Electronic Journal of Differential Equations, Vol. 2005(2005), No. 08, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

STRONG RESONANCE PROBLEMS FOR THE ONE-DIMENSIONAL p-LAPLACIAN

JIŘÍ BOUCHALA

Abstract. We study the existence of the weak solution of the nonlinear boundary-value problem

$$-(|u'|^{p-2}u')' = \lambda |u|^{p-2}u + g(u) - h(x) \quad \text{in } (0,\pi),$$
$$u(0) = u(\pi) = 0\,,$$

where p and λ are real numbers, $p > 1$, $h \in L^{p'}(0,\pi)$ ($p' = \frac{p}{p-1}$) and the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is a continuous function of the Landesman-Lazer type. Our sufficiency conditions generalize the results published previously about the solvability of this problem.

1. Introduction

We consider the boundary-value problem

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2}u + g(u) - h(x) \quad \text{in } (0,\pi), \\ u(0) = u(\pi) = 0, \end{gathered} \tag{1.1}$$

where $p > 1$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, $h \in L^{p'}(0,\pi)$ ($p' = \frac{p}{p-1}$), $\lambda \in \mathbb{R}$, and $-\Delta_p$ is the (one-dimensional) p-Laplacian, i.e. $\Delta_p u := (|u'|^{p-2}u')'$.

Problem (1.1) can be thought of as a perturbation of the homogeneous eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2}u \quad \text{in } (0,\pi),$$
$$u(0) = u(\pi) = 0\,.$$

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of $-\Delta_p$ if there exists a function $u \in W_0^{1,p}(0,\pi)$, $u \not\equiv 0$, such that

$$\int_0^\pi |u'|^{p-2}u'v'\,\mathrm{d}x = \lambda \int_0^\pi |u|^{p-2}uv\,\mathrm{d}x \quad \forall v \in W_0^{1,p}(0,\pi)\,.$$

2000 *Mathematics Subject Classification.* 34B15, 34L30, 47J30.
Key words and phrases. p-Laplacian; resonance at the eigenvalues;
Landesman-Lazer type conditions.

Submitted June 21, 2004. Published January 5, 2005.
Research supported by the Grant Agency of Czech Republic, 201/03/0671.
These results were presented (without proof) on the conference Modelling 2001 (see [1]).

The function u is then called an eigenfunction of $-\Delta_p$ corresponding to the eigenvalue λ and we write

$$u \in \ker(-\Delta_p - \lambda) \setminus \{0\}.$$

Consider the functional

$$I : W_0^{1,p}(0,\pi) \setminus \{0\} \to \mathbb{R}; \quad I(u) := \frac{\int_0^\pi |u'|^p \,\mathrm{d}x}{\int_0^\pi |u|^p \,\mathrm{d}x}$$

and the manifold

$$\mathcal{S} := \{u \in W_0^{1,p}(0,\pi) : \|u\|_{L^p(0,\pi)} = 1\}.$$

For $k \in \mathbb{N}$ let

$$\mathcal{F}_k := \{\mathcal{A} \subset \mathcal{S} : \text{there exists a continuous odd surjection } h : S^k \to \mathcal{A}\},$$

where S^k represents the unit sphere in $\mathbb{R}^k$. Next define

$$\lambda_k := \inf_{\mathcal{A} \in \mathcal{F}_k} \sup_{u \in \mathcal{A}} I(u). \tag{1.2}$$

It is known that λ_k is an eigenvalue of $-\Delta_p$ (see [3]) and that (λ_k) represents complete set of eigenvalues [4] (For any $k \in \mathbb{N}$, $\lambda_k = \big(\frac{k\pi_p}{\pi}\big)^p$, where $\pi_p := 2(p-1)^{\frac{1}{p}} \int_0^1 \frac{\mathrm{d}s}{(1-s^p)^{\frac{1}{p}}}$). Moreover, for any $k \in \mathbb{N}$ we have $0 < \lambda_k < \lambda_{k+1}$ and any corresponding eigenfunction has "the strong unique continuation property", i.e.

$$\begin{gathered} \forall v \in \ker(-\Delta_p - \lambda_k) \setminus \{0\}, \quad \|v\| = 1 : \\ (\forall \delta > 0)\,(\exists \eta(\delta) > 0) : \operatorname{meas}\{x \in (0,\pi) : |v(x)| \le \eta(\delta)\} < \delta. \end{gathered} \tag{1.3}$$

The symbol $\|\cdot\|$ indicates the norm in the Sobolev space $W_0^{1,p}(0,\pi)$, i.e.

$$\|u\| = \Big(\int_0^\pi |u'|^p \,\mathrm{d}x\Big)^{1/p}.$$

Our paper is motivated by the results in [2] and [3]. The following theorem generalizes those results for the one-dimensional problem (1.1).

