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NONLINEAR NEUMANN PROBLEMS ON BOUNDED LIPSCHITZ DOMAINS

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Abstract. We prove existence and uniqueness, up to a constant, of an entropy solution to the nonlinear and non homogeneous Neumann problem

$$\begin{gathered} -\operatorname{div}[\mathbf{a}(.,\nabla u)]+\beta(u)=f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu_{\mathbf{a}}}+\gamma(\tau u)=g \quad \text{on } \partial\Omega\,. \end{gathered}$$

Our approach is based essentially on the theory of m-accretive operators in Banach spaces, and in order preserving properties.

1. Introduction

Let Ω be a connected open bounded set in $\mathbb{R}^N$, $N\geq 3$, with a connected Lipschitz boundary $\partial\Omega$. Let $(f,g)\in L^1(\Omega)\times L^1(\partial\Omega)$ satisfy the condition $\int_\Omega f(y)dy+\int_{\partial\Omega} g(y)d\sigma(y)=0$ which is necessary and sufficient for solving classical Neumann problem in smooth bounded domains [10]. Let $\mathbf{a}(x,\xi)$ be an operator of Leray-Lions type, in the sense that $(x,\xi)\mapsto \mathbf{a}(x,\xi)$ is a Carathéodory function from $\Omega\times\mathbb{R}^N$ to $\mathbb{R}^N$, $\langle \mathbf{a}(x,\xi_1)-\mathbf{a}(x,\xi_2),\xi_1-\xi_2\rangle>0$, if $\xi_1\neq\xi_2$ and there exist some constants $p>1$, C_1, $C_2>0$ and a function $h_0\in L^{p'}(\Omega)$, $p'=\frac{p}{p-1}$, such that $\langle \mathbf{a}(x,\xi),\xi\rangle\geq C_1|\xi|^p$ and $|\mathbf{a}(x,\xi)|\leq C_2(h_0(x)+|\xi|^{p-1})$, for a.e. $x\in\Omega$ and all $\xi\in\mathbb{R}^N$, (see [12]). We discuss existence and uniqueness of a solution u for the nonlinear and non homogeneous Neumann problem

$$\begin{gathered} -\operatorname{div}[\mathbf{a}(.,\nabla u)]+\beta(u)=f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu_{\mathbf{a}}}+\gamma(\tau u)=g \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

The trace τu on $\partial\Omega$ is taken in the sense of [1]. The gradient ∇u is defined by mean of truncating, in the sense of [3], $\nabla u=DT_k u$ on every set $\{|u|\leq k\}$, $k>0$, where $T_k(r)=\max\{-k,\min(k,r)\}$, $r\in\mathbb{R}$. The normal derivative $\frac{\partial u}{\partial\nu_{\mathbf{a}}}$ related to the operator $\mathbf{a}$ may be interpreted as the trace of the inner product in $\mathbb{R}^N$ $\langle \mathbf{a}(.,\nabla u),\nu\rangle$, where ν is the outward normal derivative vector field. More precisely $\langle \mathbf{a}(.,DT_k u),\nu\rangle$ represents a.e. on $\partial\Omega$ an element of the dual space of $W^{1-\frac{1}{p},p}(\partial\Omega)\cap L^\infty(\partial\Omega)$ (see [9]), but this interpretation is not essential to our approach, since it does not appear

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explicitly in the definitions, given later, for weak solutions as well as for entropy solutions.

For a sake of simplicity, β, γ are taken as continuous non decreasing real functions everywhere defined on $\mathbb{R}$, with $\beta(0) = \gamma(0) = 0$. We may extend our approach to the case where β, γ are maximal monotone graphs in $\mathbb{R}^2$ with some compatibility conditions on their domains, as given in [17].

We prove existence and uniqueness up to a constant, of an entropy solution u to the problem (1.1), in the sense that $u : \Omega \to \mathbb{R}$ is measurable, $DT_k u \in L^p(\Omega)$, for every $k > 0$, $\beta(u) \in L^1(\Omega)$, $\gamma(\tau u) \in L^1(\partial\Omega)$, and for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, u satisfies

$$\int_\Omega \langle \mathbf{a}(., \nabla u), DT_k(u-\varphi)\rangle \le \int_\Omega (f-\beta(u))T_k(u-\varphi)+\int_{\partial\Omega}(g-\gamma(\tau u))T_k(\tau u-\varphi). \quad (1.2)$$

We cannot expect better result for uniqueness, since in the particular case where $\beta = \gamma = 0$, if u is a solution, then it is so for $u + c$, where c is an arbitrary real constant.

