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CONVERGENCE OF EIGENFUNCTION EXPANSIONS CORRESPONDING TO NONLINEAR STURM-LIOUVILLE OPERATORS

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Abstract. It is well known that the classical linear Sturm-Liouville eigenvalue problem is self-adjoint and possesses a family of eigenfunctions which form an orthonormal basis for the space L_2. A natural question is to ask if a similar result holds for nonlinear problems. In the present paper, we examine the basis property for eigenfunctions of nonlinear Sturm-Liouville equations subject to general linear, separated boundary conditions.

1. Introduction

We consider the nonlinear eigenvalue problem

$$u'' - q(x, u, \lambda)u + \lambda u = 0, \quad x \in [0, 1], \tag{1.1}$$

$$\alpha_1 u(0) + \beta_1 u'(0) = 0, \quad \alpha_2 u(1) + \beta_2 u'(1) = 0, \tag{1.2}$$

where $|\alpha_i| + |\beta_i| > 0$, for $i = 1, 2$. Here x and λ are real variables and q is a real-valued function defined on $\Omega = [0, 1] \times \mathbb{R}^2$. By an eigenfunction of (1.1)-(1.2) corresponding to an eigenvalue λ we mean a twice continuously differentiable, real-valued function $u(x)$, ($u(x) \not\equiv 0$), satisfying (1.1) on $[0, 1]$ and (1.2).

The main result is as follows.

Theorem 1.1. *Assume that*

(1) $q(x, u, \lambda)$ *is continuous on the set* Ω
(2) *There exist constants* M_0 *and* M *such that* $|q(x, u, \lambda)| \le M$ *on the set* $\Omega_0 = \{0 \le x \le 1, |u| \le M_0, -\infty < \lambda < \infty\}$
(3) *For any* λ, $-\infty < \lambda < \infty$, *and any* x, $0 \le x \le 1$, $\frac{\partial}{\partial \lambda} q(x, 0, \lambda) = 0$.

Then there exists a system $\{u_n(x)\}$ $(n = 0, 1, \dots)$ *of eigenfunctions of problem* (1.1)-(1.2) *which forms a Riesz basis for the space* $L_2(0, 1)$.

Note that when the function $q(x, u, \lambda)$ does not depend on λ, conditions (2) and (3) of Theorem 1.1 can be omitted.

Before proving Theorem 1.1 we would like to compare it with existing results in the literature. Since $q(x, 0, \lambda) = q_0(x)$ we see that problem (1.1)-(1.2) can be

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written in the operator equation form

$$Lu + N(u, \lambda) + \lambda u = 0$$

where L is the linear operator generated by the differential expression $Lu = u'' - q_0(x)u$ and boundary conditions (1.2) and $N(u, \lambda) = (q_0(x) - q(x, u, \lambda))u$. Obviously L is a self-adjoint operator with discrete spectrum and an orthonormal system of eigenfunctions which is complete in $L_2(0, 1)$. Moreover all the eigenvalues are simple. Thus in order to construct an eigenfunction system for the above spectral problem which forms a basis for $L_2(0, 1)$ one could invoke the Crandall-Rabinowitz approach, [4], and the method by Brown [2]. However to apply this approach requires strong assumptions on the nonlinear operator N, namely that $q(x, u, \lambda) \in C^1$. Here we weaken this smoothness assumption on the function $q(x, u, \lambda)$. In addition any eigenfunction system $\{u_n(x)\}$ constructed by that method lies in the neighborhood of zero, that is, $\lim_{n\to\infty} \|u_n(x)\|_{L_2(0,1)} = 0$. Thus the system $\{u_n(x)\}$ may be an unconditional basis but it cannot be a Riesz basis since in this context a Riesz basis is an almost normalized system of functions.

Nonlinear eigenvalue problems have a long history; we refer the reader to [9] and its reference list for more information.

2. Proof of Theorem 1.1

The proof divides naturally into the following parts.

(a) We use a standard method to show that, without loss of generality, we may make certain assumptions about the function $q(x, u, \lambda)$.

(b) We prove a simple technical lemma giving estimates for solutions of the initial value problem for equation (1.1).

(c) We employ a polar coordinates technique to establish the existence of eigenvalues of problem (1.1)-(1.2). Alternative methods such as fixed point theorems could have been used here. However we prefer the classical Prüfer transformation since it gives the eigenfunctions in more explicit form. In particular, we obtain the eigenfunctions with prescribed initial data and this is important later in the proof. Note that since the function $q(x, u, \lambda)$ is merely continuous it follows that solutions of initial value problems for equation (1.1) may not be unique. To overcome this obstacle we apply the generalized Kneser's theorem in the form given by Pugh in [12]. This application requires the right hand side of our system of equations to have compact support. We use cut-off functions to produce a new system of equations with compact support and show there is a solution of our new system satisfying the boundary conditions (1.2) which is a solution of problem (1.1)-(1.2). Then we use the Sturm comparison theorem to obtain a two-side estimate for the eigenvalues of problem (1.1)-(1.2).