Theorem 1.1. *Let us define*

$$F(x) := \begin{cases} \frac{p}{x} \int_0^x g(s)\, ds - g(x), & x \neq 0, \\ (p-1)g(0), & x = 0, \end{cases} \tag{1.4}$$

and set

$$\overline{F(-\infty)} = \limsup_{x \to -\infty} F(x), \quad \underline{F(+\infty)} = \liminf_{x \to +\infty} F(x),$$
$$\overline{F(+\infty)} = \limsup_{x \to +\infty} F(x), \quad \underline{F(-\infty)} = \liminf_{x \to -\infty} F(x).$$

We suppose

$$\lim_{x \to \pm\infty} \frac{g(x)}{|x|^{p-1}} = 0 \tag{1.5}$$

and

$$\begin{gathered} \forall v \in \ker(-\Delta_p - \lambda) \setminus \{0\} : \\ (p-1) \int_0^\pi h(x) v(x)\, dx < \underline{F(+\infty)} \int_0^\pi v^+(x)\, dx + \overline{F(-\infty)} \int_0^\pi v^-(x)\, dx, \end{gathered} \tag{1.6}$$

or

$$\forall v \in \ker(-\Delta_p - \lambda) \setminus \{0\} :$$
$$(p-1)\int_0^\pi h(x)v(x)\,dx > \overline{F(+\infty)}\int_0^\pi v^+(x)\,dx + \underline{F(-\infty)}\int_0^\pi v^-(x)\,dx, \tag{1.7}$$

where

$$v^+ := \max\{0, v\}, \quad v^- := \min\{0, v\}.$$

Then there exists at least one weak solution of the boundary-value problem (1.1)*; i.e. there exists* $u \in W_0^{1,p}(0,\pi)$ *such that for all* $v \in W_0^{1,p}(0,\pi)$*,*

$$\int_0^\pi |u'|^{p-2}u'v'\,dx = \lambda\int_0^\pi |u|^{p-2}uv\,dx + \int_0^\pi g(u)v\,dx - \int_0^\pi hv\,dx.$$

Note that if λ is not an eigenvalue of $-\Delta_p$ then the conditions (1.6) and (1.7) are vacuously true.

2. Preliminaries

Let

$$J_\lambda(u) := \frac{1}{p}\int_0^\pi |u'|^p\,\mathrm{d}x - \frac{\lambda}{p}\int_0^\pi |u|^p\,\mathrm{d}x - \int_0^\pi G(u)\,\mathrm{d}x + \int_0^\pi hu\,\mathrm{d}x, \tag{2.1}$$

where

$$G(t) := \int_0^t g(s)\,\mathrm{d}s.$$

It is well known that $J_\lambda \in C^1(W_0^{1,p}(0,\pi),\mathbb{R})$, and that for all $v \in W_0^{1,p}(0,\pi)$,

$$\langle J'_\lambda(u), v\rangle = \int_0^\pi |u'|^{p-2}u'v'\,\mathrm{d}x - \lambda\int_0^\pi |u|^{p-2}uv\,\mathrm{d}x - \int_0^\pi g(u)v\,\mathrm{d}x + \int_0^\pi hv\,\mathrm{d}x.$$

It follows that weak solutions of (1.1) correspond to critical points of J_λ.

The next theorem plays a fundamental role in proving that J_λ has critical points of saddle point type (see [3, 5]).

Lemma 2.1 (Deformation Lemma)**.** *Suppose that* J_λ *satisfies the Palais-Smale condition, i.e. if* (u_n) *is a sequence of functions in* $W_0^{1,p}(0,\pi)$ *such that* $(J_\lambda(u_n))$ *is bounded in* $\mathbb{R}$ *and* $J'_\lambda(u_n) \to 0$ *in* $(W_0^{1,p}(0,\pi))^*$*, then* (u_n) *has a subsequence that is strongly convergent in* $W_0^{1,p}(0,\pi)$*. Let* $c \in \mathbb{R}$ *be a regular value of* J_λ *and let* $\bar\varepsilon > 0$*. Then there exists* $\varepsilon \in (0,\bar\varepsilon)$ *and a continuous one-parameter family of homeomorphisms,* $\phi : W_0^{1,p}(0,\pi) \times \langle 0,1\rangle \to W_0^{1,p}(0,\pi)$*, with the properties:*

(i) *if* $t = 0$ *or* $|J_\lambda(u) - c| \ge \bar\varepsilon$*, then* $\phi(u,t) = u$*,*
(ii) *if* $J_\lambda(u) \le c + \varepsilon$*, then* $J_\lambda(\phi(u,1)) \le c - \varepsilon$*.*

3. Proof of main Theorem

The proof is divided into four lemmas. First we prove that functional J_λ satisfies the Palais-Smale condition, and in the next steps we prove our theorem separately for situations: $\lambda < \lambda_1$, $\lambda_k < \lambda < \lambda_{k+1}$ and $\lambda = \lambda_k$.