Later on, for uniqueness, we will take in (1.2) the test function φ in a class larger than $\mathcal{C}_0^\infty(\mathbb{R}^N)$.

To the best of our knowledge, the Non homogeneous case $g \ne 0$, with a double nonlinearity $\beta(u)$ and $\gamma(\tau u)$, even in the Quasilinear case where $\operatorname{div}[\mathbf{a}(x, \nabla u)] = \Delta u$, is new. The homogeneous case $g = 0$ has been discussed by many authors. See e.g. [2], [8]. For the nonlinear problem, with particular β and γ, we refer the reader to [1] for the case $\beta(u) = u$, and to [15] for $\beta = 0$ and $\gamma(\tau u) = \lambda\tau u$. In all these approaches, the boundary condition is a part of the definition of the operator's domain. This is no longer possible in our situation. For this reason, to investigate the non homogeneous quasilinear Neumann problem in a half-space, we used in [17] a matrix operator A on a product space. This had been extended in [18] to the problem,

$$\begin{gathered} -\operatorname{div}[\mathbf{a}(., \nabla u)] + \beta(u) = f \quad \text{in } \mathbb{R}^N \setminus \partial\Omega \\ [\frac{\partial u}{\partial \nu_{\mathbf{a}}}] + \gamma(\tau u) = g \quad \text{on } \partial\Omega \\ [u] = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \quad (1.3)$$

Where Ω is given as previously, $[\frac{\partial u}{\partial \nu_{\mathbf{a}}}]$ and $[u]$ are respectively the jump of the normal derivative $\frac{\partial u}{\partial \nu_{\mathbf{a}}}$ and of the trace τu across $\partial\Omega$.

In the present, $X_1 = L^1(\Omega) \times L^1(\partial\Omega)$ and A is an operator related to the problem

$$\begin{gathered} -\operatorname{div}[\mathbf{a}(., \nabla u)] = f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu_{\mathbf{a}}} = g \quad \text{on } \partial\Omega, \end{gathered} \quad (1.4)$$

in the sense that $A(u, \tau u) = (f, g)$, if u is an entropy solution to (1.4). A_1 is the restriction of A to X_1.

Our approach is based essentially on the theory of m-accretive operators in Banach spaces and the following order preserving properties:

If $F_i = (f_i, g_i) \in L^1(\Omega) \times L^1(\partial\Omega)$, $i = 1, 2$, satisfy the conditions $\int_\Omega f_i(y)dy + \int_{\partial\Omega} g_i(y)d\sigma(y) = 0$, $A(u_i, \tau u_i) = (f_i, g_i)$ and $\varphi = \operatorname{sign}_0(u_1-u_2)$ and $\psi = \operatorname{sign}_0(\tau u_1-$

$\tau u_2)$, then

$$\int_\Omega (f_1 - f_2)\varphi + \int_{\Omega\cap\{u_1=u_2\}} |f_1 - f_2| + \int_{\partial\Omega} (g_1 - g_2)\psi + \int_{\partial\Omega\cap\{\tau u_1=\tau u_2\}} |g_1 - g_2| \geq 0 \tag{1.5}$$

If in addition, $(u_i, \tau u_i) \in X_1$, $i = 1, 2$ and $A_1(u_i, \tau u_i) = (f_i, g_i)$, then for every $\varphi \in \operatorname{sign}(u_1 - u_2)$ and every $\psi \in \operatorname{sign}(\tau u_1 - \tau u_2)$, we have

$$\int_\Omega (f_1 - f_2)\varphi + \int_{\partial\Omega} (g_1 - g_2)\psi \geq 0, \tag{1.6}$$

where

$$\operatorname{sign}(r) = \begin{cases} r/|r| & \text{if} \quad r \neq 0 \\ [-1, 1] & \text{if} \quad r = 0, \end{cases} \qquad \operatorname{sign}_0(r) = \begin{cases} r/|r| & \text{if} \quad r \neq 0 \\ 0 & \text{if} \quad r = 0. \end{cases}$$

Note that the main difficulty of the problem here as well as in [18] is that the domain of the operator A in not necessary included in $L^1 + L^\infty$.