(d) We prove that the constructed eigenfunction system of problem (1.1)-(1.2) divided by a suitable number p is quadratically close to a complete orthonormal eigenfunction system of the self-adjoint eigenvalue problem for the linearized equation subject to the same boundary conditions. In this part of the proof we use well known asymptotic formulae for eigenvalues of the linear Sturm-Liouville operator, an integral representation for solutions of the initial-value problem, and λ-independent relations between the L_∞ and L_2-norms of eigenfunctions and their derivatives [13].

(e) We apply well known theorems of functional analysis to establish the basis property for the constructed eigenfunction system.

One can see that the arguments in steps a), b) and e) are short and simple, while those in steps c) and d) are more complicated.

(a) Let us consider the following eigenvalue problem for the linearized equation

$$u'' - q_0(x)u + \lambda u = 0 \tag{2.1}$$

with boundary conditions (1.2). From [8], all eigenvalues of problem (2.1)-(1.2) are real and, moreover, there exists a smallest eigenvalue, $\lambda_0^{(0)}$. We set $\lambda^* = 2M + 1 - \lambda_0^{(0)}$ and $\tilde{q}(x, u, \lambda) = q(x, u, \lambda - \lambda^*) + \lambda^*$. Clearly the function $\tilde{q}(x, u, \lambda)$ satisfies all the conditions of the theorem with the inequality in condition (2) replaced by the inequality $M - \lambda_0^{(0)} + 1 \le \tilde{q}(x, u, \lambda) \le 3M - \lambda_0^{(0)} + 1$. We denote $M_1 = 3M + |\lambda_0^{(0)}| + 1$. Since

$$\tilde{u}_n'' - \tilde{q}(x, \tilde{u}_n, \tilde{\lambda}_n)\tilde{u}_n + \tilde{\lambda}_n \tilde{u}_n = \tilde{u}_n'' - q(x, \tilde{u}_n, \tilde{\lambda}_n - \lambda^*)\tilde{u}_n + (\tilde{\lambda}_n - \lambda^*)\tilde{u}_n$$

it follows that an eigenfunction $\tilde{u}_n(x)$ of the problem

$$\begin{gathered} \tilde{u}'' - \tilde{q}(x, \tilde{u}, \tilde{\lambda})\tilde{u} + \tilde{\lambda}\tilde{u} = 0, \\ \alpha_1 \tilde{u}(0) + \beta_1 \tilde{u}'(0) = 0, \quad \alpha_2 \tilde{u}(1) + \beta_2 \tilde{u}'(1) = 0 \end{gathered}$$

corresponding to an eigenvalue $\tilde{\lambda}_n$ is an eigenfunction of problem (1.1)-(1.2) corresponding to the eigenvalue $\lambda_n = \tilde{\lambda}_n - \lambda^*$. Thus without loss of generality we may assume that the function $q(x, u, \lambda)$ satisfies the inequalities

$$|q(x, u, \lambda) - q_0(x)| \le 2M, \qquad |q(x, u, \lambda)| \le M_1 \tag{2.2}$$

on the set Ω_0 and that the smallest eigenvalue of problem (2.1), (1.2), $\lambda_0^{(0)}$, is $2M + 1$. Also without loss of generality we may assume that $|\alpha_1| + |\beta_1| \le 1$. We will assume later that $\lambda > 0$ and set $\mu = \sqrt{\lambda}$.

(b) Let $\eta(t) \in C^\infty(R)$ be a cut off function

$$\eta(t) = \begin{cases} 1 & \text{if } |t| \le \frac{M_0}{2} \\ 0 & \text{if } |t| \ge M_0, \end{cases}$$

and $0 \le \eta(t) \le 1$. Consider the initial-value problem

$$\begin{gathered} u'' - q(x, u, \lambda)\eta(u)u + \lambda u = 0, \quad x \in [0, 1], \\ u(0) = b\beta_1, \qquad u'(0) = -b\alpha_1, \end{gathered} \tag{2.3}$$

where b is an arbitrary number; if $\beta_1 = 0$ then we set $u(0) = 0$ and $u'(0) = b\mu$.