Lemma 3.1. *Let us assume* (1.5) *and (*(1.6) *or* (1.7)*). Then the functional* J_λ *satisfies the Palais-Smale condition.*

Proof. We will start with the proof that any Palais-Smale sequence is bounded in $W_0^{1,p}(0,\pi)$. Suppose, by contradiction, that (u_n) is a sequence of functions in $W_0^{1,p}(0,\pi)$ such that

$$(J_\lambda(u_n)) \text{ is bounded in } \mathbb{R}, \tag{3.1}$$

$$J'_\lambda(u_n) \to 0 \text{ in } \left(W_0^{1,p}(0,\pi)\right)^*, \tag{3.2}$$

$$\|u_n\| \to +\infty. \tag{3.3}$$

Due to the reflexivity of $W_0^{1,p}(0,\pi)$ and the compact embeding

$$W_0^{1,p}(0,\pi) \hookrightarrow\hookrightarrow C^0(<0,\pi>),$$

there exists $v \in W_0^{1,p}(0,\pi)$ such that (up to subsequences)

$$v_n := \frac{u_n}{\|u_n\|} \rightharpoonup v \quad \text{(i.e., weakly) in } W_0^{1,p}(0,\pi), \tag{3.4}$$

$$v_n \to v \quad \text{(i.e., strongly) in } C^0\big(\langle 0,\pi\rangle\big). \tag{3.5}$$

From (3.2), (3.3) and (3.4), we have

$$\begin{aligned} &\frac{\langle J'_\lambda(u_n), v_n - v\rangle}{\|u_n\|^{p-1}} \\ &= \int_0^\pi |v_n'|^{p-2} v_n' (v_n - v)' \, \mathrm{d}x - \lambda \int_0^\pi |v_n|^{p-2} v_n (v_n - v) \, \mathrm{d}x \\ &\quad - \int_0^\pi \frac{g(u_n)}{\|u_n\|^{p-1}} (v_n - v) \, \mathrm{d}x + \int_0^\pi \frac{h}{\|u_n\|^{p-1}} (v_n - v) \, \mathrm{d}x \to 0, \end{aligned} \tag{3.6}$$

and since the last three terms approach 0 (here we need the assumption (1.5)), we have

$$\int_0^\pi |v_n'|^{p-2} v_n' (v_n - v)' \, \mathrm{d}x \to 0.$$

It follows from here, (3.4) and from the Hölder inequality that

$$\begin{aligned} 0 &\leftarrow \int_0^\pi |v_n'|^{p-2} v_n' (v_n - v)' \, \mathrm{d}x - \int_0^\pi |v'|^{p-2} v' (v_n - v)' \, \mathrm{d}x \\ &= \int_0^\pi |v_n'|^p \, \mathrm{d}x - \int_0^\pi |v_n'|^{p-2} v_n' v' \, \mathrm{d}x - \int_0^\pi |v'|^{p-2} v' v_n' \, \mathrm{d}x + \int_0^\pi |v'|^p \, \mathrm{d}x \\ &\geq \|v_n\|^p - \|v_n\|^{p-1}\|v\| - \|v\|^{p-1}\|v_n\| + \|v\|^p \\ &= (\|v_n\|^{p-1} - \|v\|^{p-1})(\|v_n\| - \|v\|) \geq 0 \end{aligned} \tag{3.7}$$

which implies

$$\|v_n\| \to \|v\|. \tag{3.8}$$

The uniform convexity of $W_0^{1,p}(0,\pi)$ then yields

$$v_n \to v \text{ in } W_0^{1,p}(0,\pi), \quad \|v\| = 1. \tag{3.9}$$

It follows from (3.2) and (3.3) that, for any $w \in W_0^{1,p}(0,\pi)$,

$$\begin{aligned} \frac{\langle J'_\lambda(u_n), w\rangle}{\|u_n\|^{p-1}} &= \int_0^\pi |v_n'|^{p-2} v_n' w' \, \mathrm{d}x - \lambda \int_0^\pi |v_n|^{p-2} v_n w \, \mathrm{d}x \\ &\quad - \int_0^\pi \frac{g(u_n)}{\|u_n\|^{p-1}} w \, \mathrm{d}x + \int_0^\pi \frac{h}{\|u_n\|^{p-1}} w \, \mathrm{d}x \to 0. \end{aligned}$$

