ON CLASSICAL SOLUTIONS OF THE RELATIVISTIC VLASOV-KLEIN-GORDON SYSTEM

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Abstract. We consider a collisionless ensemble of classical particles coupled with a Klein-Gordon field. For the resulting nonlinear system of partial differential equations, the relativistic Vlasov-Klein-Gordon system, we prove local-in-time existence of classical solutions and a continuation criterion which says that a solution can blow up only if the particle momenta become large. We also show that classical solutions are global in time in the one-dimensional case.

1. Introduction

In kinetic theory one often considers collisionless ensembles of classical particles which interact only by fields which they create collectively. This situation is commonly referred to as the mean field limit of a many particle system. Such systems have been studied extensively. In the case of non-relativistic, gravitational or electrostatic fields the corresponding system of partial differential equations is the Vlasov-Poisson system, in the case of relativistic electrodynamics it is the Vlasov-Maxwell system and in the case of general relativistic gravity the Vlasov-Einstein system.

On the other hand the coupling of a single particle to a classical or quantum field has been studied. In case of the Maxwell field this is a classical problem [1], but the actual dynamics and asymptotics of such systems is still an active area of research [6, 7, 8, 9]. In [5] the case of a single classical particle coupled to a quantum mechanical Klein-Gordon field was investigated.

In the present paper we consider a collisionless ensemble of particles moving at relativistic speeds, coupled to a Klein-Gordon field. This is a natural generalization of the one-particle situation just described. Let $f = f(t, x, v) \geq 0$ denote the density of the particles in phase space, $\rho = \rho(t, x)$ their density in space, and $u = u(t, x)$ a scalar Klein-Gordon field; $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, and $v \in \mathbb{R}^3$ denote time, position, and momentum, respectively. The system then reads as follows:

$$\partial_t f + \hat{v} \cdot \partial_x f - \partial_x u \cdot \partial_v f = 0, \tag{1.1}$$

$$\partial_t^2 u - \Delta u + u = -\rho, \tag{1.2}$$

$$\rho(t,x) = \int f(t,x,v)\,dv. \tag{1.3}$$

Here we have set all physical constants as well as the rest mass of the particles to unity, and

$$\hat{v} = \frac{v}{\sqrt{1+|v|^2}} \tag{1.4}$$

denotes the relativistic velocity of a particle with momentum v. This system is called the relativistic Vlasov-Klein-Gordon system. It is supplemented by initial data

$$f(0) = \mathring{f},\ u(0) = \mathring{u}_1,\quad \partial_t u(0) = \mathring{u}_2. \tag{1.5}$$

The study of this system was initiated in [11] where the existence of global weak solutions for initial data satisfying a size restriction was proved. This size restriction was necessary because the energy of the system is indefinite so that conservation of energy does not lead to a-priory bounds for general data. A natural next step in the study of the Vlasov-Klein-Gordon system is the existence theory of classical solutions, locally and if possible globally in time. This is the topic of the present investigation.

Another motivation for studying this system of partial differential equations is an intrinsically mathematical one. Since the field equation is hyperbolic the system resembles the relativistic Vlasov-Maxwell system, for which the quest for global-in-time classical general solutions is still open. One might hope that studying related systems can help in understanding these open problems more thoroughly. In fact in this work we follow the general outline of the existence proof of Glassey and Strauss [3] for the Vlasov-Maxwell system. Note, however, that the existence theory of weak solutions of the two systems is quite different [2, 11, 13].

The paper proceeds as follows. In Section 2 we prove some a-priori estimates necessary for the proof of our main result. These estimates rely on representation formulas for the first and second order derivatives of the Klein-Gordon field u, cf. Lemmas 2.1 and 2.2. Note that in contrast to the corresponding parts in [3] we also need to bound the mixed second order derivatives of the field. In Section 3 we prove our main results, a local-in-time existence and uniqueness result for classical solutions and a continuation criterion which says that such solutions can blow up in finite time only if the support of f in momentum space becomes unbounded, cf. Thms 3.1 and 3.3. We also discuss conservation of mass, energy and momentum. In Section 4 we briefly show that the continuation criterion is indeed satisfied in the one-dimensional situation where $x, v \in \mathbb{R}$ so that we obtain global classical solutions in that case. Finally some material on the (inhomogeneous) Vlasov equation is collected in an appendix as well as some unpleasant technical aspects of the proof of the local existence result, which often have been omitted in the treatment of related systems.

2. A-priori estimates

Although our notation is mostly standard or self-explaining we mention the following conventions: For a function $h = h(t,x,v)$ or $h = h(t,x)$ we denote for given t by $h(t)$ the corresponding function of the remaining variables. For a function h depending on the variables x, v we denote its gradient by $\partial_{(x,v)}h$. By $\|\,.\,\|_p$ we denote the usual L^p-norm for $p \in [1, \infty]$. The subscript c in function spaces refers to compactly supported functions. Sometimes we write $z = (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$.

One main ingredient of our analysis are representation formulas for u and its derivatives, which will allow us to establish the necessary a-priori bounds. As point of departure we recall that the solution of (1.2) is given by

$$u(t,x) = u_{\text{hom}}(t,x) + u_{\text{inh}}(t,x), \quad t \geq 0, \ x \in \mathbb{R}^3, \tag{2.1}$$

where

$$
\begin{aligned}
u_{\text{hom}}(t,x) = {} & \frac{1}{4\pi t^2} \int_{|x-y|=t} \overset{\circ}{u}_1(y)\, dS_y - \frac{1}{4\pi t^2} \int_{|x-y|=t} (\partial_x \overset{\circ}{u}_1)(y) \cdot y\, dS_y \\
& - \frac{1}{8\pi} \int_{|x-y|=t} \overset{\circ}{u}_1(y)\, dS_y - \frac{1}{4\pi} \int_{|x-y|\leq t} \overset{\circ}{u}_1(y) \left(\frac{J_1(\xi)}{\xi} \right)' \frac{t}{\xi}\, dy \\
& + \frac{1}{4\pi t} \int_{|x-y|=t} \overset{\circ}{u}_2(y)\, dS_y - \frac{1}{4\pi} \int_{|x-y|\leq t} \overset{\circ}{u}_2(y) \frac{J_1(\xi)}{\xi}\, dy,
\end{aligned}
$$

is the solution of the homogeneous Klein-Gordon equation with initial data as in (1.5) and $\xi := \sqrt{t^2 - |x-y|^2}$, and

$$u_{\text{inh}}(t,x) = -\frac{1}{4\pi} \int_0^t \int_{|x-y|=t-s} \rho(s,y)\, dS_y \frac{ds}{t-s} + \frac{1}{4\pi} \int_0^t \int_{|x-y|\leq t-s} \rho(s,y) \frac{J_1(\xi)}{\xi}\, dy\, ds$$

with $\xi := \sqrt{(t-s)^2 - |x-y|^2}$ is the solution of the inhomogeneous Klein-Gordon equation with vanishing initial data, cf. [12] or [14]; J_k denotes the Bessel function. To derive formulas for the derivatives of u the differential operators

$$S = \partial_t + \hat{v} \cdot \partial_x, \quad T = -\omega \partial_t + \partial_x; \qquad \omega = \frac{x-y}{|x-y|},$$

which are adapted to our system and have first been introduced in [3] in connection with the Vlasov-Maxwell system turn out to be useful.