Lemma 2.1. *If $\lambda \ge 1$ and b be given, then any solution $u(x)$ of problem* (2.3) *on* $[0, 1]$ *satisfies*

$$|u(x)| \le |b| e^{M_1}.$$

Proof. It follows from [1] that a solution $u(x)$ of problem (2.3) exists on $[0, 1]$. By [3, Ch. 3, Sec. 6, Th. 6.4], any solution of (2.3) satisfies the integral equation

$$u(x) = u(0)\cos\mu x + u'(0)\frac{\sin\mu x}{\mu} + \frac{1}{\mu}\int_0^x \sin\mu(x - t) q(t, u(t), \lambda)\eta(u(t))u(t)dt. \tag{2.4}$$

From inequality (2.2) it follows that

$$|q(t, u(t), \lambda)|\eta(u(t)) \le M_1 \quad \forall (t, u, \lambda) \in \Omega.$$

From this it follows that

$$|u(x)| \le |R(x)| + \frac{M_1}{\mu}\int_0^x |u(t)|dt$$

where $R(x)$ is the sum of two first terms on the right-hand side of (2.4). From the last inequality and Gronwall's lemma, [1], we see that

$$|u(x)| \le |R(x)| + \frac{M_1}{\mu}\int_0^x e^{\frac{M_1}{\mu}(x-t)}|R(t)|dt. \tag{2.5}$$

Since $|R(x)| \le b$ it follows from (2.5) that

$$|u(x)| \le |b|e^{M_1}.$$

Let

$$0 \le |b| \le b_0 \tag{2.6}$$

where $b_0 = \min\big(1, \frac{M_0}{2e^{M_1}}\big)$. It follows from Lemma 2.1 that $|u(x)| \le \frac{M_0}{2}$.

From this and the definition of the cut off function $\eta(t)$ it follows that if condition (2.6) is satisfied, then any solution of problem (2.3) is a solution of initial-value problem

$$\begin{gathered} u'' - q(x,u,\lambda)u + \lambda u = 0, \quad x \in [0,1], \\ u(0) = b\beta_1, u'(0) = -b\alpha_1; \end{gathered} \tag{2.7}$$

if $\beta_1 = 0$ then $u(0) = 0$ and $u'(0) = b\mu$. □

(c) Now we consider two linear eigenvalue problems

$$\begin{gathered} u'' + Q_1(x,\lambda,d)u = 0, \\ \alpha_1 u(0) + \beta_1 u'(0) = 0, \\ \alpha_2 u(1) + \beta_2 u'(1) = 0 \end{gathered} \tag{2.8}$$

and

$$\begin{gathered} u'' + Q_2(x,\lambda,d)u = 0, \\ \alpha_1 u(0) + \beta_1 u'(0) = 0, \\ \alpha_2 u(1) + \beta_2 u'(1) = 0 \end{gathered} \tag{2.9}$$

where $Q_1(x,\lambda,d) = -q(x,0,\lambda) - d + \lambda$, $Q_2(x,\lambda,d) = -q(x,0,\lambda) + d + \lambda$, and d is an arbitrary positive number. Let $\theta_1(x,\lambda,d)$ be a solution of the equation $\theta_i' = \cos^2\theta_i + Q_i(x,\lambda,d)\sin^2\theta_i$, moreover, $\theta_i(0,\lambda,d) = \arctan(-\frac{\alpha_1}{\beta_1})$ if $\beta_1 \ne 0$ and $\theta_i(0,\lambda,d) = 0$ $(i = 1,2)$ if $\beta_1 = 0$. By the oscillation theorem, [6], problem (2.8) has eigenvalues $\lambda_0^{(1)}(d) < \lambda_1^{(1)}(d) < \dots$, and problem (2.9) has eigenvalues $\lambda_0^{(2)}(d) < \lambda_1^{(2)}(d) < \dots$. Clearly, it follows that $\lambda_n^{(1)}(d) = \lambda_n^{(2)}(d) + 2d$, $\lambda_0^{(2)}(2M) = 1$. From [6] it also follows that

$$\theta_i(\lambda_n^{(i)}(d), 1, d) = \arctan(-\frac{\alpha_2}{\beta_2}) + \pi n, \tag{2.10}$$

where $n = 0, 1, \dots$. Here and throughout the remaining part of the paper we assume that $\arctan(-\frac{\alpha_2}{\beta_2}) = \pi$ if $\beta_2 = 0$.

Lemma 2.2. *For each $b \ne 0$ satisfying condition* (2.6) *and n $(n = 0, 1, \dots)$ there exists an eigenvalue $\lambda_n(b)$ of problem* (1.1)-(1.2) *satisfying*

$$\lambda_n^{(2)}(2M) \le \lambda_n(b) \le \lambda_n^{(1)}(2M). \tag{2.11}$$

Moreover, the corresponding eigenfunction $u_n(x,\lambda_n(b))$ is a solution of initial-value problem (2.7).