Now the last two terms approach zero. Hence for all $w \in W_0^{1,p}(0,\pi)$:

$$\int_0^\pi |v_n'|^{p-2}v_n'w'\,\mathrm{d}x - \lambda\int_0^\pi |v_n|^{p-2}v_n w\,\mathrm{d}x \to 0. \tag{3.10}$$

It is known [3] that the maps $A,\ B : W_0^{1,p}(0,\pi) \to \big(W_0^{1,p}(0,\pi)\big)^*$;

$$\langle Au, w\rangle := \int_0^\pi |u'|^{p-2}u'w'\,\mathrm{d}x, \quad \langle Bu, w\rangle := \int_0^\pi |u|^{p-2}uw\,\mathrm{d}x$$

are continuous, and therefore from (3.9) and (3.10) we have

$$\int_0^\pi |v'|^{p-2}v'w'\,\mathrm{d}x = \lambda\int_0^\pi |v|^{p-2}vw\,\mathrm{d}x, \quad \forall w \in W_0^{1,p}(0,\pi)$$

and

$$v \in \ker(-\Delta_p - \lambda)\setminus\{0\}, \quad \|v\| = 1.$$

The boundedness of $(J_\lambda(u_n))$, $J_\lambda'(u_n) \to 0$, and $\|u_n\| \to \infty$ imply

$$\begin{aligned}\frac{\langle J_\lambda'(u_n), u_n\rangle - pJ_\lambda(u_n)}{\|u_n\|} &= \int_0^\pi \frac{pG(u_n) - g(u_n)u_n}{\|u_n\|}\,\mathrm{d}x - (p-1)\int_0^\pi h\frac{u_n}{\|u_n\|}\,\mathrm{d}x\\ &= \int_0^\pi F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x - (p-1)\int_0^\pi h\frac{u_n}{\|u_n\|}\,\mathrm{d}x \to 0.\end{aligned}$$

Hence

$$\lim\int_0^\pi F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x = (p-1)\int_0^\pi hv\,\mathrm{d}x. \tag{3.11}$$

Now we assume (1.6) (the other case (1.7) is treated similarly). It follows

$$\underline{F(+\infty)} > -\infty \quad\text{and}\quad \overline{F(-\infty)} < +\infty.$$

For arbitrary $\varepsilon > 0$ set

$$c_\varepsilon := \begin{cases}\underline{F(+\infty)} - \varepsilon & \text{if } \underline{F(+\infty)} \in \mathbb{R},\\ \frac{1}{\varepsilon} \text{ if } \underline{F(+\infty)} = +\infty;\end{cases}$$

$$d_\varepsilon := \begin{cases}\overline{F(-\infty)} + \varepsilon & \text{if } \overline{F(-\infty)} \in \mathbb{R},\\ -\frac{1}{\varepsilon} & \text{if } \overline{F(-\infty)} = -\infty.\end{cases}$$

Then for any $\varepsilon > 0$ there exists $K > 0$ such that

$$F(t) \ge c_\varepsilon \ \text{ for any } \ t > K, \quad F(t) \le d_\varepsilon \ \text{ for any } \ t < -K. \tag{3.12}$$

On the other hand, the continuity of F on $\mathbb{R}$ implies that for any $K > 0$ there exists $c(K) > 0$ such that

$$|F(t)| \le c(K) \quad \text{for any } t \in \langle -K, K\rangle. \tag{3.13}$$

Let us choose $\varepsilon > 0$ and consider the corresponding $K > 0$ and $c(K) > 0$ given by (3.12) and (3.13), respectively. Set

$$\int_0^\pi F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x = A_{K,n} + B_{K,n} + C_{K,n} + D_{K,n} + E_{K,n}, \tag{3.14}$$

where

$$A_{K,n} = \int_{\substack{\{x\in(0,\pi):\\ |u_n(x)|\le K\}}} F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x, \quad B_{K,n} = \int_{\substack{\{x\in(0,\pi):\\ u_n(x)>K,\\ v(x)>0\}}} F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x,$$

$$C_{K,n} = \int_{\substack{\{x\in(0,\pi):\\ u_n(x)>K,\\ v(x)\le 0\}}} F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x, \quad D_{K,n} = \int_{\substack{\{x\in(0,\pi):\\ u_n(x)<-K,\\ v(x)<0\}}} F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x,$$

$$E_{K,n} = \int_{\substack{\{x\in(0,\pi):\\ u_n(x)<-K,\\ v(x)\ge 0\}}} F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x\,.$$