Lemma 2.1 (Representation of ∂u). *Suppose $u \in \mathcal{C}^2$ is a solution of the Klein-Gordon equation (1.2) with ρ given by (1.3) for some $f \in \mathcal{C}^1$. Then*

$$\partial_k u(t,x) = F_0^k(t,x) + F_S^k(t,x) + F_T^k(t,x) + F_R^k(t,x) + F_J^k(t,x), \quad k \in \{1,2,3,t\}$$

where F_0^k is a linear functional of the initial data only, and

$$F_S^k(t,x) = -\frac{1}{4\pi} \int_{|x-y|\leq t} \int \frac{\omega_k}{1+\omega \cdot \hat{v}} (Sf)(t-|x-y|,y,v)\, dv \frac{dy}{|x-y|},$$

$$F_T^k(t,x) = \frac{1}{4\pi} \int_{|x-y|\leq t} \int a^k(\omega,\hat{v}) f(t-|x-y|,y,v)\, dv \frac{dy}{|x-y|^2},$$

$$F_R^k(t,x) = \frac{1}{8\pi} \int_{|x-y|\leq t} \rho(t-|x-y|,y)\omega_k\, dy,$$

$$F_J^k(t,x) = -\frac{1}{4\pi} \int_0^t \int_{|x-y|\leq t-s} \rho(s,y) \frac{J_2(\xi)}{\xi^2} (x_k - y_k)\, dy\, ds,$$

$$F_S^t(t,x) = -\frac{1}{4\pi} \int_{|x-y|\leq t} \int \frac{1}{1+\omega \cdot \hat{v}} (Sf)(t-|x-y|,y,v)\, dv \frac{dy}{|x-y|},$$

$$F_T^t(t,x) = \frac{1}{4\pi} \int_{|x-y|\leq t} \int a^t(\omega,\hat{v}) f(t-|x-y|,y,v)\, dv \frac{dy}{|x-y|^2},$$

$$F_R^t(t,x) = \frac{1}{8\sqrt{2}\pi} \int_{|x-y|\leq t} \rho(t-|x-y|,y)\, dy,$$

$$F_J^t(t,x) = -\frac{1}{4\pi}\int_0^t\int_{|x-y|\le t-s}\rho(s,y)\frac{J_2(\xi)}{\xi^2}(t-s)dyds.$$

Here $\xi = \sqrt{(t-s)^2 - |x-y|^2}$ and the kernels a^k and a^t are

$$a^k(\omega,\hat{v}) = -\frac{\hat{v}_k}{1+\omega\cdot\hat{v}} - \frac{\omega_k}{(1+|v|^2)(1+\omega\cdot\hat{v})^2}, \quad a^t(\omega,\hat{v}) = \frac{|\hat{v}|^2 + \omega\cdot\hat{v}}{(1+\omega\cdot\hat{v})^2}.$$

Proof. The proof is a straightforward calculation using

$$\partial_{x_k} = \frac{\omega_k}{1+\omega\cdot\hat{v}}\,S + \sum_{j=1}^3\left(\delta_{jk} - \frac{\omega_k\hat{v}_j}{1+\omega\cdot\hat{v}}\right)T_j, \quad \partial_t = \frac{1}{1+\omega\cdot\hat{v}}\left(S - \hat{v}\cdot T\right)$$

where $\delta_{jj} = 1$ and $\delta_{jk} = 0$ for $j \ne k$. Terms involving only the initial data are collected in F_0^k and terms involving Tf are integrated by parts, using the identity $(Tf)(t-|x-y|,y,v) = \partial_y(f(t-|x-y|,y,v))$. $\qquad\square$

Next we turn to the second order derivatives.

Lemma 2.2 (Representation of $\partial^2 u$). *Suppose $u \in \mathcal{C}^2$ is a solution of the Klein-Gordon equation (1.2) with ρ given by (1.3) for some $f \in \mathcal{C}^2$. Then we have for $k,\ell \in \{1,2,3,t\}$*

$$\partial_{k\ell}u(t,x) \quad = \quad F_0^{k\ell} + F_{SS}^{k\ell} + F_{ST}^{k\ell} + F_{TS}^{k\ell} + F_{TT}^{k\ell} + F_{RS}^{k\ell} + F_{RT}^{k\ell} + F_{JR}^{k\ell} + F_{JJ}^{k\ell},$$

where $F_0^{k\ell}$ are linear functionals of the initial data only, and

$$F_{SS}^{k\ell} = \frac{-1}{4\pi}\int_{|x-y|\le t}\int c^{k\ell}(\omega,\hat{v})(S^2 f)(t-|x-y|,y,v)\,dv\frac{dy}{|x-y|}, \quad |c^{k\ell}| \le \frac{C}{(1+\omega\cdot\hat{v})^2}$$

$$F_{ST}^{k\ell} = \frac{1}{4\pi}\int_{|x-y|\le t}\int b_1^{k\ell}(\omega,\hat{v})(Sf)(t-|x-y|,y,v)\,dv\frac{dy}{|x-y|^2}, \quad |b_1^{k\ell}| \le \frac{C}{(1+\omega\cdot\hat{v})^3}$$

$$F_{TS}^{k\ell} = \frac{1}{4\pi}\int_{|x-y|\le t}\int b_2^{k\ell}(\omega,\hat{v})(Sf)(t-|x-y|,y,v)\,dv\frac{dy}{|x-y|^2}, \quad |b_2^{k\ell}| \le \frac{C}{(1+\omega\cdot\hat{v})^3}$$

$$F_{TT}^{k\ell} = \frac{-1}{4\pi}\int_{|x-y|\le t}\int a^{k\ell}(\omega,\hat{v})f(t-|x-y|,y,v)\,dv\frac{dy}{|x-y|^3}, \quad \int_{|\omega|=1}a^{k\ell}(\omega,\hat{v})d\omega = 0,$$

$$F_{RS}^{k\ell} = \frac{1}{8\pi}\int_{|x-y|\le t}\int d^{k\ell}(\omega,\hat{v})(Sf)(t-|x-y|,y,v)\,dv\,dy, \quad |d^{k\ell}| \le \frac{C}{1+\omega\cdot\hat{v}}$$

$$F_{RT}^{k\ell} = \frac{1}{8\pi}\int_{|x-y|\le t}\int e^{k\ell}(\omega,\hat{v})f(t-|x-y|,y,v)\,dv\frac{dy}{|x-y|}, \quad |e^{k\ell}| \le \frac{C}{(1+\omega\cdot\hat{v})^2}$$

$$F_{JR}^{k\ell} = \frac{-1}{32\pi}\Big(1 - \frac{\sqrt{2}-1}{\sqrt{2}}(\delta_{kt}+\delta_{\ell t}-\delta_{kt}\delta_{\ell t})\Big)\int_{|x-y|\le t}\rho(t-|x-y|,y)\omega_k\omega_\ell|x-y|\,dy,$$

$$F_{JJ}^{k\ell} = \frac{2(\delta_{kt}+\delta_{\ell t}-\delta_{kt}\delta_{\ell t})-1}{4\pi}$$

$$\times \int_0^t\int_{|x-y|\le t-s}\rho(s,y)\Big(\frac{J_3(\xi)}{\xi^3}(x_k-y_k)(x_\ell-y_\ell) + (1-\delta_{kt}-\delta_{\ell t})\frac{J_2(\xi)}{\xi^2}\Big)dy\,ds.$$

Here we have set $x_t = t$, $y_t = s$, and $\omega_t = 1$ for notational convenience.