Proof. Let $u(x,\lambda)$ be a solution of problem (2.3). Using the classical Prüfer transformation [6],

$$u(x) = r(x)\sin\varphi(x), \quad u'(x) = r(x)\cos\varphi(x)$$

it follows that for any λ problem (2.3) is equivalent the system of equations

$$\begin{gathered}\varphi' = \cos^2\varphi + [\Lambda - q(x, r\sin\varphi, \Lambda)\eta(r\sin\varphi)]\sin^2\varphi \\ r' = \frac{1}{2} r\,[1-\Lambda + q(x, r\sin\varphi,\Lambda)\eta(r\sin\varphi)]\sin 2\varphi \\ \Lambda' = 0\end{gathered} \tag{2.12}$$

with initial conditions

$$\varphi(0) = \varphi_0, \quad r(0) = r_0, \quad \Lambda(0) = \lambda \tag{2.13}$$

where $\varphi_0 = \arctan(-\frac{\alpha_1}{\beta_1})$ and $r_0 = |b|\sqrt{\alpha_1^2+\beta_1^2}$ if $\beta_1 \neq 0$ while $\varphi_0 = 0$ and $r_0 = |b|\mu$ if $\beta_1 = 0$.

Let us consider also the system of equations

$$\begin{gathered}\varphi' = \big[\cos^2\varphi + (\Lambda - q(x, r\sin\varphi,\Lambda)\eta(r\sin\varphi)\sin^2\varphi\big]\,\tilde\eta(\varphi, r,\Lambda) \\ r' = \frac{1}{2}\,[r(1-\Lambda + q(x, r\sin\varphi,\Lambda)\eta(r\sin\varphi)\sin 2\varphi]\,\tilde\eta(\varphi,r,\Lambda) \\ \Lambda' = 0\end{gathered} \tag{2.14}$$

subject to the same initial conditions (2.13), where $\tilde\eta(\varphi,r,\Lambda) = \eta_0(\varphi)\eta_0(r)\eta_0(\Lambda)$ and where the cut off function $\eta_0(t) = 1$ if $|t| \le H$, $\eta_0(t) = 0$ if $|t| \ge H+1$, $0 \le \eta_0(t) \le 1$, $\eta_0(t) \in C^\infty(R)$, H is an arbitrary number such that $H \ge \mu e^{1+\lambda+M_1} + \pi$.

It is easy to see that right hand side of system (2.14) has compact support. From inequality (2.2) if follows that for any x,

$$|q(x, r\sin\varphi,\Lambda)|\eta(r\sin\varphi) \le M_1. \tag{2.15}$$

From (2.15) and the first of equations (2.14) it follows that $|\varphi'| \le 1+\lambda+M_1$. Since $|\varphi(0)| < \pi$ we have $|\varphi(x)| \le 1+\pi+\lambda+M_1$ for $0 \le x \le 1$.

From the second of equations (2.14) it follows that

$$r(x) = r(0)\exp\big(\frac{1}{2}\int_0^x h(t)dt\big),$$

where

$$h(t) = [1-\Lambda + q(t, r(t)\sin\varphi(t),\Lambda)\eta(r(t)\sin\varphi(t))\sin 2\varphi(t)]\,\tilde\eta(\varphi(t), r(t),\Lambda)dt.$$

Evaluating right-hand part of the last equation we obtain

$$r(x) \le r(0)e^{\frac{1}{2}(1+\lambda+M_1)} \le |b|\mu e^{\frac{1}{2}(1+\lambda+M_1)}.$$

By the definitions of H and $\eta(\varphi,r,\Lambda)$, any solution of the Cauchy problem (2.14)-(2.13) is a solution of the Cauchy problem (2.12)-(2.13).

Let $(\widetilde\varphi(x), \tilde r(x), \wedge)$ be an arbitrary solution of Cauchy problem (2.14)-(2.13) for an arbitrary λ, $1 \le \lambda \le \lambda_n^{(1)}(2M)$, where $H = \sqrt{\lambda_n^{(1)}(2M)}e^{1+\lambda_n^{(1)}(2M)+M_1} + \pi$, and b satisfies condition (2.6). Since any solution of Cauchy problem (2.14)-(2.13) is a solution of Cauchy problem (2.12)-(2.13) we have

$$\widetilde\varphi' = \cos^2\widetilde\varphi + (\lambda - q(x,\tilde r\sin\widetilde\varphi,\lambda)\eta(\tilde r\sin\widetilde\varphi))\sin^2\widetilde\varphi\,.$$

Moreover, the function $\tilde{u}(x) = \tilde{r}(x)\sin\widetilde{\varphi}(x)$ is a solution of problem (2.3). Then by Lemma 2.1, $|\tilde{u}(x)| \le \frac{M_0}{2}$; therefore, $\eta(\tilde{u}(x)) = 1$. By virtue of (2.2) we have inequality

$$Q_1(x,\lambda,2M) \le -q(x,\tilde{u}(x),\lambda)\eta(\tilde{u}(x)) + \lambda \le Q_2(x,\lambda,2M).$$

From the last inequality and the comparison theorem, [6] it follows that

$$\theta_1(x,\lambda,2M) \le \widetilde{\varphi}(x,\lambda) \le \theta_2(x,\lambda,2M).$$