Before estimating these integrals we claim that for any $K > 0$ the following assertions are true:

$$\begin{gathered}
\lim_{n\to\infty} \operatorname{meas}\{x\in(0,\pi):\ u_n(x) > K \text{ and } v(x)\le 0\} = 0,\\
\lim_{n\to\infty} \operatorname{meas}\{x\in(0,\pi):\ u_n(x) < -K \text{ and } v(x)\ge 0\} = 0,\\
\lim_{n\to\infty} \operatorname{meas}\{x\in(0,\pi):\ u_n(x) \le K \text{ and } v(x)> 0\} = 0,\\
\lim_{n\to\infty} \operatorname{meas}\{x\in(0,\pi): u_n(x) \ge -K \text{ and } v(x)< 0\} = 0
\end{gathered}$$

cf. (1.3) and (3.5). We are now ready to estimate the integrals from (3.14).

$$\begin{aligned}
&|A_{K,n}| \le \frac{c(K)K\pi}{\|u_n\|} \to 0,\\
&B_{K,n} \ge c_\varepsilon\Big(\int_{\substack{\{x\in(0,\pi):\\ v(x)>0\}}} v_n\,\mathrm{d}x - \int_{\substack{\{x\in(0,\pi):\\ u_n(x)\le K,\\ v(x)>0\}}} v_n\,\mathrm{d}x\Big) \to c_\varepsilon \int_{\substack{\{x\in(0,\pi):\\ v(x)>0\}}} v(x)\,\mathrm{d}x,\\
&C_{K,n} \ge c_\varepsilon \int_{\substack{\{x\in(0,\pi):\\ u_n(x)>K,\\ v(x)\le 0\}}} v_n\,\mathrm{d}x \to 0,\\
&D_{K,n} \ge d_\varepsilon\Big(\int_{\substack{\{x\in(0,\pi):\\ v(x)<0\}}} v_n\,\mathrm{d}x - \int_{\substack{\{x\in(0,\pi):\\ u_n(x)\ge -K,\\ v(x)<0\}}} v_n\,\mathrm{d}x\Big) \to d_\varepsilon \int_{\substack{\{x\in(0,\pi):\\ v(x)<0\}}} v(x)\,\mathrm{d}x,\\
&E_{K,n} \ge d_\varepsilon \int_{\substack{\{x\in(0,\pi):\\ u_n(x)<-K,\\ v(x)\ge 0\}}} v_n\,\mathrm{d}x \to 0.
\end{aligned}$$

Hence (see (3.14)), for any $\varepsilon > 0$,

$$\begin{aligned}
\liminf \int_0^\pi F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x &= \liminf\ (A_{K,n} + B_{K,n} + C_{K,n} + D_{K,n} + E_{K,n})\\
&\ge c_\varepsilon \int_{\substack{\{x\in(0,\pi):\\ v(x)>0\}}} v(x)\,\mathrm{d}x + d_\varepsilon \int_{\substack{\{x\in(0,\pi):\\ v(x)<0\}}} v(x)\,\mathrm{d}x,
\end{aligned}$$

which together with (3.11) implies

$$(p-1)\int_0^\pi h(x)v(x)\,\mathrm{d}x \ge \underline{F(+\infty)}\int_0^\pi v^+(x)\,\mathrm{d}x + \overline{F(-\infty)}\int_0^\pi v^-(x)\,\mathrm{d}x,$$

contradicting (1.6). This proves that (u_n) is bounded.

The rest of the proof is very easy. If the sequence (u_n), which is bounded in $W_0^{1,p}(0,\pi)$, satisfies conditions (3.1) and (3.2), then there exists $u \in W_0^{1,p}(0,\pi)$

such that (passing to subsequences)

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(0,\pi), \quad u_n \to u \text{ in } C^0(\langle 0,\pi\rangle).$$

It follows from here, (3.2) and (1.5) that

$$\begin{aligned}\lim\langle J'_\lambda(u_n), u_n - u\rangle &= \lim \int_0^\pi |u'_n|^{p-2}u'_n(u_n-u)'\,\mathrm{d}x - \lambda\int_0^\pi |u_n|^{p-2}u_n(u_n-u)\,\mathrm{d}x \\ &\quad - \int_0^\pi g(u_n)(u_n-u)\,\mathrm{d}x + \int_0^\pi h(u_n-u)\,\mathrm{d}x \\ &= \lim \int_0^\pi |u'_n|^{p-2}u'_n(u_n-u)'\,\mathrm{d}x = 0\end{aligned}$$

which implies $\|u_n\| \to \|u\|$ (cf. (3.7)). The uniform convexity of $W_0^{1,p}(0,\pi)$ then yields $u_n \to u$ in $W_0^{1,p}(0,\pi)$. The proof is complete. □

Lemma 3.2. *Let us assume* (1.5) *and let* $\lambda < \lambda_1$. *Then there exists at least one weak solution of* (1.1).