Proof. The proof is a long and tedious calculation similar to the previous lemma. The critical part is, of course, to prove that the kernels $a^{k\ell}$ appearing in the singular TT-terms vanish when integrated over the unit sphere. To give some flavor of the

respective calculations we outline them in the case of the (most complicated) kernel $a^{k\ell}$ for $1 \le k, \ell \le 3$. We have

$$
a^{k\ell} = -3 \frac{\omega_\ell \cdot \hat{v}_k + \omega_k \cdot \hat{v}_\ell}{(1 + |v|^2)(1 + \omega \cdot \hat{v})^3} - \frac{3\omega_k \omega_\ell}{(1 + |v|^2)^2(1 + \omega \cdot \hat{v})^4}
$$
$$
- \frac{2\hat{v}_k \hat{v}_\ell}{(1 + \omega \cdot \hat{v})^2} + \frac{\delta_{k\ell}}{(1 + |v|^2)(1 + \omega \cdot \hat{v})^2}
$$
$$
=: a_1^{k\ell} + a_2^{k\ell} + a_3^{k\ell} + a_4^{k\ell}.
$$

Using $\partial_{v_i} \left((\sqrt{1 + |v|^2} + \omega \cdot v)^{-2} \right) = -2(\hat{v}_i + \omega_i) \left(\sqrt{1 + |v|^2} + \omega \cdot v \right)^{-3}$ we find

$$
v_k \int_{|\omega|=1} \frac{\omega_\ell \, d\omega}{(\sqrt{1 + |v|^2} + \omega \cdot v)^3} = -\frac{v_k}{2} \partial_{v_\ell} \int_{|\omega|=1} \frac{d\omega}{(\sqrt{1 + |v|^2} + \omega \cdot v)^2}
$$
$$
- v_k \hat{v}_\ell \int_{|\omega|=1} \frac{d\omega}{(\sqrt{1 + |v|^2}(1 + \omega \cdot \hat{v}))^3} = -4\pi v_k v_\ell.
$$

Hence $\int_{|\omega|=1} a_1^{k\ell} \, d\omega = 24\pi v_k v_\ell$. By the identity

$$
\partial_{v_k} \left(\partial_{v_\ell} \frac{1}{(\sqrt{1 + |v|^2} + \omega \cdot v)^2} + \frac{2\hat{v}_\ell}{(\sqrt{1 + |v|^2} + \omega \cdot v)^3} \right) = 6 \frac{\omega_\ell(\hat{v}_k + \omega_k)}{(\sqrt{1 + |v|^2} + \omega \cdot v)^4}
$$

we obtain

$$
- \int_{|\omega|=1} \frac{\omega_\ell \hat{v}_k \, d\omega}{(\sqrt{1 + |v|^2} + \omega \cdot v)^4}
$$
$$
+ \frac{1}{6} \partial_{v_k} \int_{|\omega|=1} \left(\partial_{v_\ell} \frac{1}{(\sqrt{1 + |v|^2} + \omega \cdot v)^2} + \frac{2\hat{v}_\ell}{(\sqrt{1 + |v|^2} + \omega \cdot v)^3} \right) d\omega
$$
$$
= \frac{16\pi}{3} v_\ell v_k + \frac{4\pi}{3} \delta_{\ell k},
$$

which implies $\int_{|\omega|=1} a_2^{k\ell} \, d\omega = -16\pi v_k v_\ell - 4\pi \delta_{k\ell}$. The remaining two terms can be integrated directly to yield

$$
\int_{|\omega|=1} a_3^{k\ell} \, d\omega = -8\pi v_k v_\ell, \quad \int_{|\omega|=1} a_4^{k\ell} \, d\omega = 4\pi \delta_{k\ell}.
$$

Summing up we obtain the desired result

$$
\int_{|\omega|=1} a^{k\ell} \, d\omega = 0.
$$

The computations for the kernels

$$
a^{tk} = \frac{2\hat{v}_k(\omega \cdot \hat{v} + |\hat{v}|^2)}{(1 + \omega \cdot \hat{v})^3} + \frac{3\omega_k(\omega \cdot \hat{v} + |\hat{v}|^2)}{(1 + |v|^2)(1 + \omega \cdot \hat{v})^4} - \frac{\hat{v}_k}{(1 + |v|^2)(1 + \omega \cdot \hat{v})^3}
$$
$$
a^{tt} = \frac{1}{(1 + \omega \cdot \hat{v})^4} \left(3|\hat{v}|^4 - (\omega \cdot \hat{v})^2|\hat{v}|^2 - |\hat{v}|^2 + 3(\omega \cdot \hat{v})^2 + 4\omega \cdot \hat{v}|\hat{v}|^2 \right)
$$

$(1 \le k \le 3)$ along the same lines. $\square$

Note that in contrast to the corresponding analysis of the Vlasov-Maxwell system [3] we also need to provide the mixed second order derivatives of the field. The fact that the averaging property of corresponding the TT-kernel holds is a pleasant surprise in so far as the mixed derivatives do not appear in the field equation.

We can now use these formulas to derive the necessary estimates on the derivatives of u. Constants denoted by C are positive and may change their value from line to line. If they depend on the initial data this is explicitly mentioned.

Lemma 2.3 (Estimates on ∂f, ∂u, $\partial^2 u$).

(i) *Suppose that $f \in \mathcal{C}^1$ is a solution of the Vlasov equation*

$$Sf(t,x,v) = F(t,x)\partial_v f(t,x,v), \qquad f(0,x,v) = \overset{\circ}{f}(x,v), \qquad t \in [0,T[,$$

for some $F \in \mathcal{C}^1$. Then we have, with $z = (x,v)$,

$$\|\partial_z f(t)\|_\infty \leq \|\partial_z \overset{\circ}{f}\|_\infty + C \int_0^t (1 + \|\partial_x F(s)\|_\infty)\|\partial_z f(s)\|_\infty ds.$$

(ii) *In addition, assume there exists an increasing function $P(t)$ such that $f(t,x,v) = 0$ for $|v| \geq P(t)$, and suppose u is the solution (2.1) of the Klein-Gordon equation for $t \in [0,T[$. Then $u \in \mathcal{C}^2$, and*

$$\|\partial_{(t,x)}u(t)\|_\infty \leq C\Big((1+t)^5(1+P(t))^5 + \int_0^t (t-s)(1+P(s))^6\|F(s)\|_\infty ds\Big),$$

where the constant C depends on the norms of the initial data.

(iii) *On any bounded time interval on which F and P are bounded we have*

$$\|\partial_{(t,x)}^2 u(t)\|_\infty \leq C\Big(1 + \log^*(\sup_{0 \leq s \leq t} \|\partial_x f(s)\|_\infty) + \int_0^t \|\partial_{(t,x)}F(s)\|_\infty ds\Big),$$

where $\log^*(s) = s$ for $0 \leq s \leq 1$ and $\log^*(s) = 1 + \log(s)$ for $1 \geq s$, and C depends on the time interval, the bounds for P and F, and the initial data.

Proof. The assertion in (i) follows directly from (5.3) below with $g = 0$ and $G(t,z) = (\hat{v}, F(t,x))$. As to (ii), the estimate of ∂u is a straightforward consequence of Lemma 2.1 by using $Sf = F\partial_v f$ and getting rid of the v derivatives via integration by parts. Then we estimate each term individually using $\|f(t)\|_\infty = \|\overset{\circ}{f}\|_\infty$, cf. (5.2) with $g \equiv 0$, the estimate

$$\frac{1}{1 + \omega \cdot \hat{v}} \leq 2(1 + |v|^2),$$

and the fact that the v-integration is only over a finite ball of radius $P(t)$.

To prove (iii) we use the same procedure with Lemma 2.2 replacing Lemma 2.1. First we need to get rid of the second derivative (i.e., the $S^2 f$ term in $F_{SS}^{k\ell}$). So let us first assume $Sf \in \mathcal{C}^1$ and consider

$$F_{SS}^{k\ell} = \frac{-1}{4\pi} \int_{|x-y| \leq t} \int \int c^{k\ell}(\omega, \hat{v})(S^2 f)(t - |x-y|, y, v)\, dv \frac{dy}{|x-y|}.$$

Observe that

$$S^2 f = (SF) \cdot \partial_v f + F \cdot \partial_v(F \cdot \partial_v f) - Fc(v)\partial_x f, \quad c_{jk}(v) = \partial_{v_j}\hat{v}_k.$$

Inserting this into $F_{SS}^{k\ell}$ and removing all v derivatives using integration by parts we end up with an expression involving only first order derivatives of f, which also holds without the $Sf \in \mathcal{C}^1$ assumption by a standard approximation argument, in particular, $u \in \mathcal{C}^2$.

