s^{(k)}_q < t^{(k)}_q < \dots$ and $s^{(k)}_q \to \infty$ as $q \to \infty$, such that

$$y(s^{(k)}_q,\phi_k) = \beta, \quad y(t^{(k)}_q,\phi_k) = \frac{\beta}{k+1}, \tag{2.16}$$

$$\frac{\beta}{k+1} < y(t,\phi_k) < \beta, \quad t \in (s^{(k)}_q, t^{(k)}_q). \tag{2.17}$$

By Lemma 2.4, for a given integer $k > 0$, there is a $T^{(k)} > 0$ such that $x_i(t,\phi_k) \leq M_x$ ($i = 1,2$) and $y(t,\phi_k) \leq M_y$ for all $t \geq T^{(k)}$. Further, there is a constant $\sigma^{(k)} > 0$ such that

$$H^{(k)}_1 \int_{-\infty}^{-\sigma^{(k)}} k(s)\,ds < M_y,$$

where $H_1^{(k)} = \sup\{y(t+s,\phi_k) : t \geq 0, s \leq 0\}$. Because of $s_q^{(k)} \to \infty$ as $q \to \infty$, there is a positive integer $K_1^{(k)}$ such that $s_q^{(k)} > T^{(k)} + \sigma^{(k)}$ as $q \geq K_1^{(k)}$. For any $t \geq T^{(k)} + \sigma^{(k)}$, we have

$$\begin{aligned}\frac{dy(t,\phi_k)}{dt} &\geq y(t,\phi_k)\big[-g(t) - q(t)\int_{-\sigma^{(k)}}^{0} k_{22}(s)y(t+s,\phi_k)\,ds \\ &\quad - q(t)\int_{-\infty}^{-\sigma^{(k)}} k_{22}(s)y(t+s,\phi_k)\,ds\big] \\ &\geq y(t,\phi_k)\big[-g(t) - 2q(t)M_y\big].\end{aligned}$$

Integrating the above inequality from $s_q^{(k)}$ to $t_q^{(k)}$, for any $q \geq K_1^{(k)}$ we get

$$y(t_q^{(k)},\phi_k) \geq y(s_q^{(k)},\phi_k)\exp\int_{s_q^{(k)}}^{t_q^{(k)}} [-g(t) - 2q(t)M_y]dt.$$

Consequently

$$t_q^{(k)} - s_q^{(k)} \geq \frac{\ln(k+1)}{r_1},\ q \geq K_1^{(k)}, \tag{2.18}$$

where $r_1 = \max_{t\geq 0}\{|g(t)| + 2M_y q(t)\}$. By (2.11), there are constants $P > 0$ and r such that for any $a \geq P$ we have

$$\int_0^a [-g(t) + h(t)\int_{-\infty}^0 k_{21}(s)x_1^*(t+s)\,ds - 2h(t)\varepsilon_0 - 2q(t)\varepsilon_0]dt > r.$$

Obviously, for any k there is $K_2^{(k)} > K_1^{(k)}$ such that for all $q \geq K_2^{(k)}$,

$$H_1^{(k)}\int_{-\infty}^{T^{(k)}-s_q^{(k)}} k(s)\,ds < \frac{1}{2}\beta. \tag{2.19}$$

Further, we can choose a constant σ_0 such that

$$H_2\int_{-\infty}^{-\sigma_0} k(s)\,ds < \frac{1}{2}\beta, \tag{2.20}$$

where $H_2 = M_y + \max_{t\geq 0}\{x_1^*(t) + x_2^*(t)\}$. by (2.18), there is a integer $N_1 > 0$ such that $t_q^{(k)} - s_q^{(k)} > \sigma_0$ for all $k \geq N_1, q \geq K_2^{(k)}$. For any $k \geq N_1, q \geq K_2^{(k)}$ and