Proof. Assumption (1.5) and the variational characterization of λ_1 yield: For all $u \in W_0^{1,p}(0,\pi)$ and all $\varepsilon > 0$ there exists $c > 0$ such that

$$\begin{aligned}J_\lambda(u) &= \frac{1}{p}\int_0^\pi |u'|^p\,\mathrm{d}x - \frac{\lambda_1}{p}\int_0^\pi |u|^p\,\mathrm{d}x + \frac{\lambda_1-\lambda}{p}\int_0^\pi |u|^p\,\mathrm{d}x \\ &\quad - \int_0^\pi G(u)\,\mathrm{d}x + \int_0^\pi hu\,\mathrm{d}x \\ &\geq \frac{\lambda_1-\lambda}{p}\int_0^\pi |u|^p\,\mathrm{d}x - c\int_0^\pi |u|\,\mathrm{d}x - \frac{\varepsilon}{p}\int_0^\pi |u|^p\,\mathrm{d}x - \int_0^\pi |hu|\,\mathrm{d}x \\ &\geq \frac{\lambda_1-\lambda-\varepsilon}{p}\|u\|^p_{L^p(0,\pi)} - c\|u\|_{L^1(0,\pi)} - \|h\|_{L^{p'}(0,\pi)}\|u\|_{L^p(0,\pi)}.\end{aligned}$$

Hence the functional J_λ is bounded from bellow on $W_0^{1,p}(0,\pi)$. It follows from this and from Lemma 3.1 that J_λ attains its global minimum on $W_0^{1,p}(0,\pi)$ [6, Corollary 2.5]. □

Lemma 3.3. *Let us assume* (1.5) *and (*(1.6) *or* (1.7)*). Let there exists* $k \in \mathbb{N}$ *such that* $\lambda_k < \lambda < \lambda_{k+1}$. *Then there exists at least one weak solution of* (1.1).

Proof. Let $m \in (\lambda_k, \lambda)$ and let $\mathcal{A} \in \mathcal{F}_k$ be such that

$$\sup_{u\in\mathcal{A}} I(u) \leq m$$

(see Section 1 for $\mathcal{F}_k$). Then (we again need (1.5)): For all $u \in \mathcal{A}$, all $t > 0$ and all $\varepsilon > 0$, there exists $c > 0$ such that

$$\begin{aligned}J_\lambda(tu) &= \frac{1}{p}t^p\Big(\int_0^\pi |u'|^p\,\mathrm{d}x - \lambda\int_0^\pi |u|^p\,\mathrm{d}x\Big) - \int_0^\pi G(tu)\,\mathrm{d}x + t\int_0^\pi hu\,\mathrm{d}x \\ &\leq \frac{1}{p}t^p(m-\lambda)\|u\|^p_{L^p(0,\pi)} + ct\|u\|_{L^1(0,\pi)} \\ &\quad + \frac{\varepsilon}{p}t^p\|u\|^p_{L^p(0,\pi)} + t\|h\|_{L^{p'}(0,\pi)}\|u\|_{L^p(0,\pi)} \\ &= \frac{1}{p}t^p(m-\lambda+\varepsilon)\|u\|^p_{L^p(0,\pi)} + t\big(c\|u\|_{L^1(0,\pi)} + \|h\|_{L^{p'}(0,\pi)}\|u\|_{L^p(0,\pi)}\big),\end{aligned}$$

and

$$\lim_{t\to+\infty} J_\lambda(tu) = -\infty \quad \forall u \in \mathcal{A}\,. \tag{3.15}$$

Now we continue similarly as in [3]. Let

$$\mathcal{E}_{k+1} := \{u \in W_0^{1,p}(0,\pi) : \int_0^\pi |u'|^p \,\mathrm{d}x \ge \lambda_{k+1} \int_0^\pi |u|^p \,\mathrm{d}x\},$$

and notice that for all $u \in \mathcal{E}_{k+1}$, all $\varepsilon > 0$ there exists $c > 0$ such that

$$J_\lambda(u) \ge \frac{1}{p}\,(\lambda_{k+1} - \lambda - \varepsilon)\,\|u\|^p_{L^p(0,\pi)} - c\|u\|_{L^1(0,\pi)} - \|h\|_{L^{p'}(0,\pi)}\|u\|_{L^p(0,\pi)}.$$