Now we can estimate each term as before except for the TT-kernel which is the most critical one, i.e.,

$$F_{TT}^{k\ell} = \frac{1}{4\pi} \int_{|x-y|\leq t} \int a^{k\ell}(\omega,\hat{v})f(t-|x-y|,y,v)\,dv\,\frac{dy}{|x-y|^3}$$

$$= \int_0^t \frac{1}{t-s} \int_{|\omega|=1} \int a^{kl}(\omega,\hat{v})f(s,x+(t-s)\omega,v)dv\,d\omega\,ds,$$

where we split the s-integral into two parts, I over $[0, t-\tau]$ and II over $[t-\tau, t]$. The first of these can be estimated directly by $C\log(t/\tau)$. For the second one we use the averaging property of $a^{k\ell}$ to write it as

$$II = \int_{t-\tau}^t \frac{1}{t-\tau} \int_{|\omega|=1} \int a^{k\ell}(\omega,\hat{v})\Big(f(s,x+(t-s)\omega,v) - f(s,x,v)dvd\omega\Big)ds.$$

Hence by the mean value theorem

$$|II| \leq C\,\tau \sup_{t-\tau\leq s\leq t} \|\partial_x f(s)\|_\infty.$$

Summing up we obtain the estimate

$$|F_{TT}^{k\ell}(t)| \leq C(\log(t/\tau) + \tau N(t)), \quad 0 \leq t \leq T,$$

where $N(t) = \sup_{0\leq s\leq t} \|\partial_x f(s)\|_\infty$. For $N(t) \leq t^{-1}$ we can choose the optimal value $\tau = N(t)^{-1}$, otherwise we choose $\tau = t$, and this yields

$$|F_{TT}^{k\ell}(t)| \leq C\log^*(tN(t)).$$

Combining these estimates the remaining claim follows. $\qquad\square$

3. EXISTENCE OF CLASSICAL SOLUTIONS

We now have collected all ingredients to show existence and uniqueness of local classical solutions of the relativistic Vlasov-Klein-Gordon system.

Theorem 3.1 (Local existence of classical solutions). *Let $\mathring{f} \in \mathcal{C}_c^1(\mathbb{R}^6)$, $\mathring{u}_1 \in \mathcal{C}_b^3(\mathbb{R}^3)$, $\mathring{u}_2 \in \mathcal{C}_b^2(\mathbb{R}^3)$. Then there exists a unique classical solution*

$$f \in \mathcal{C}^1([0,T[\times\mathbb{R}^6), \; u \in \mathcal{C}^2([0,T[\times\mathbb{R}^3)$$

of the relativistic Vlasov-Klein-Gordon system (1.1)–(1.3) for some $T > 0$, satisfying the initial conditions (1.5). Moreover,

$$f(t,x,v) = 0 \text{ for } |x| \geq \mathring{R} + t \text{ or } |v| \geq P(t)$$

where $\mathring{R}$ is determined by $\mathring{f}$ and P is a positive continuous function on $[0,T[$.

Proof. We begin with the *uniqueness* part which relies only on Lemma 2.1. Let $(f^{(1)}, u^{(1)})$, $(f^{(2)}, u^{(2)})$ be two solutions satisfying the same initial conditions, and on any compact time interval $[0, T_0]$ on which both solutions exist define $f = f^{(1)} - f^{(2)}$ and $u = u^{(1)} - u^{(2)}$. Then u satisfies (1.2) with (1.3), and f satisfies

$$Sf = \partial_x u^{(1)} \partial_v f^{(1)} - \partial_x u^{(2)} \partial_v f^{(2)} = \partial_x u \partial_v f^{(1)} + \partial_x u^{(2)} \partial_v f.$$

Proceeding on a term-by-term basis using the representation from Lemma 2.1 and replacing the Sf term via the Vlasov equation and integrating by parts we obtain the estimate

$$\|\partial_x u(t)\|_\infty \leq C \int_0^t \Big(\|f(s)\|_\infty + \|\partial_x u(s)\|_\infty \|f^{(1)}(s)\|_\infty + \|\partial_x u^{(2)}(s)\|_\infty \|f(s)\|_\infty \Big)\,ds.$$

Using the boundedness of $f^{(1)}$ and $\partial_x u^{(2)}$ this implies

$$\|\partial_x u(t)\|_\infty \le C \int_0^t (\|f(s)\|_\infty + \|\partial_x u(s)\|_\infty)\, ds.$$

On the other hand, by (5.2)

$$f(s,x,v) = \int_0^t (\partial_x u \, \partial_v f^{(1)})(s, Z^{(2)}(s,t,x,v))\, ds,$$

where $Z^{(2)}(s,t,x,v)$ is the solution of the characteristic system corresponding to $u^{(2)}$. Hence we obtain, using the boundedness of $\partial_v f^{(1)}$,

$$\|f(t)\|_\infty \le C \int_0^t \|\partial_x u(s)\|_\infty ds.$$

Combining both estimates gives

$$\|f(t)\|_\infty + \|\partial_x u(t)\|_\infty \le C \int_0^t (\|f(s)\|_\infty + \|\partial_x u(s)\|_\infty)\, ds$$

and Gronwall's lemma implies $f(t) = u(t) = 0$, proving uniqueness.

Next we turn to *existence*. To this end we set up an iterative scheme and prove its convergence to a solution. Let $f^{(0)} = \mathring{f}$, $u^{(0)} = \mathring{u}_1$ and define $f^{(n)} \in \mathcal{C}^1([0,\infty[\times\mathbb{R}^6))$, $u^{(n)} \in \mathcal{C}^2([0,\infty[\times\mathbb{R}^3)$ recursively via

$$S f^{(n)} - \partial_x u^{(n-1)} \partial_v f^{(n)} = 0, \quad f^{(n)}(0) = \mathring{f}$$

and

$$\partial_t^2 u^{(n)} - \Delta u^{(n)} + u^{(n)} = -\rho^{(n)}, \quad u^{(n)}(0) = \mathring{u}_1, \quad \partial_t u^{(n)}(0) = \mathring{u}_2$$

where

$$\rho^{(n)}(t,x) = \int f^{(n)}(t,x,v)\, dv;$$

notice that $f^{(n)}$ satisfies a support estimate as the one asserted for the solution f, but for all $t \ge 0$ and with the increasing function

$$P^{(n)}(t) := \sup\left\{ |v| \mid f^{(n)}(\tau,x,v) \ne 0,\ 0 \le \tau \le t,\ x \in \mathbb{R}^3 \right\}$$

instead of P.

Step 1 (Uniform bounds on $P^{(n)}$ and $\partial u^{(n)}$):
By Lemma 2.3 (ii) and (5.2) we have

$$\|\partial_{(t,x)} u^{(n)}(t)\|_\infty \le C\Big((1+t)^5(1+P^{(n)})^5 + \int_0^t s^2 P^{(n)}(s)^6 \|\partial_{(t,x)} u^{(n-1)}(s)\|_\infty ds\Big)$$

and

$$P^{(n)}(t) \le \mathring{P} + \int_0^t \|\partial_x u^{(n-1)}(s)\|_\infty ds,$$

where $\mathring{P}$ bounds the v-support of the data.
Define $Q^{(n)}(t) := \max_{0 \le k \le n} \|\partial_{(t,x)} u^{(k)}(t)\|_\infty$. Then

$$Q^{(n)}(t) \le C\Big((1+t)^5(1+P^{(n)}(t))^5 + \int_0^t (t-s)(1+P^{(n)}(s))^6 Q^{(n)}(s)\, ds\Big),$$

and by Gronwall's lemma,

$$Q^{(n)}(t) \le C\Big((1+t)^5(1+P^{(n)}(t))^5 \exp\big(t^2(1+P^{(n)}(t))^6\big)\Big).$$

If we insert this into the estimate for $P^{(n)}$ we find that

$$P^{(n)}(t) \leq \mathring{P} + C \int_0^t (1+s)^5 (1+P^{(n)}(s))^5 \exp\left(s^2(1+P^{(n)}(s))^6\right) ds, \; t \geq 0. \quad (3.1)$$

Hence by induction, $P^{(n)}(t) \leq P(t)$, $n \in \mathbb{N}$, $t \in [0, T[$, where P is the maximal solution of the integral equation corresponding to (3.1), which exists on some time interval $[0, T[$ whose length is determined by $\mathring{P}$ and the norms of the initial data entering the constant C. In addition, $\partial_{(t,x)} u^{(n)}(t)$ are bounded in terms of $P(t)$ on that interval.