From this and (2.10) it follows that

$$\widetilde{\varphi}(1,\lambda_n^{(2)}(2M)) \le \arctan(-\frac{\alpha_2}{\beta_2}) + \pi n \le \widetilde{\varphi}(1,\lambda_n^{(1)}(2M)). \tag{2.16}$$

Further we consider Cauchy problem (2.14)-(2.13). It is clear that the set of initial conditions

$$P = \{0,\varphi_0,r_0,\lambda\}$$

where $\lambda_n^{(2)}(2M) \le \lambda \le \lambda_n^{(1)}(2M)$ is a connected compact set in R^4. Hence by the generalised Kneser's theorem, [12], the set $K_1(P) = \bigcup_{p\in P} K_1(p)$ where $K_1(p)$ is the funnel section at the point $x = 1$ of the set of solutions of system (2.14) subject to the initial conditions $p = (0,\varphi_0,r_0,\lambda)$ is a nonempty connected compact set in R^3. From (2.16) it follows that the set $K_1(P)$ contains points above and below the plane $\varphi = \arctan(-\frac{\alpha_2}{\beta_2}) + \pi n$, therefore the set $K_1(P)$ has a point of intersection with this plane. Therefore, for some $\lambda = \lambda_n(b)$ satisfying (2.11), there exists a solution $(\varphi_n(x,\lambda), r_n(x,\lambda), \lambda_n(b))$ to the Cauchy problem (2.14)-(2.13) such that

$$\varphi_n(1,\lambda_n(b)) = \arctan(-\frac{\alpha_2}{\beta_2}) + \pi n. \tag{2.17}$$

Now solutions of problem (2.14)-(2.13) are solutions of problem (2.12)-(2.13), problem (2.12)-(2.13) is equivalent problem (2.3), and any solution of problem (2.3) is a solution of problem (2.7). Hence for any b satisfying condition (2.6) there is a function $\lambda_n(b)$ satisfying inequality (2.11) and a corresponding solution

$$u_n(x,\lambda_n(b)) = r_n(x,\lambda_n(b))\sin\varphi_n(x,\lambda_n(b))$$

of problem (2.7). From (2.17) it follows that the function $u_n(x,\lambda_n(b))$ satisfies boundary conditions (1.2); that is, the function $u_n(x,\lambda_n(b))$ is an eigenfunction of problem (1.1)-(1.2) corresponding to the eigenvalue $\lambda_n(b)$. □

(d) Let $\{\mathring{u}_n(x)\}$ be a complete orthonormal system of eigenfunctions in $L_2(0,1)$ of the linear self-adjoint problem (2.1)-(1.2), $\lambda_n^{(0)}$ be the corresponding eigenvalues, and $\mu_n^{(0)} = \sqrt{\lambda_n^{(0)}}$, for $n = 0,1,\dots$. Above it was shown that $\lambda_n^{(0)} \ge 2M+1$. From [13] it follows that $\max_{0\le x\le 1} |\mathring{u}_n(x)| \le C_0$, $\max_{0\le x\le 1} |\mathring{u}_n'(x)| \le \mu_n^{(0)} C_0$; therefore, $\mathring{u}_n(0) = \beta_1\gamma_n$ and $\mathring{u}_n'(0) = \zeta_n\mu_n^{(0)}$, where $0 \le |\beta_1\gamma_n|, |\zeta_n| \le C_0$ and $|\beta_1\gamma_n| + |\zeta_n| > 0$. If $\beta_1 \ne 0$ it follows from boundary condition (1.2) that $\zeta_n = \frac{-\alpha_1\gamma_n}{\mathring{\mu}_n}$ and therefore $\gamma_n \ne 0$. If $\beta_1 = 0$ then $\zeta_n \ne 0$. By [6] for any $d > 0$

$$\lambda_n^{(2)}(d) \le \lambda_n^{(0)} \le \lambda_n^{(1)}(d). \tag{2.18}$$

Lemma 2.3. *There exist $p \ne 0$ and an eigensystem $\{u_n(x,\lambda_n,p)\}$, $n = 0,1,\dots$ for problem (1.1)-(1.2) satisfying initial conditions $u_n(0,\lambda_n,p) = p\gamma_n\beta_1$, $u_n'(0,\lambda_n,p) =$*

$-p\gamma_n\alpha_1$ *if* $\beta_1 \neq 0$ *and initial conditions* $u_n(0,\lambda_n,p) = 0$, $u_n'(0,\lambda_n,p) = p\zeta_n\sqrt{\lambda_n}$ *if* $\beta_1 = 0$ *such that*

$$\sum_{n=0}^{\infty} \Big\| \frac{u_n(x,\lambda_n,p)}{p} - \overset{\circ}{u}_n(x) \Big\|^2_{L_2(0,1)} < 1.$$