$t \in [s_q^{(k)} + \sigma_0, t_q^{(k)}]$, by (2.17), (2.19) and (2.20) we have

$$\begin{aligned}
\frac{dx_1(t,\phi_k)}{dt} &= a(t)x_2(t,\phi_k) - b(t)x_1(t,\phi_k) - d(t)x_1^2(t,\phi_k) \\
&\quad - p(t)x_1(t,\phi_k)\int_{-\infty}^{T^{(k)}} k_{12}(u-t)y(u,\phi_k)\,du \\
&\quad - p(t)x_1(t,\phi_k)\int_{T^{(k)}}^{s_q^{(k)}} k_{12}(u-t)y(u,\phi_k)\,du \\
&\quad - p(t)x_1(t,\phi_k)\int_{s_q^{(k)}}^{t} k_{12}(u-t)y(u,\phi_k)du \\
&\geq a(t)x_2(t,\phi_k) - b(t)x_1(t,\phi_k) - d(t)x_1^2(t,\phi_k) \\
&\quad - p(t)x_1(t,\phi_k)H_1^{(k)}\int_{-\infty}^{T^{(k)}-t} k_{12}(s)\,ds \\
&\quad - p(t)x_1(t,\phi_k)M_y\int_{-\infty}^{s_q^{(k)}-t} k_{12}(s)\,ds \\
&\quad - p(t)x_1(t,\phi_k)\beta\int_{-\infty}^{0} k_{12}(s)\,ds \\
&\geq a(t)x_2(t,\phi_k) - b(t)x_1(t,\phi_k) - d(t)x_1^2(t,\phi_k) - 2p(t)x_1(t,\phi_k)\beta \\
&= a(t)x_2(t,\phi_k) - [b(t)+2\beta p(t)]x_1(t,\phi_k) - d(t)x_1^2(t,\phi_k),
\end{aligned}$$

$$\frac{dx_2(t,\phi_k)}{dt} = c(t)x_1(t,\phi_k) - f(t)x_2^2(t,\phi_k).$$

Let $x_\beta(t) = (x_{1\beta}(t), x_{2\beta}(t))$ be the solution of (2.12) for $\alpha = \beta$ with the initial condition $x_\beta(s_q^{(k)} + \sigma_0) = x(s_q^{(k)} + \sigma_0, \phi_k)$. Then by the vector comparison theorem, it follows that

$$x_i(t,\phi_k) \geq x_{i\beta}(t), \quad i = 1,2,\ t \in [s_q^{(k)} + \sigma_0, t_q^{(k)}]. \tag{2.21}$$

From $\lim_{q\to\infty} s_q^{(k)} = \infty$ and Lemmas 2.4 and 2.5, we obtain that for any k there is a $K_3^{(k)} > K_2^{(k)}$ such that for any $q \geq K_3^{(k)}$,

$$\delta_x \leq x_i(s_q^{(k)} + \sigma_0, \phi_k) \leq M_x, \quad i = 1,2.$$

For the parameter $\alpha = \beta$, Equation (2.12) has a globally asymptotically stable positive ω-periodic solution $x_\beta^*(t) = (x_{1\beta}^*(t), x_{2\beta}^*(t))$. ¿From the periodicity of (2.12) we know that the periodic solution $x_\beta^*(t)$ also is globally uniformly asymptotically stable. Hence, there is a $T_5 > P$, and T_5 is independent of any k and q, such that

$$x_{i\beta}(t) > x_{i\beta}^*(t) - \frac{1}{2}\varepsilon_0$$

for all $t \geq T_5 + s_q^{(k)} + \sigma_0$ and $q \geq K_3^{(k)}$. Consequently, by (2.13),

$$x_{i\beta}(t) > x_i^*(t) - \varepsilon_0 \tag{2.22}$$

for all $t \geq T_5 + s_q^{(k)} + \sigma_0$ and $q \geq K_3^{(k)}$. By(2.18), there is a $N_2 \geq N_1$ such that $t_q^{(k)} - s_q^{(k)} \geq 2W$ for all $k \geq N_2$ and $q \geq K_3^{(k)}$, where $W \geq T_5 + \sigma_0$. Hence, from (2.21) and (2.22) we finally obtain

$$x_i(t,\phi_k) \geq x_i^*(t) - \varepsilon_0. \tag{2.23}$$

for all $t \in [W + s_q^{(k)}, t_q^{(k)}]$, $k \geq N_2$ and $q \geq K_3^{(k)}$. Since, for any $t \in [W + s_q^{(k)} + \sigma_0, t_q^{(k)}]$, $k \geq N_2$ and $q \geq K_3^{(k)}$, by (2.23), (2.19) and (2.20), we have