For the rest of the proof we now argue on a bounded, arbitrary, but fixed time interval $[0, T_0] \subset [0, T[$. Constants denoted by C may now depend on T_0 and the bounds established in Step 1.

Step 2 (Uniform bounds on $\partial^2 u^{(n)}$ and $\partial_z f^{(n)}$):
By Lemma 2.3 (i) and (iii),

$$\|\partial_z f^{(n)}(t)\|_\infty \leq C\Big(1 + \int_0^t \big(1 + \|\partial_{(t,x)}^2 u^{(n-1)}(s)\|_\infty \|\partial_z f^{(n)}(s)\|_\infty\big) ds\Big), \quad (3.2)$$

$$\|\partial_{(t,x)}^2 u^{(n)}(t)\|_\infty \leq C\Big(\big(1 + \log^*(\sup_{0 \leq t \leq T_0} \|\partial_x f^{(n)}(t)\|_\infty) + \int_0^t \|\partial_{(t,x)}^2 u^{(n-1)}(s)\|_\infty ds\Big). \quad (3.3)$$

Applying Gronwall's lemma to (3.2) yields

$$\|\partial_z f^{(n)}(t)\|_\infty \leq C \exp\Big(C \int_0^t (1 + \|\partial_{(t,x)}^2 u^{(n-1)}(s)\|_\infty) ds\Big). \quad (3.4)$$

Inserting this into (3.3) we obtain

$$\|\partial_{(t,x)}^2 u^{(n)}(t)\|_\infty \leq C + C \int_0^t (1 + \|\partial_{(t,x)}^2 u^{(n-1)}(s)\|_\infty) ds,$$

and by induction,

$$\|\partial_{(t,x)}^2 u^{(n)}(t)\|_\infty \leq C e^{CT_0}.$$

The bound on $\|\partial_z f^{(n)}(t)\|_\infty$ now follows from (3.4).

Step 3 (Uniform Cauchy property of $f^{(n)}(t)$ and $\partial_x u^{(n)}(t)$):
Introduce

$$f^{m,n} = f^{(m)} - f^{(n)}, \qquad u^{m,n} = u^{(m)} - u^{(n)}$$

and note that

$$Sf^{m,n} = (\partial_x u^{(n-1)})\partial_v f^{m,n} + \partial_x u^{m-1,n-1}\partial_v f^{(m)}.$$

As in the uniqueness part we derive the estimates

$$\|\partial_x u^{m,n}(t)\|_\infty \leq C \int_0^t (\|\partial_x u^{m-1,n-1}(s)\|_\infty + \|f^{m,n}(s)\|_\infty) ds$$

and

$$\|f^{m,n}(t)\|_\infty \leq C \int_0^t \|\partial_x u^{m-1,n-1}(s)\|_\infty ds.$$

Combining these we obtain

$$\|\partial_x u^{m,n}(t)\|_\infty \leq C \int_0^t \|\partial_x u^{m-1,n-1}(s)\|_\infty ds,$$

and hence by induction,

$$\|\partial_x u^{m,n}(t)\|_\infty \le C \frac{(Ct)^k}{k!}, \quad k = \min(m,n).$$

So $\partial_x u^{(n)}(t)$ is a uniform Cauchy sequence, and the same is true for $f^{(n)}(t)$.

Step 4 (Uniform Cauchy property of $\partial_z f^{(n)}(t)$ and $\partial^2_{(t,x)} u^{(n)}(t)$):

We begin by establishing a bound on $\|\partial^2_{(t,x)} u^{m,n}(t)\|_\infty$. Using the representation of Lemma 2.2, proceeding on a term-by-term basis and again using a splitting as in the uniqueness part we obtain

$$\|\partial^2_{(t,x)} u^{m,n}(t)\|_\infty \le C \int_0^t \left(\|\partial^2_{(t,x)} u^{m-1,n-1}(s)\|_\infty + \|\partial_z f^{m,n}(s)\|_\infty \right) ds. \qquad (3.5)$$

Here the critical TT-kernel is estimated as in the proof of Lemma 2.3 letting $\tau \to t$. For the other terms we use the boundedness assertions as well as the Cauchy properties already obtained for the lower derivatives.

Next we prove that the characteristics converge uniformly. Writing $Z^{(n)}(s,t,z) = (X^{(n)}(s,t,z), V^{(n)}(s,t,z))$, $z = (x,v)$, where we omit the arguments if there is no danger of misinterpretation, we find

$$\Big| \frac{d}{ds} (X^{(n)} - X^{(m)}) \Big| \le |V^{(n)} - V^{(m)}|,$$

$$\begin{aligned}
\Big| \frac{d}{ds} (V^{(n)} - V^{(m)}) \Big| &= |\partial_x u^{(n-1)}(X^{(n)}) - \partial_x u^{(m-1)}(X^{(m)})| \\
&\le |\partial_x u^{(n-1)}(X^{(n)}) - \partial_x u^{(n-1)}(X^{(m)})| \\
&\quad + |\partial_x u^{(n-1)}(X^{(m)}) - \partial_x u^{(m-1)}(X^{(m)})| \\
&\le C |X^{(n)} - X^{(m)}| + \delta_{m,n}.
\end{aligned}$$

Here the expression $\delta_{m,n}$ converges to zero if $m,n \to \infty$ by the Cauchy property of $\partial_x u^{(n)}$ and we have used the boundedness of $\partial^2_x u^{(n)}$. Combining these two estimates and again using Gronwall's lemma we obtain the claimed convergence of the characteristics, i.e., $Z^{(n)}(s,t,z)$ converges uniformly for $0 \le s \le T_0$; the convergence is also uniform w.r.t. the parameters t and z.

Writing $Z^{(n)}(s)$ for $Z^{(n)}(s,t,z)$ the analog of equation (5.3) for iterates and vanishing g implies

$$\partial_z f(t,z) = \partial_z \mathring{f}(Z^{(n)}(0)) - \int_0^t \partial_z f^{(n)}(s, Z^{(n)}(s)) \, \partial^2_x u^{(n-1)}(s, X^{(n)}(s)) \, ds,$$

and hence

$$\begin{aligned}
|\partial_z f^{m,n}(t,z)| \le {} & |\partial_z \mathring{f}(Z^{(m)}(0)) - \partial_z \mathring{f}(Z^{(n)}(0))| \\
& + \int_0^t \Big| \partial_z f^{(m)}(s, Z^{(m)}(s)) \, \partial^2_x u^{(m-1)}(s, X^{(m)}(s)) \\
& \quad - \partial_z f^{(n)}(s, Z^{(n)}(s)) \, \partial^2_x u^{(n-1)}(s, X^{(n)}(s)) \Big| \, ds.
\end{aligned}$$