The inequality (1.5) is applied to the proof of uniqueness for the nonlinear perturbation $(\beta(u), \gamma(\tau u))$, in the problem (1.1) which leads to the uniqueness of the solution u up to a constant, while (1.6) is applied to the proof of existence of a solution to (1.1), and mainly when β and γ are possibly, mulivalued maximal monotone graphs in $\mathbb{R}^2$.

Following [2] and [8], the inequalities (1.5) and (1.6) may be interpreted as properties of maximum principle type or order preserving properties in the sense of [5] related to an operator whose domain is not necessary included in $L^1(\Omega)$.

For the sequel, we proceed as follows: In Section 2, we collect some properties of functional spaces and traces. In Section 3, we prove that operator A is one-to-one (modulo constants) and A_1 is m-completely accretive on X_1. Section 4 is devoted to establish order preserving properties (1.5) and (1.6). In Section 5, we discuss existence and uniqueness for the entropy solution to (1.1).

2. Preliminaries and notations

Let $\mathcal{M}(\Omega)$ be the space of classes of Borel measurable real valued functions on Ω, equipped with the topology of the convergence in measure

$$d(f, g) = \int_\Omega \frac{|f - g|(x)}{1 + |f - g|(x)} dx.$$

For $r > 0$, we consider the functional $\mathcal{N}_r$ and the Marcinkiewicz space $M^r(\Omega)$,

$$\mathcal{N}_r(u) = \big[\sup_{\lambda>0} \lambda^r |\{|u| > \lambda\}|\big]^{1/r}, \quad \text{if } u \in \mathcal{M} \text{ and } M^r(\Omega) = \{u \in \mathcal{M} : \mathcal{N}_r(u) < \infty\}$$

If $r > 1$ and $\mathcal{B}$ is the Borel family of subsets of Ω or $\partial\Omega$, then $\mathcal{N}_r$ is equivalent to the norm,

$$\|u\|_{M^r} = \sup_{K\in\mathcal{B},\ |K|>0} \frac{1}{|K|^{\frac{1}{r'}}} \int_K |u(x)| d\mu(x), \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

$(M^r(\Omega), \|\ \|_{M^r})$, (respect: $(M^r(\partial\Omega)), \|\ \|_{M^r})$, is a Banach space and the inclusion $M^r \subset L^q$ holds for all r, q, $1 \leq q < r$. (see [4]).

We recall from [18] that, for an arbitrary $r > 0$, $\mathcal{N}_r(u+v) \leq 2^{1/r}[\mathcal{N}_r(u)+\mathcal{N}_r(v)]$, if $0 < q < r$, then $u \in M^r$ if and only if $|u|^q \in M^{\frac{r}{q}}$ and M^r is closed subspace of $\mathcal{M}$. In particular M^r is complete for the topology of convergence in measure.

Following [3], the gradient by mean of truncating ∇u is a measurable function

$$V:\Omega\to\mathbb{R}^N \text{ such that } V=DT_ku,\quad \text{a.e. on } \Omega_k=\{|u|\le k\},\ k>0. \tag{2.1}$$

and the set $\mathcal{T}^{1,p}(\Omega)$ is given by

$$\mathcal{T}^{1,p}(\Omega)=\{u\in\mathcal{M}(\Omega), \text{ such that } DT_ku\in L^p(\Omega), \text{ for every } k>0\}.$$

Note that in view of the identity $\mathbf{1}_{\{|u|<k\}}\nabla u=DT_ku$, the notation ∇u becomes superfluous on the set $\{|u|<k\}$, where $T_ku=u$. For this reason $\mathbf{1}_{\{|u|<k\}}\nabla u$ will be noted simply $\mathbf{1}_{\{|u|<k\}}Du$.