Hence

$$\alpha := \inf\{J_\lambda(u) : u \in \mathcal{E}_{k+1}\} \in \mathbb{R}. \tag{3.16}$$

From (3.15) and (3.16) we see that there exists $T > 0$ such that

$$\gamma := \max\{J_\lambda(tu) : u \in \mathcal{A} \;\text{ and }\; t \in \langle T, +\infty)\} < \alpha.$$

The rest of the proof can be copied from [3]. If we define

$$T\mathcal{A} := \{tu \in W_0^{1,p}(0,\pi) : u \in \mathcal{A} \text{ and } t \in \langle T, +\infty)\},$$
$$\Gamma := \{h \in C^0(B_k, W_0^{1,p}(0,\pi)) : h|_{S^k} \text{ is an odd map into } T\mathcal{A}\},$$

where

$$B_k := \{x = (x_1, \dots, x_k) \in \mathbb{R}^k : \|x\|_{\mathbb{R}^k} = \sqrt{x_1^2 + \dots + x_k^2} \le 1\},$$

then we can prove that Γ is nonempty and if $h \in \Gamma$ then $h(B_k) \cap \mathcal{E}_{k+1} \ne \emptyset$.

Moreover, from Deformation Lemma then follows that

$$c := \inf_{h\in\Gamma} \sup_{x\in B_k} J_\lambda(h(x))$$

is a critical value of J_λ. Indeed, assume by contradiction, that c is a regular value of J_λ. It is clear that $c \ge \alpha$. Now we consider arbitrary $\bar\varepsilon > 0$ such that $\bar\varepsilon < c - \gamma$ and we apply Deformation Lemma. We get a deformation ϕ and a corresponding $\varepsilon > 0$. By definition of c there is an $h \in \Gamma$ such that

$$\sup_{x\in B_k} J_\lambda(h(x)) < c + \varepsilon.$$

Now when

$$\tilde h(x) := \phi(h(x), 1),$$

we obtain

$$\tilde h \in \Gamma,\ \forall x \in B_k : J_\lambda(\tilde h(x)) = J_\lambda(\phi(h(x), 1)) \le c - \varepsilon,$$

it is a contradiction to the definition of c. □

Lemma 3.4. *Let us assume* (1.5) *and (*(1.6) *or* (1.7)*). Let there exists $k \in \mathbb{N}$ such that $\lambda = \lambda_k$. Then there exists at least one weak solution of* (1.1)*.*

Proof. At first we assume (1.6). Let (μ_n) be a sequence in $(\lambda_k, \lambda_{k+1})$ such that $\mu_n \searrow \lambda_k$. Now thanks to Lemma 3.3, we have: For all $n \in \mathbb{N}$ there exists $c_n \in \mathbb{R}$ and $u_n \in W_0^{1,p}(0,\pi)$ such that

$$J'_{\mu_n}(u_n) = 0 \;\text{ and }\; J_{\mu_n}(u_n) = c_n \ge \alpha_n := \inf\{J_{\mu_n}(u) : u \in \mathcal{E}_{k+1}\}.$$

It follows from (1.5) and from the monotonousness of (μ_n) that for all $n \in \mathbb{N}$, all $u \in \mathcal{E}_{k+1}$ and all $\varepsilon > 0$, there exists $c > 0$ such that

$$\begin{aligned}
&J_{\mu_n}(u)\\
&\geq \frac{1}{p}(\lambda_{k+1} - \mu_n)\|u\|^p_{L^p(0,\pi)} - c\|u\|_{L^1(0,\pi)} - \frac{\varepsilon}{p}\|u\|^p_{L^p(0,\pi)} - \|h\|_{L^{p'}(0,\pi)}\|u\|_{L^p(0,\pi)}\\
&\geq \frac{1}{p}(\lambda_{k+1} - \mu_1 - \varepsilon)\|u\|^p_{L^p(0,\pi)} - c\|u\|_{L^1(0,\pi)} - \|h\|_{L^{p'}(0,\pi)}\|u\|_{L^p(0,\pi)},
\end{aligned}$$

and so the sequence (c_n) is bounded below.