Proof. Let p be an arbitrary number satisfying

$$0 < |p| < \frac{b_0|\beta_1|}{C_0} \text{ if } \beta_1 \neq 0 \quad \text{and} \quad 0 < |p| < \frac{b_0}{C_0} \text{ if } \beta_1 = 0. \tag{2.19}$$

Denote $p_n = p\gamma_n$ if $\beta_1 \neq 0$ and $p_n = p\zeta_n$ if $\beta_1 = 0$. It is easy to see that in both cases $0 < |p_n| < b_0$. Notice that $|p_n| \leq \frac{C_0}{|\beta_1|}|p|$ if $\beta_1 \neq 0$ and $|p_n| \leq C_0|p|$ if $\beta_1 = 0$.

For every $n = 0, 1, \dots$, let $\lambda_n(p_n)$ be an eigenvalue of problem (1.1)-(1.2) satisfying inequality (2.16) and let $u_n(x,\lambda_n(p_n))$ be a corresponding eigenfunction satisfying the initial conditions $u_n(0,\lambda_n(p_n)) = p_n\beta_1$, $u_n'(0,\lambda_n(p_n)) = -p_n\alpha_1$ if $\beta_1 \neq 0$ and satisfying the initial conditions $u_n(0,\lambda_n(p_n)) = 0$, $u_n'(0,\lambda_n(p_n)) = p_n\sqrt{\lambda_n(p_n)}$ if $\beta_1 = 0$. From inequalities (2.11)-(2.18) it follows that $|\lambda_n(p_n) - \lambda_n^{(0)}| \leq 4M$. From the last inequality and the asymptotic formulae $\mu_n^{(0)} = \pi n + O(n^{-1})$ if $\beta_1 \neq 0, \beta_2 \neq 0$ or $\beta_1 = \beta_2 = 0$ and $\mu_n^{(0)} = \pi(n+1/2) + O(n^{-1})$ if $\beta_1 = 0, \beta_2 \neq 0$ given in [8] it follows that there exists $N_0 > 0$ such that for all $n > N_0$

$$|\mu_n(p_n) - \mu_n^{(0)}| \leq \frac{4M}{\mu_n(p_n) + \mu_n^{(0)}} \leq \frac{5M}{n}, \tag{2.20}$$

where $\mu_n(p_n) = \sqrt{\lambda_n(p_n)}$ and $\mu_n^0 = \sqrt{\lambda_n^0}$. Thus there exists $N_1 > N_0$ such that for all $n > N_1$

$$\mu_n^{(0)} > \frac{n}{2}, \quad \mu_n(p_n) > \frac{n}{2}. \tag{2.21}$$

Let $\phi(x,\lambda), \psi(x,\lambda)$ be the fundamental system of solutions of equation (2.1) satisfying $\phi(0,\lambda) = 0$, $\phi_x'(0,\lambda) = \mu$, $\psi(0,\lambda) = 1$, $\psi_x'(0,\lambda) = 0$. Now

$$\phi(x,\lambda) = \sin\mu x + O(\frac{1}{\mu}), \quad \psi(x,\lambda) = \cos\mu x + O(\frac{1}{\mu}) \tag{2.22}$$

uniformly in x, $0 \leq x \leq 1$; see [9]. Therefore,

$$\begin{gathered} |\phi(x,\lambda)| \leq C_1, \quad |\psi(x,\lambda)| \leq C_1 \\ |\phi(x,\lambda_2) - \phi(x,\lambda_1)| \leq |\mu_2 - \mu_1| + O(\frac{1}{\mu_1}) + O(\frac{1}{\mu_2}), \\ |\psi(x,\lambda_2) - \psi(x,\lambda_1)| \leq |\mu_2 - \mu_1| + O(\frac{1}{\mu_1}) + O(\frac{1}{\mu_2}). \end{gathered} \tag{2.23}$$

uniformly in x, $0 \leq x \leq 1$. From [11] it follows that the functions $\phi(x,\lambda)$ and $\psi(x,\lambda)$ are continuous with partial derivatives in (x,λ) and hence

$$\begin{aligned} |\phi(x,\lambda_2) - \phi(x,\lambda_1)| &\leq C_2(\widehat{\lambda})|\lambda_2 - \lambda_1|, \\ |\psi(x,\lambda_2) - \psi(x,\lambda_1)| &\leq C_2(\widehat{\lambda})|\lambda_2 - \lambda_1|. \end{aligned} \tag{2.24}$$

for $0 \leq x \leq 1$, $1 \leq \lambda_1, \lambda_2 \leq \widehat{\lambda}$, for any $\widehat{\lambda} \geq 1$ where $C_2(\widehat{\lambda})$ depends on $\widehat{\lambda}$. Let $W(x,\lambda)$ be the Wronskian for functions $\phi(x,\lambda)$ and $\psi(x,\lambda)$. We denote

$$K(x,t,\lambda) = \frac{\psi(x,\lambda)\phi(t,\lambda) - \phi(x,\lambda)\psi(t,\lambda)}{W(t,\lambda)}.$$