The first term vanishes in the limit $m, n \to \infty$, and we split the second term into the following four parts:

$$
\begin{aligned}
\int_0^t \Big(&|\partial_z f^{(m)}(s, Z^{(m)}(s)) - \partial_z f^{(m)}(s, Z^{(n)}(s))| \; |\partial_x^2 u^{(m-1)}(s, X^{(m)}(s))| \\
&+ |\partial_z f^{(m)}(s, Z^{(n)}(s))| \; |\partial_x^2 u^{(m-1)}(s, X^{(m)}(s)) - \partial_x^2 u^{(m-1)}(s, X^{(n)}(s))| \\
&+ |\partial_z f^{(m)}(s, Z^{(n)}(s)) - \partial_z f^{(n)}(s, Z^{(n)}(s))| \; |\partial_x^2 u^{(m-1)}(s, X^{(n)}(s))| \\
&+ |\partial_z f^{(n)}(s, Z^{(n)}(s))| \; |\partial_x^2 u^{(m-1)}(s, X^{(n)}(s)) - \partial_x^2 u^{(n-1)}(s, X^{(n)}(s))| \Big) \; ds.
\end{aligned}
\tag{3.6}
$$

The first two vanish in the limit $m, n \to \infty$ by Lemma 3.2 below while the latter two may be estimated due to the boundedness of $\partial_x^2 u^{(n)}$ and $\partial_z f^{(n)}$. Hence

$$
\|\partial_z f^{m,n}(t, z)\|_\infty \le \delta_{m,n} + C \int_0^t \left(\|\partial_z f^{m,n}(s)\|_\infty + \|\partial_x^2 u^{m-1,n-1}(s)\|_\infty \right) \; ds,
$$

where again $\delta_{m,n} \to 0$ if $m, n \to \infty$. Another application of Gronwall's lemma gives

$$
\|\partial_z f^{n,m}(t, z)\|_\infty \le \delta_{m,n} + C \int_0^t \|\partial_x^2 u^{m-1,n-1}(s)\|_\infty ds.
\tag{3.7}
$$

We finally insert the last inequality into equation (3.5) to obtain

$$
\|\partial_{(t,x)}^2 u^{m,n}\|_\infty \le \delta_{m,n} + C \int_0^t \|\partial_{(t,x)}^2 u^{m-1,n-1}(s)\|_\infty ds
$$

and, by iteration for each $l \in \mathbb{N}$

$$
\|\partial_{(t,x)}^2 u^{m,n}\|_\infty \le \delta_{m,n} \; e^C + \frac{BC^l T_0^l}{l!},
$$

on $[0, T_0]$, where B is a bound for $\|\partial_{(t,x)}^2 u^m\|_\infty$. This proves the required Cauchy property for $\|\partial_{(t,x)}^2 u^n\|_\infty$ while the one for $\|\partial_z f^n(s)\|_\infty$ follows from equation (3.7), thereby finishing the proof. $\qquad\square$

It remains to show that the first two terms in (3.6) indeed vanish if $m, n \to \infty$, which amounts to showing that $\partial_z f^{(n)}$ as well as $\partial_{(t,x)}^2 u^{(n)}$ are uniformly continuous, with modulus of continuity uniform in n. One could be tempted to use the mean value theorem but suitable estimates on the second order derivatives of $f^{(n)}$ and the third order derivatives of $u^{(n)}$ are not available. We begin by defining

$$
\delta_n(t, \eta) := \sup \left\{ |\partial_z f^{(n)}(t, z) - \partial_z f^{(n)}(t, z')| \mid z, z' \in \mathbb{R}^6, \; |z - z'| \le \eta \right\},
$$

$$
\theta_n(t, \eta) := \sup \left\{ |\partial_{(t,x)}^2 u^{(n-1)}(t, x) - \partial_{(t,x)}^2 u^{(n-1)}(t, y)| \mid x, y \in \mathbb{R}^3, \; |x - y| \le \eta \right\}.
$$

Note that both δ_n and θ_n are bounded uniformly in n. Our desired result now is

Lemma 3.2. *On any time interval $[0, T_0]$ on which the iterates satisfy the bounds established in Steps 1 and 2 in the proof of Thm. 3.1 the following is true: For all $\varepsilon > 0$ there exist $\eta_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$*

$$
\delta_n(t, \eta_0), \; \theta_n(t, \eta_0) \le \varepsilon, \; t \in [0, T_0].
$$

We defer the rather technical proof of this lemma to Appendix 6. We now establish a continuation criterion for the solutions obtained in Thm. 3.1:

Theorem 3.3 (Continuation criterion)*. Let (f, u) be a solution of the relativistic Vlasov-Klein-Gordon system on $[0, T[$ as in Theorem 3.1. Then the function*

$$P(t) := \sup \left\{ |v| \ | f(\tau, x, v) \neq 0, \ 0 \leq \tau \leq t, \ x \in \mathbb{R}^3 \right\}$$

is bounded on $[0, T[$ iff $\|\partial_x u(t)\|_\infty$ is bounded on $[0, T[$. Moreover, if T is chosen maximally then any of these bounds implies that the solution is global, i.e., $T = \infty$.

Proof. To prove that a bound on $\partial_x u$ implies a bound on P we integrate the v-component of the characteristic system. For the reverse direction we note that we can estimate $\partial_x u$ in terms of P exactly as we did for the iterates in Step 1 of the proof of Thm. 3.1, using Lemma 2.1.

Assume now that T is chosen maximally, that P is bounded on $[0, T[$, and $T < \infty$. For any $t_0 \in [0, T[$ we can use the arguments from the proof of Thm. 3.1 to show that a solution with data $f(t_0)$, $u(t_0)$, $\partial_t u(t_0)$ prescribed at $t = t_0$ exists on some time interval $[t_0, t_0 + \delta[$, except that there is one technical catch here: $u(t_0)$, $\partial_t u(t_0)$ are not sufficiently regular to qualify as initial data in the context of Thm. 3.1. But since we already have the solution on $[0, t_0]$ we can define the iterates used to obtain the extended solution as follows: For (f^0, u^0) we take a global extension of the existing solution with the required regularity and with $\|f^{(0)}(t)\|_\infty$, $\|\partial_z f^{(0)}(t)\|_\infty$, $\|\partial_{(t,x)} u^{(0)}\|_\infty$, $\|\partial_{(t,x)}^2 u^{(0)}\|_\infty$ bounded in t. Given the $(n-1)$st iterate we define the nth iterate exactly as before on $[0, \infty[$. Then all these iterates coincide with the solution on $[0, t_0]$, the data term in the formulas for the field, from which the loss of derivatives arises, is the one determined by the data at $t = 0$, and it is straight forward to repeat the arguments from the proof of Thm. 3.1 to extend the solution to some time interval $[0, t_0 + \delta[$. The crucial point now is that the uniform bound on the momenta implies that $\delta > 0$ can be chosen independently of t_0, cf. (3.1) and the lines that follow. For t_0 close enough to T this contradicts the maximality of T. $\qquad\square$

To conclude this section we briefly discuss conservation laws for the solutions just obtained. Conservation of mass takes the ususal form, i.e.,

$$\partial_t \rho + \operatorname{div} j = 0$$

where the current is defined by $j(t, x) := \int \hat{v} f(t, x, v) dv$. The same holds true for the energy of the system (cf. [11], (1.5)): in local form, we have

$$\partial_t e + \operatorname{div} \eta = 0$$

where the density resp. the flux is defined by $e(t, x) = \int \sqrt{1 + v^2} f dv + \frac{1}{2}((\partial_t u)^2 + (\partial_x u)^2 + u^2) + \rho u$ resp. $\eta(t, x) = \int f v dv + u j - \partial_t u \partial_x u$. On the other hand, conservation of momentum seems not to be available in the expected form.