We apply also the sets $\mathcal{T}^{1,p}_{\rm tr}(\Omega)$ introduced in [1, Theorem 3.1] as being the subset of functions in $\mathcal{T}^{1,p}(\Omega)$ for which a generalized notion of trace may be defined. More precisely $u\in\mathcal{T}^{1,p}_{\rm tr}(\Omega)$ if $u\in\mathcal{T}^{1,p}(\Omega)$ and there exist a sequence $(u_n)_n$ in $W^{1,p}(\Omega)$ and a measurable function v on $\partial\Omega$ such that

$$\begin{gathered} u_n\to u\quad\text{a.e. in }\Omega,\\ DT_k(u_n)\to DT_k(u)\quad\text{in } L^1(\Omega),\\ (\tau u_n)_n\to v\quad\text{a.e. on }\partial\Omega\,. \end{gathered} \tag{2.2}$$

Therefore, we set $\widetilde{\tau}u=v$ the trace of u. The operator $\tilde\tau$ satisfies the following properties

(i) If $u\in W^{1,p}(\Omega)$, then $\tau T_ku=T_k\tau u$, for every $k>0$ and $\tilde\tau u=\tau u$ a.e. on $\partial\Omega$.
(ii) If $u\in\mathcal{T}^{1,p}_{\rm tr}(\Omega)$, then $\tau(T_ku)=T_k(\tilde\tau u)$, for all $k>0$.
(iii) $\widetilde{\tau}(u-\varphi)=\widetilde{\tau}u-\tau\varphi$, if $\varphi\in W^{1,p}(\Omega)$ and $u\in\mathcal{T}^{1,p}_{\rm tr}(\Omega)$.

In the sequel, $\widetilde{\tau}u$ is noted simply τu.

Lemma 2.1. *Let be given $\delta>0$ and p such that $1<p<N$. If $p_1=\frac{N(p-1)}{N-p}$, $p_2=\frac{N(p-1)}{N-1}$ and $u\in\mathcal{T}^{1,p}_{tr}(\Omega)$ such that the trace $\tau u\in L^1(\partial\Omega)$ and u satisfies*

$$\frac1k\int_{\{|u|<k\}}|Du|^pdx\le\delta,\ \textit{for every } k>0, \tag{2.3}$$

then we have

(i) *$u\in M^{p_1}(\Omega)$ and there exists a constant $C_3=C_3(N,p,\Omega,\delta)$ such that*
$$|\{|u|>k\}|\le C_3k^{-p_1}\quad\textit{for every } k>0, \tag{2.4}$$
(ii) *$\nabla u\in M^{p_2}(\Omega)$ and there exists a constant $C_4=C_4(N,p,\Omega,\delta)$ such that*
$$|\{|\nabla u|>k\}|\le C_4k^{-p_2},\quad\textit{for every } k>0. \tag{2.5}$$

Proof. (i) We denote by $\overline{v}=\frac{1}{|\partial\Omega|}\int_{\partial\Omega}v$ the mean of any measurable function v, when it exits and we select $k_0>2|\overline{u}|$. If $k\ge k_0$, then,

$$\begin{aligned} |\{|u|\ge k\}|&=|\{|T_ku|=k\}|\\ &\le|\{|T_ku|+\frac k2\ge k+|\overline{u}|\}|\\ &\le\big|\big\{|T_ku|+\frac k2\ge k+|\overline{T_ku}|\big\}\big|\\ &\le\big|\big\{|T_ku-\overline{T_ku}|\ge\frac k2\big\}\big|\\ &\le\big(\frac2k\|T_ku-\overline{T_ku}\|_{p^*}\big)^{p^*}, \end{aligned}$$

where $p^* = \frac{Np}{N-p}$.

The last estimation follows from Hölder inequality. Indeed if $\Omega'_k = \{|T_k u - \overline{T_k u}| \geq \frac{k}{2}\}$, then

$$\frac{k}{2}|\Omega'_k| \leq \int_{\Omega'_k} |T_k u - \overline{T_k u}| \leq \|T_k u - \overline{T_k u}\|_{p^*} |\Omega'_k|^{1-\frac{1}{p^*}}$$

Then applying [20, page 191], there exists a constant $C = C(N, p, \Omega)$ such that

$$|\{|u| \geq k\}| \leq C(\|DT_k u\|_p)^{p^*} (\frac{k}{2})^{-p^*} \leq 2^{p^*} C \delta^{\frac{p*}{p}} k^{\frac{p^*}{p} - p^*}, \quad \text{if } k \geq k_0.$$

Hence, (2.4) follows if we select for example, $C_3 = \max\{2^{p^*} C \delta^{\frac{p*}{p}}, k_0^{p_1} |\Omega|\}$.
(ii) The same proof of [3, lemma 4.2] apply here, taking into account that the constant C_4 depends on Ω. $\square$

3. Accretive operators and entropy solutions

We define the Banach spaces, $X_r = L^r(\Omega) \times L^r(\partial\Omega)$, $r \geq 1$, $X_\infty = L^\infty(\Omega) \times L^\infty(\partial\Omega)$ and the measure space $\mathcal{X} = (\Omega \cup \partial\Omega, \mathcal{B}_\Omega \cup \mathcal{B}_{\partial\Omega}, dx \oplus d\sigma)$.