An easy calculation gives $W(0,\lambda) = -\mu$. From this and Liouville's formula it follows that for any x, $W(x,\lambda) = -\mu$. From the last equality and estimates (2.23) it follows that

$$|K(x,t,\lambda)| \le \frac{2C_1^2}{\mu}. \tag{2.25}$$

It is easy to see that

$$\mathring{u}_n(x) = \beta_1\gamma_n\psi(x,\mathring{\lambda}_n) + \zeta_n\phi(x,\mathring{\lambda}_n). \tag{2.26}$$

By [3, Ch. 3, Sec. 6, Th. 6.4] solutions $u_n(x,\lambda_n(p_n))$ of the initial value problem (2.7) satisfy

$$\begin{aligned} &u_n(x,\lambda_n(p_n)) \\ &= p\,[\beta_1\gamma_n\psi(x,\lambda_n(p_n)) + \zeta_n\phi(x,\lambda_n(p_n))] \\ &\quad + \int_0^x K(x,t,\lambda_n(p_n))(q(t,u_n(t,\lambda_n(p_n)),\lambda_n(p_n)) - q_0(t))u_n(t,\lambda_n(p_n))dt. \end{aligned} \tag{2.27}$$

Now, we establish an upper bound for $u_n(x,\lambda_n(p_n))$. Clearly

$$|p(\beta_1\gamma_n\psi(x,\lambda_n(p_n)) + \zeta_n\phi(x,\lambda_n(p_n)))| \le C_3|p|. \tag{2.28}$$

From Lemma 2.1 and estimates for the p_n it follows that for any p satisfying condition 2.19, $|u_n(x,\lambda_n(p_n))| \le \frac{M_0}{2}$. Thus from (2.2) it follows that

$$|q(t,u_n(x,\lambda_n(p_n)),\lambda_n(p_n)) - q_0(t)| \le 2M. \tag{2.29}$$

Applying Gronwall's lemma to (2.27) using (2.25), (2.28), and (2.29), we see that

$$|u_n(x,\lambda_n(p_n))| \le C_4|p|. \tag{2.30}$$

From (2.20), (2.21), (2.22), (2.23), (2.29), and (2.30) it follows that for $n > N_1$

$$\begin{aligned} &\Big|\frac{u_n(x,\lambda_n(p_n))}{p} - \mathring{u}_n(x)\Big| \\ &\le |\beta_1\gamma_n|\big|\psi(x,\lambda_n(p_n)) - \psi(x,\mathring{\lambda}_n)\big| + |\zeta_n|\big|\phi(x,\lambda_n(p_n)) - \phi(x,\mathring{\lambda}_n)\big| \\ &\quad + \frac{1}{p}\int_0^x |K(x,t,\lambda_n(p_n))||q(t,u_n(t,\lambda_n(p_n)),\lambda_n(p_n)) - q_0(t)||u_n(t,\lambda_n(p_n))|dt \\ &\le \frac{\widetilde{C}}{n}. \end{aligned}$$

Clearly there exists a number $N_2 > N_1$ such that

$$\widetilde{C}^2\sum_{n=N_2}^{\infty}\frac{1}{n^2} < \frac{1}{2}.$$

From the last two inequalities and any p satisfying condition 2.19 it follows that

$$\sum_{n=N_2}^{\infty}\Big|\frac{u_n(x,\lambda_n(p_n))}{p} - \mathring{u}(x)\Big|^2 < \frac{1}{2}. \tag{2.31}$$

Consider the case $0 \le n \le N_2 - 1$. From (2.11) it follows that

$$\lambda_n(p_n) \le \lambda_n^{(1)}(2M) \le \lambda_{N_2}^{(1)}(2M). \tag{2.32}$$

Let ε be an arbitrary positive number. From condition 1) of Theorem 1.1 it follows that there exists δ, $0<\delta<1$, such that for $|u|<\delta$, $1\le\lambda\le\lambda_{N_2}^{(1)}(2M)$, $0\le t\le 1$,

$$|q(t,u,\lambda)-q_0(t)|<\varepsilon. \tag{2.33}$$

Let p satisfy the supplementary condition

$$0<|p|<\frac{\delta}{C_4+1}.$$

Then from (2.30) we obtain $|u_n(x,\lambda_n(p_n))|<\delta$. From the last inequality, (2.32), and (2.33) it follows that

$$Q_1(x,\lambda_n(p_n),\varepsilon)\le -q(x,u_n(x,\lambda_n(p_n)),\lambda_n(p_n))+\lambda_n(p_n)\le Q_2(x,\lambda_n(p_n),\varepsilon).$$