4. The one-dimensional case

In this section we illustrate that our continuation criterion from Thm. 3.3 holds and hence classical solutions are global in the one dimensional case, where $x, v \in \mathbb{R}$. To do so we first need to derive the representation formulas for u and its derivatives in this situation. The standard trick to do this is to observe that $u(t, x)$ solves the Klein-Gordon equation (1.2) iff $w(t, x, \xi) = u(t, x)e^{-i\xi}$ solves the wave equation

$$(\partial_t^2 - \partial_x^2 - \partial_\xi^2)w = -e^{-i\xi}\rho$$

with initial data transformed accordingly. This leads to

$$u(t,x) = u_{\text{hom}}(t,x) + u_{\text{inh}}(t,x), \quad t \geq 0, \ x \in \mathbb{R},$$

where $u_{\text{hom}}(t,x)$ is the solution of the homogeneous equation and depends only on the initial data for u, and

$$u_{\text{inh}}(t,x) = -\frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} \rho(s,y) \, J_0(\sqrt{(t-s)^2 - |x-y|^2}) \, dy \, ds.$$

Hence

$$
\begin{aligned}
\partial_x u_{\text{inh}}(t,x) = {}& -\frac{1}{2} \int_0^t \left(\rho(s, x+(t-s)) - \rho(s, x-(t-s)) \right) ds \\
& -\frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} \rho(s,y) \frac{J_1(\sqrt{(t-s)^2 - |x-y|^2})}{\sqrt{(t-s)^2 - |x-y|^2}} (x-y) \, dy \, ds.
\end{aligned}
\tag{4.1}
$$

Assume now that we have a (local) solution of the Vlasov-Klein-Gordon system in the one-dimensional case, with $f(t)$ compactly supported for all t. As before,

$$P(t) := \sup \{ |v| \mid f(\tau, x, v) \neq 0, \ 0 \leq \tau \leq t, \ x \in \mathbb{R} \}.$$

Since f is constant along characteristics of the Vlasov equation,

$$\rho(t,x) \leq 2 \|\mathring{f}\|_\infty P(t).$$

Integrating the Vlasov equation w.r.t. x and v implies that $\|f(t)\|_1 = \|\rho(t)\|_1 = \|\mathring{f}\|_1$, and since $J_1(\xi)/\xi$ is bounded on $\xi > 0$, (4.1) implies that

$$\|\partial_x u(t)\|_\infty \leq C(1 + t^2 + tP(t)).$$

Integrating the v-component of the characteristic system implies that

$$P(t) \leq \mathring{P} + C(1+t)^3 + C \int_0^t sP(s) \, ds \tag{4.2}$$

so that by Gronwall's lemma P can not blow up on any bounded time interval.

Indeed, these estimates could be repeated for iterates defined as in the proof of Thm. 3.1. Controlling first order derivatives of $f^{(n)}$ and second order ones of $u^{(n)}$ would be much easier than in the three dimensional case, as should be obvious from comparing Eqn. (4.1) with Lemma 2.1. Hence:

Theorem 4.1. *Let $\mathring{f} \in \mathcal{C}_c^1(\mathbb{R}^2)$, $\mathring{u}_1 \in \mathcal{C}_b^3(\mathbb{R})$, $\mathring{u}_2 \in \mathcal{C}_b^2(\mathbb{R})$. Then there exists a unique classical solution*

$$f \in \mathcal{C}^1([0, \infty[\times\mathbb{R}^2), \quad u \in \mathcal{C}^2([0, \infty[\times\mathbb{R})$$

of the one-dimensional relativistic Vlasov-Klein-Gordon system, satisfying the initial conditions (1.5). If P denotes the solution of the integral equation corresponding to (4.2) then

$$f(t,x,v) = 0 \quad \text{for } |x| \geq \mathring{R} + t \text{ or } |v| \geq P(t)$$

with $\mathring{R}$ determined by $\mathring{f}$.

5. Appendix: Some facts on the Vlasov equation

In this appendix we collect some useful facts on the (inhomogeneous) Vlasov equation for easy reference. As before, we combine x and v to one variable $z = (x, v)$, and we consider the initial value problem

$$\partial_t f(t, z) + G(t, z)\partial_z f(t, z) = g(t, z), \quad f(0, z) = \mathring{f}(z),$$

where $\mathring{f}, G \, g \in \mathcal{C}^1$ and G is such that the solutions of the corresponding characteristic system

$$\dot{z}(s) = G(s, z(s))$$

exist on the time interval on which G is defined. Denote by $s \mapsto Z(s, t, z)$ the solution corresponding to the initial condition $Z(t, t, z) = z$ and recall that

$$Z(s, t, z) = Z(s, r, Z(r, t, z))$$

and

$$\partial_t Z(s, t, z) + \partial_z Z(s, t, z)G(t, z) = 0;$$

the second equation follows from the first by differentiating $z_0 = Z(s, t, Z(t, s, z_0))$ with respect to t and then choosing $z_0 = Z(s, t, z)$. The partial derivative $\partial_z Z(s, t, z)$ satisfies the first variational equation

$$\partial_z \dot{Z}(s, t, z) = (\partial_z G)(s, Z(s, t, z)) \, \partial_z Z(s, t, z),$$

with $\partial_z Z(s, s, z) = \mathbb{I}$ the unit matrix, or equivalently,

$$\partial_z Z(s, t, z) = \mathbb{I} + \int_t^s (\partial_z G)(r, Z(r, t, z))\partial_z Z(r, t, z)dr.$$

In addition, there is also the less obvious equation

$$\partial_z Z(s, t, z) = \mathbb{I} + \int_t^s \partial_z Z(s, r, Z(r, s, z))(\partial_z G)(r, Z(r, t, z)) \, dr, \qquad (5.1)$$

which holds because the right hand side solves the first variational equation. However, note that the integrands in the last two equations are not equal.

We now apply these results to the Vlasov equation. Clearly the solution is given by

$$f(t, z) = \mathring{f}(Z(0, t, z)) + \int_0^t g(s, Z(s, t, z)) \, ds. \qquad (5.2)$$

Moreover, $\partial_z f(t, z)$ exists and satisfies

$$\partial_z f(t, z) = (\partial_z \mathring{f})(Z(0, t, z)) - \int_0^t (\partial_z f)(s, Z(s, t, z)(\partial_z G)(s, Z(s, t, z))ds$$
$$+ \int_0^t (\partial_z g)(s, Z(s, t, z) \, ds. \qquad (5.3)$$

This can be seen by a straightforward calculation using equation (5.1). We should remark that the usual way of deriving this equation is by differentiating the Vlasov equation with respect to z and viewing the resulting equation as an inhomogeneous Vlasov equation for $\partial_z f$. This requires $\mathring{f}, g \in \mathcal{C}^2$ which was not necessary here.

6. Proof of Lemma 3.2

We split the proof into several steps, following the approach used in [10, pp. 78–90] in the case of the Vlasov-Maxwell system. Throughout we argue on a time interval $[0, T_0]$ on which the iterates satisfy all the bounds established in Steps 1 and 2 of the proof of Thm. 3.1.

Lemma 6.1. *(Estimate on θ_n in terms of θ_{n-1}, δ_{n-1})*
There exists $C > 0$ such that $\forall \varepsilon > 0 \ \exists \eta_0 > 0 \ \forall \eta \in [0, \eta_0] \ \forall t \in [0, T_0] \ \forall n \in \mathbb{N}$,

$$\theta_n(t, \eta) \leq \varepsilon + C \left(\eta + \int_0^t \big(\theta_{n-1}(s, \eta) + \delta_{n-1}(s, \eta) \big) ds \right).$$

Proof. Using once more the representation formulas of Lemma 2.2 we proceed as in Steps 1 and 2 of the proof of Theorem 3.1. $\square$

Lemma 6.2. *(Estimate on δ_n in terms of θ_n)*
There exists $C > 0$ such that $\forall \varepsilon > 0 \ \exists \eta_0 > 0 \ \forall \eta \in [0, \eta_0] \ \forall t \in [0, T_0] \ \forall n \in \mathbb{N}$,

$$\delta_n(t, \eta) \leq \varepsilon + C \int_0^t \theta_n(s, \eta) ds.$$

To prove Lemma 6.2 we need some additional information on the derivatives of the characteristics which is provided by the following lemma.