$$\begin{aligned}
\frac{dy(t,\phi_k)}{dt} &\geq y(t,\phi_k)[-g(t) + h(t)\int_{-\sigma_0}^{0} k_{21}(s)x_1(t+s,\phi_k)\,ds \\
&\quad - q(t)\int_{-\infty}^{T^{(k)}} k_{22}(u-t)y(u,\phi_k)du - q(t)\int_{T^{(k)}}^{s_q^{(k)}} k_{22}(u-t)y(u,\phi_k)du \\
&\quad - q(t)\int_{s_q^{(k)}}^{t} k_{22}(u-t)y(u,\phi_k)du] \\
&\geq y(t,\phi_k)[-g(t) + h(t)\int_{-\sigma_0}^{0} k_{21}(s)(x_1^*(t+s,\phi_k) - \varepsilon_0)\,ds \\
&\quad - q(t)H_1^{(k)}\int_{-\infty}^{T^{(k)}-t} k_{22}(s)\,ds - q(t)M_y\int_{-\infty}^{s_q^{(k)}-t} k_{22}(s)\,ds \\
&\quad - q(t)\beta\int_{-\infty}^{0} k_{22}(s)\,ds] \\
&\geq y(t,\phi_k)[-g(t) + h(t)\int_{-\infty}^{0} k_{21}(s)x_1^*(t+s,\phi_k)\,ds \\
&\quad - 2h(t)\varepsilon_0 - q(t)\frac{1}{2}\beta - q(t)\frac{1}{2}\beta - q(t)\beta] \\
&\geq y(t,\phi_k)[-g(t) + h(t)\int_{-\infty}^{0} k_{21}(s)x_1^*(t+s,\phi_k)\,ds - 2h(t)\varepsilon_0 - 2q(t)\varepsilon_0]\,.
\end{aligned}$$

Integrating from $W + s_q^{(k)} + \sigma_0$ to $t_q^{(k)}$ for any $k \geq N_2$ and $q \geq K_3^{(k)}$ we obtain

$$\begin{aligned}
y(t_q^{(k)},\phi_k) &\geq y(W + s_q^{(k)} + \sigma_0, \phi_k)\exp\int_{W+s_q^{(k)}+\sigma_0}^{t_q^{(k)}} \Big[-g(t) \\
&\quad + h(t)\int_{-\infty}^{0} k_{21}(s)x_1^*(t+s,\phi_k)\,ds - 2h(t)\varepsilon_0 - 2q(t)\varepsilon_0\Big]dt.
\end{aligned}$$

Hence, by (2.16) and (2.17) we finally have

$$\begin{aligned}
\frac{\beta}{k+1} &\geq \frac{\beta}{k+1}\exp\int_{W+s_q^{(k)}+\sigma_0}^{t_q^{(k)}} \big[-g(t) + h(t)\int_{-\infty}^{0} k_{21}(s)x_1^*(t+s,\phi_k)\,ds \\
&\quad - 2h(t)\varepsilon_0 - 2q(t)\varepsilon_0\big]dt \\
&> \frac{\beta}{k+1}\,,
\end{aligned}$$

$t_q^{(k)} - (W + s_q^{(k)} + \sigma_0) \geq T_5 > P$, which leads to a contradiction. This completes the proof. □

Proof of main Theorem. The sufficiency of this theorem now follows from Lemmas 2.4 2.5 2.6 2.7. We thus only need to prove the necessity of theorem. Suppose that

$$\int_0^{\omega}[-g(t) + h(t)\int_{-\infty}^{0} k_{12}(s)x_1^*(t+s)\,ds]dt \leq 0.$$

We will show that $\lim_{t\to\infty} y(t) = 0$. In fact, we know that for any given $0 < \varepsilon < 1$, there exist $\varepsilon_1 < \varepsilon$ and $\varepsilon_0 > 0$ such that

$$\begin{aligned}&\int_0^\omega [-g(t) + h(t)\int_{-\infty}^0 k_{12}(s)(x_1^*(t+s) + \varepsilon_1)\,ds - \frac{1}{2}q(t)l\varepsilon]dt \\ &\le \varepsilon_1 \int_0^\omega h(t)dt - \frac{1}{2}l\varepsilon\int_0^\omega q(t)dt < -\varepsilon_0,\end{aligned} \tag{2.24}$$

where $l = \int_{-\infty}^0 k_{22}(s)\exp(e^m s)\,ds$, $e(t) = -g(t) + h(t)\int_{-\infty}^0 k_{21}(s)x_1^*(t+s)\,ds + h(t)\varepsilon_1$. Since

$$\begin{aligned}\dot{x}_1 &\le a(t)x_2 - b(t)x_1 - d(t)x_1^2,\\ \dot{x}_2 &= c(t)x_1 - f(t)x_2^2,\end{aligned}$$