From this and the comparison theorem, [6], we obtain

$$\theta_1(1,\lambda_n(p_n),\varepsilon)\le\varphi_n(1,\lambda_n(p_n))\le\theta_2(1,\lambda_n(p_n),\varepsilon).$$

Moreover, we have the equality

$$\theta_1(1,\lambda_n^{(1)}(\varepsilon),\varepsilon)=\varphi_n(1,\lambda_n(p_n))=\theta_2(1,\lambda_n^{(2)}(\varepsilon),\varepsilon)=\arctan(-\frac{\alpha_2}{\beta_2})+\pi n.$$

From the last two relations it follows that

$$\begin{gathered}\theta_1(1,\lambda_n(p_n),\varepsilon)\le\theta_1(1,\lambda_n^{(1)}(\varepsilon),\varepsilon),\\ \theta_2(1,\lambda_n^{(2)}(\varepsilon),\varepsilon)\le\theta_2(1,\lambda_n(p_n),\varepsilon).\end{gathered}$$

From this and the monotonicity of the functions $\theta_i(1,\lambda,\varepsilon)$, [6], it follows that

$$\lambda_n^{(2)}(\varepsilon)\le\lambda_n(p_n)\le\lambda_n^{(1)}(\varepsilon).$$

From the last inequality and (2.18) we obtain

$$|\lambda_n(p_n)-\lambda_n^{(0)}|\le 2\varepsilon. \tag{2.34}$$

From (2.24), (2.25), (2.30), (2.33), and (2.34) it follows that

$$\begin{aligned}&\Big|\frac{u_n(x,\lambda_n(p_n))}{p}-\mathring{u}_n(x)\Big|\\ &\le|\beta_1\gamma_n|\big|\psi(x,\lambda_n(p_n))-\psi(x,\mathring{\lambda}_n)\big|+|\zeta_n|\big|\phi(x,\lambda_n(p_n))-\phi(x,\mathring{\lambda}_n)\big|\\ &\quad+\frac{1}{p}\int_0^x|K(x,t,\lambda_n(p_n))||u_n(t,\lambda_n(p_n))||q(t,u_n(t,\lambda_n(p_n)),\lambda_n(p_n))-q_0(t)|dt\\ &\le C_5\varepsilon\,,\end{aligned}$$

where the constant C_5 does not depend on ε.

Choosing $\varepsilon=1/(C_5\sqrt{2N_2})$ we obtain

$$\sum_{n=0}^{N_2-1}\Big|\frac{u_n(x,\lambda_n(p_n))}{p}-\mathring{u}_n(x)\Big|^2<\frac{1}{2}.$$

From the last inequality and (2.31) it follows that

$$\sum_{n=0}^{\infty}\Big\|\frac{u_n(x,\lambda_n(p_n))}{p}-\mathring{u}_n(x)\Big\|^2_{L_2(0,1)}<1.$$

□

(e) Obviously, the system $\{\overset{\circ}{u}_n(x)\}$ forms an orthonormal basis for the space $L_2(0,1)$. From the last inequality it follows that the eigenfunction systems $\{u_n(x,\lambda_n(p_n))/p\}$ and $\{u_n(x,\lambda_n(p_n))\}$ are a Bari basis and a Riesz basis, respectively [5, 7].

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References

[1] Y. N. Bibikov, *Ordinary Differential Equations*, Nauka, 1991.
[2] K. J. Brown, *A completeness theorem for a nonlinear problem*, Proc. Edinburgh Math. Soc. 19 (1974), 169-172.
[3] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.
[4] M. G. Crandall and P. H. Rabinowith, *Bifurcation from Simple Eigenvalues*, J. Funct. Anal. 8 (1971), 321-340.
[5] I. Ts. Gokhberg and M. G. Krein, *Introduction to the Theory of Linear Not Self-Adjoint Operators*, Nauka, 1965.
[6] P. Hartman, *Ordinary differential equations*, New York-London-Sydney, 1964.
[7] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1984.
[8] A. G. Kostuchenko and I. S. Sargsjan, *Eigenvalue distribution*, Nauka, 1979.
[9] A. P. Makhmudov, *Nonlinear eigenvalue problems and bifurcation theory*, Baku, 1996.
[10] M. A. Naimark, *Linear Differential Operators*, Ungar, 1967.
[11] L. S. Pontryagin, *Ordinary Differential Equations*, Nauka, 1965.
[12] C. Pugh, *Funnel Sections*, J. Diff. Equat. 19 (1975), 270-295.
[13] V. V. Tikhomirov, *Sharp estimates for the regular solutions of the one-dimensional Schrödinger operator with a spectral parameter*, Soviet. Math. Dokl. 28, No. 3, 722-725 (1983); translation from Dokl. Akad. Nauk SSSR 273, No. 4, 807-810, (Russian) (1983).

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