Lemma 6.3 (Estimates on the derivatives of the characteristics). (i) *There exists $C > 0$ such that $\forall s, t \in [0, T_0] \ \forall z, z' \in \mathbb{R}^6 \ \forall n \in \mathbb{N}$*

$$|Z^{(n)}(s, t, z) - Z^{(n)}(s, t, z')| \leq C|z - z'|$$

(ii) *There exists $C > 0$ such that $\forall s, t \in [0, T_0], \ s \leq t, \ \forall z, z' \in \mathbb{R}^6 \ \forall n \in \mathbb{N}$*

$$|\partial_z Z^{(n)}(s, t, z) - \partial_z Z^{(n)}(s, t, z')|$$
$$\leq C|z - z'| + \int_s^t \left| \partial_x^2 u^{(n-1)}(\tau, X^{(n)}(\tau, t, z)) - \partial_x^2 u^{(n-1)}(\tau, X^{(n)}(\tau, t, z')) \right| d\tau.$$

Proof. Part (i) follows immediately from equation (5.1). As to (ii), we only treat the terms involving x-derivatives; the ∂_v-terms may be estimated in the same fashion. So we start with the term $|\partial_{x_i} X^{(n)}(s, t, z) - \partial_{x_i} X^{(n)}(s, t, z')|$, $1 \leq i \leq 3$. By an elementary calculation, cf. [10, pp. 83–85], we obtain

$$|\partial_s \partial_{x_i} X^{(n)}(s, t, z) - \partial_s \partial_{x_i} X^{(n)}(s, t, z')|$$
$$\leq 2|\partial_{x_i} V^{(n)}(s, t, z) - \partial_{x_i} V^{(n)}(s, t, z')| \tag{6.1}$$
$$+ 5|\partial_{x_i} V^{(n)}(s, t, z) - \partial_{x_i} V^{(n)}(s, t, z')| \, |V^{(n)}(s, t, z) - V^{(n)}(s, t, z')|.$$

Using

$$\partial_s \partial_{x_i} V^{(n)}(s, t, z) = -\partial_{x_i} \partial_x u^{(n-1)}(s, X^{(n)}(s, t, z)) \, \partial_{x_i} X^{(n)}(s, t, z)$$

we obtain

$$|\partial_s \partial_{x_i} V^{(n)}(s, t, z) - \partial_s \partial_{x_i} V^{(n)}(s, t, z')|$$
$$\leq |\partial_x^2 u^{(n-1)}(s, X^{(n)}(s, t, z)) - \partial_x^2 u^{(n-1)}(s, X^{(n)}(s, t, z'))| \, |\partial_{x_i} X^{(n)}(s, t, z)| \tag{6.2}$$
$$+ |\partial_x^2 u^{(n-1)}(s, X^{(n)}(s, t, z'))| \, |\partial_{x_i} X^{(n)}(s, t, z) - \partial_{x_i} X^{(n)}(s, t, z')|.$$

Using (i) and equations (6.1) and (6.2), we obtain

$$|\partial_{x_i} Z^{(n)}(s, t, z) - \partial_{x_i} Z^{(n)}(s, t, z')|$$

$$\leq C\Big(|z - z'| + \int_s^t |\partial_{x_i} Z^{(n)}(\tau, t, z) - \partial_{x_i} Z^{(n)}(\tau, t, z')| d\tau$$

$$+ \int_s^t |\partial_x^2 u^{(n-1)}(\tau, X^{(n)}(\tau, t, z)) - \partial_x^2 u^{(n-1)}(\tau, X^{(n)}(\tau, t, z'))| d\tau \Big).$$

Another appeal to Gronwall's lemma completes the proof of the lemma. $\qquad\square$

Proof of Lemma 6.2. We only treat the $\partial_v f1(n)$-terms; the $\partial_x f^{(n)}$-terms can be estimated analogously. Again writing $z = (x, v)$ respectively $z' = (y, w)$ we find

$$\partial_v f^{(n)}(t, z) - \partial_v f^{(n)}(t, z')|$$

$$\leq |\partial_z \overset{\circ}{f}(Z^{(n)}(0, t, z))| \, |\partial_v Z^{(n)}(0, t, z) - \partial_v Z^{(n)}(0, t, z')| \qquad (6.3)$$

$$+ |\partial_v Z^{(n)}(0, t, z')| \, |\partial_z \overset{\circ}{f}(Z^{(n)}(0, t, z)) - \partial_z \overset{\circ}{f}(Z^{(n)}(0, t, z'))|.$$

The second term in the above estimate is bounded by ε for large n, $t \in [0, T_0]$ and $|z - z'|$ suitably small by the fact that $\partial_z Z^{(n)}$ is uniformly (in n) bounded, the uniform continuity of the data $\partial_z \overset{\circ}{f}$ and Lemma 6.3 (i).

By Lemma 6.3 (ii) and the fact that the characteristics are uniformly bounded the first term in equation (6.3) may be estimated by

$$C_1 |z - z'| + C_1 \int_0^t \left| \partial_x^2 u^{(n-1)}(\tau, X^{(n)}(\tau, t, z)) - \partial_x^2 u^{(n-1)}(\tau, X^{(n)}(\tau, t, z')) \right| d\tau.$$

Denote by C_2 the maximum of C_1 and C from Lemma 6.3 (i). Set $C = (\lceil 2C_2 \rceil + 1)C_1$ where $\lceil r \rceil$ denotes the smallest integer bigger or equal to r. We claim that this constant verifies the assertion. Indeed, let $\varepsilon > 0$, and $\eta \in [0, \eta_0]$ with $\eta_0 := \varepsilon/(2C_1 + 1)$. Let $z, z' \in \mathbb{R}^6$ with $|z - z'| \leq \eta$, $n \in \mathbb{N}$ and $\tau, t \in [0, T_0]$. Then by Lemma 6.3 (i)

$$|Z^{(n)}(\tau, t, z) - Z^{(n)}(\tau, t, z')| \leq 2C_2 \eta.$$

Together with the fact that $\theta_n(\tau, k\eta) = k\theta_n(\tau, \eta)$ for all $k \in \mathbb{N}$ this gives

$$|\partial_x^2 u^{(n-1)}(\tau, X^{(n)}(\tau, t, z)) - \partial_x^2 u^{(n-1)}(\tau, X^{(n)}(\tau, t, z'))|$$
$$\leq \theta_n(\tau, 2C_2 \eta) \leq \theta_n(\tau, (1 + \lceil 2C_2 \rceil)\eta)$$
$$\leq (1 + \lceil 2C_2 \rceil)\theta_n(\tau, \eta).$$

Summing up we obtain

$$|\partial_v f^{(n)}(t, z) - \partial_v f^{(n)}(t, z')| \leq \varepsilon + C \int_0^t \theta_n(\tau, \eta) \, d\tau.$$

$\qquad\square$

Proof of Lemma 3.2. Combining Lemma 6.1 and Lemma 6.2 we obtain:

$$\forall \varepsilon > 0 \, \exists \eta_0 > 0 \, \forall \eta \in [0, \eta_0], \ t \in [0, T_0], \ n \in \mathbb{N} : \ \theta_n(t, \eta) \leq C\Big(\varepsilon + \int_0^t \theta_{n-1}(s, \eta)ds\Big),$$

which by iteration shows that

$$\forall \varepsilon > 0 \, \exists \eta_0 > 0 \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0, \ t \in [0, T] : \ \theta_n(t, \eta_0) \leq \varepsilon.$$

The claim concerning δ_n now immediately follows from Lemma 6.2. $\qquad\square$

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