for all $t \ge 0$. Let $\overline{x}(t) = (\overline{x}_1(t), \overline{x}_2(t))$ be the solution of (2.1) with initial condition $\overline{x}(0) = x(0)$. By the vector comparison theorem we obtain $x_i(t) \le \overline{x}_i(t)$ $(i = 1, 2)$, $t \ge 0$. Obviously, by the global asymptotic stability of $x^*(t)$, there is a $T_6 > 0$ such that $\overline{x}_i(t) \le x_i^*(t) + \frac{1}{2}\varepsilon_1$ $(i = 1, 2)$ for all $t \ge T_6$. Hence, we have

$$x_i(t) \le x_i^*(t) + \frac{1}{2}\varepsilon_1 \quad (i = 1, 2) \tag{2.25}$$

for all $t \ge T_6$. Choose a constant $\tau_1 > 0$ such that

$$H_0 \int_{-\infty}^{-\tau_1} k(s)\,ds < \frac{1}{2}\varepsilon_1 \tag{2.26}$$

$$\int_{-\tau_1}^0 k_{22}(s)\exp(e^m s)\,ds > \frac{1}{2}l. \tag{2.27}$$

For any $t \ge T_6 + \tau_1$, by (2.25) and (2.26) we have

$$\begin{aligned}\dot{y} &\le y\big[-g(t) + h(t)\int_{-\tau_1}^0 k_{21}(s)x_1(t+s)\,ds + h(t)\int_{-\infty}^{-\tau_1} k_{21}(s)x_1(t+s)\,ds\big]\\ &\le y\big[-g(t) + h(t)\int_{-\tau_1}^0 k_{21}(s)(x_1^*(t+s) + \frac{1}{2}\varepsilon_1)\,ds + \frac{1}{2}h(t)\varepsilon_1\big]\\ &\le ye(t)\end{aligned}$$

Hence, by (2.27), for any $t \ge t + s \ge T_6 + \tau_1$ we obtain

$$\begin{aligned}\dot{y} &\le y\big[e(t) - q(t)\int_{-\tau_1}^0 k_{22}(s)y(t+s)\,ds\big]\\ &\le y\big[e(t) - q(t)\int_{-\tau_1}^0 k_{22}(s)\exp(e^m s)\,ds\,y\big]\\ &\le y\big[e(t) - \frac{1}{2}lq(t)y\big].\end{aligned}$$

If $y(t) \ge \varepsilon$ for all $t \ge T_6 + 2\tau_1$, then we have

$$\dot{y} \le y[e(t) - \frac{1}{2}lq(t)\varepsilon]. \tag{2.28}$$

Consequently, by (2.24) we obtain

$$y(t) \le y(T_6 + 2\tau_1)\exp\int_{T_6+2\tau_1}^t [e(u) - \frac{1}{2}lq(u)\varepsilon]du \to 0$$

as $t \to \infty$, which leads to a contradiction. Hence, there is a $t_1 \geq T_6 + 2\tau_1$ such that $y(t_1) < \varepsilon$.

Let $M(\varepsilon) = \max_{t\geq 0}\{|e(t)| + \frac{1}{2}lq(t)\varepsilon\}$. We know that $M(\varepsilon)$ is bounded for $\varepsilon \in [0, 1]$. We then show that

$$y(t) \leq \varepsilon \exp(M(\varepsilon)\omega), \quad t \geq t_1. \tag{2.29}$$

Otherwise, there are $t_3 > t_2 > t_1$ such that $y(t_3) > \varepsilon \exp(M(\varepsilon)\omega)$, $y(t_2) = \varepsilon$ and $y(t) > \varepsilon$ for all $t \in (t_2, t_3]$. Let $p \geq 0$ be an integer such that $t_3 \in (t_2 + p\omega, t_2 + (p+1)\omega]$. Then from (2.28) we have

$$\begin{aligned}
\varepsilon \exp(M(\varepsilon)\omega) &< y(t_3) \\
&\leq y(t_2) \exp \int_{t_2}^{t_3} [e(t) - \frac{1}{2}lq(t)\varepsilon]dt \\
&= \varepsilon \exp(\int_{t_2}^{t_2+p\omega} + \int_{t_2+p\omega}^{t_3})[e(t) - \frac{1}{2}lq(t)\varepsilon]dt \\
&< \varepsilon \exp(\int_{t_2+p\omega}^{t_3} [e(t) - \frac{1}{2}lq(t)\varepsilon]dt) \\
&\leq \varepsilon \exp(M(\varepsilon)\omega).
\end{aligned}$$

This leads to a contradiction. Hence, inequality (2.29) holds. Further, by the arbitrariness of ε we obtain $y(t) \to 0$ as $t \to \infty$. This completes the proof.

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