Now we prove that the corresponding sequence of critical points, (u_n), is bounded. Suppose, by contradiction, that $\|u_n\| \to +\infty$. Then we can assume that there exists

$$v \in \ker(-\Delta_p - \lambda_k) \setminus \{0\} \tag{3.17}$$

such that (up to subsequences)

$$\frac{u_n}{\|u_n\|} \to v \quad \text{in } W^{1,p}_0(0,\pi). \tag{3.18}$$

Because (c_n) is bounded from below, it follows from (1.6), (3.17) and (3.18) that

$$\begin{aligned}
0 &\leq \liminf \frac{pc_n}{\|u_n\|}\\
&\leq \limsup \frac{pc_n}{\|u_n\|}\\
&= \limsup \frac{pJ_{\mu_n}(u_n) - \langle J'_{\mu_n}(u_n), u_n\rangle}{\|u_n\|}\\
&= \limsup \Big(- \frac{p\int_0^\pi G(u_n)\,\mathrm{d}x - \int_0^\pi g(u_n)u_n\,\mathrm{d}x}{\|u_n\|} + (p-1)\int_0^\pi h\frac{u_n}{\|u_n\|}\,\mathrm{d}x\Big)\\
&= -\liminf \Big(\frac{p\int_0^\pi G(u_n)\,\mathrm{d}x - \int_0^\pi g(u_n)u_n\,\mathrm{d}x}{\|u_n\|}\Big) + (p-1)\int_0^\pi hv\,\mathrm{d}x\\
&= -\liminf \Big(\int_0^\pi F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x\Big) + (p-1)\int_0^\pi hv\,\mathrm{d}x < 0.
\end{aligned}$$

This is a contradiction, therefore (u_n) is bounded. Thus there will be a subsequence of critical points that converges to the desired solution.

Now we assume (1.7). Because for $\lambda = \lambda_1$ our assertion was proved in [2], we focus on $k > 1$. Let (μ_n) be a sequence in $(\lambda_{k-1}, \lambda_k)$ such that $\mu_n \nearrow \lambda_k$. We can find (similarly as in [3]) a sequence (u_n) of critical points associated with the functionals J_{μ_n} such that the sequence $c_n := J_{\mu_n}(u_n)$ is decreasing, i.e.

$$J'_{\mu_n}(u_n) = 0, \quad J_{\mu_n}(u_n) = c_n \geq c_{n+1}.$$

Now we are going to prove that (u_n) is bounded. Suppose, by contradiction, $\|u_n\| \to \infty$. Then there exists $v \in \ker(-\Delta_p - \lambda_k) \setminus \{0\}$ such that (up to subsequence)

$\frac{u_n}{\|u_n\|} \to v$ and

$$
\begin{aligned}
0 &\ge \limsup \frac{pc_n}{\|u_n\|} \ge \liminf \frac{pc_n}{\|u_n\|} \\
&= \liminf \frac{pJ_{\mu_n}(u_n) - \langle J'_{\mu_n}(u_n), u_n\rangle}{\|u_n\|} \\
&= \liminf \Big(-\frac{p\int_0^\pi G(u_n)\,\mathrm{d}x - \int_0^\pi g(u_n)u_n\,\mathrm{d}x}{\|u_n\|} + (p-1)\int_0^\pi h\frac{u_n}{\|u_n\|}\,\mathrm{d}x\Big) \\
&= -\limsup \Big(\frac{p\int_0^\pi G(u_n)\,\mathrm{d}x - \int_0^\pi g(u_n)u_n\,\mathrm{d}x}{\|u_n\|}\Big) + (p-1)\int_0^\pi hv\,\mathrm{d}x \\
&= -\limsup \Big(\int_0^\pi F(u_n)\frac{u_n}{\|u_n\|}\,\mathrm{d}x\Big) + (p-1)\int_0^\pi hv\,\mathrm{d}x > 0,
\end{aligned}
$$

which is a contradiction. Now it is a simple matter to show that, by passing to a subsequence, we obtain a critical point of J_{λ_k} in the limit. □

Acknowledgements. The author is indebted to Professor Pavel Drábek for his valuable comments.

References

[1] J. Bouchala: *Resonance problems for p-Laplacian*, Mathematics and Computers in Simulation 61 (2003), 599–604.

[2] J. Bouchala and P. Drábek: *Strong resonance for some quasilinear elliptic equations*, Journal of Mathematical Analysis and Applications 245 (2000), 7–19.

[3] P. Drábek and S. B. Robinson: *Resonance Problems for the p-Laplacian*, Journal of Functional Analysis 169 (1999), 189–200.

[4] M. Cuesta: *On the Fučík Spectrum of the Laplacian and the p-Laplacian*, Proceedings of Seminar in Differential Equations, University of West Bohemia in Pilsen (2000), 67–96.

[5] M. Struwe: *Variational methods*, A series of modern surveys in mathematics 34, Springer, Berlin (2000).

[6] M. Willem: *Minimax theorems*, Birkhäuser, Boston (1996).

Jiří Bouchala

Department of Applied Mathematics VŠB-Technical University Ostrava, Czech republic

E-mail address: `jiri.bouchala@vsb.cz`