For $(U, V) \in X_r \times X_{r'}$, $\frac{1}{r} + \frac{1}{r'} = 1$, $U = (u_1, u_2)$, $V = (v_1, v_2)$, we use the notation

$$UV = (u_1 v_1, u_2 v_2) \quad \text{and} \quad \int_{\mathcal{X}} UV = \int_\Omega u_1 v_1 + \int_{\partial\Omega} u_2 v_2$$

The spaces X_r, $r \geq 1$, and X_∞ are equipped respectively with the norms

$$\|F\|_r = \Big[\int_{\mathcal{X}} (|f|^r, |g|^r)\Big]^{1/r} = \Big[\int_\Omega |f(x)|^r dx + \int_{\partial\Omega} |g(x)|^r d\sigma(x)\Big]^{1/r},$$

for $F = (f, g) \in X_r$ and

$$\|F\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| + \operatorname*{ess\,sup}_{x\in\partial\Omega} |g(x)|,$$

for $F = (f, g) \in X_\infty$. Let us recall the classical sets

$$\mathcal{P}_0 = \{p : \mathbb{R} \to \mathbb{R}, p \text{ Lipschitz, odd, non decreasing and } p' \text{ has a compact support}\},$$

$$\mathcal{J}_0 = \{j : \mathbb{R} \to \mathbb{R}, j \text{ is convex, lower semi-continuous, with } \min j = j(0) = 0\}.$$

Definition 3.1. If A_1 is a mapping from $D(A_1) \subset X_1$ to X_1, then A_1 is said to be is **m-accretive in** X_1, if the resolvent $J_\lambda^{A_1} = (I + \lambda A_1)^{-1}$ satisfies,

$$J_\lambda^{A_1} \text{ is a contraction everywhere defined in } X_1, \text{ for every } \lambda > 0.$$

$X_1 = L^1(\mathcal{X})$ is a normal Banach space in the sense of [5, page 53]. If $U_i \in D(A_1)$, $F_i \in X_1$, are given such that, $A_1 U_i = F_i$, $i = 1, 2$ and $p \in \mathcal{P}_0$, then

$$(A_1 U_1 - A_1 U_2) p(U_1 - U_2) \in L^1(\mathcal{X}).$$

Therefore, the condition

$$\int_{\mathcal{X}} ((A_1 U_1 - A_1 U_2) p(U_1 - U_2))^+ \geq \int_{\mathcal{X}} ((A_1 U_1 - A_1 U_2) p(U_1 - U_2))^-$$

is equivalent to

$$\int_{\mathcal{X}} (A_1 U_1 - A_1 U_2) p(U_1 - U_2) \geq 0,$$

This leads to the next definition [5, proposition 2.2].

Definition 3.2. A_1 is **m-completely accretive** in X_1, if A_1 is m-accretive and verifies one of the following equivalent conditions,

$$\int_{\mathcal{X}} j(J_\lambda^{A_1}U_1 - J_\lambda^{A_1}U_2) \le \int_{\mathcal{X}} j(U_1 - U_2), \text{ for all } U_1, U_2 \in X_1,\ \lambda > 0 \text{ and } j \in \mathcal{J}_0. \tag{3.1}$$

$$\int_{\mathcal{X}} (A_1U_1 - A_1U_2)p(U_1 - U_2) \ge 0, \text{ for all } U_1, U_2 \in D(A_1) \text{ and } p \in \mathcal{P}_0. \tag{3.2}$$

As a consequence, if $A_1(u_i, \tau u_i) = (f_i, g_i)$, $i = 1, 2$, then by selecting $p(r) = mT_{\frac{1}{m}}(r)$ in (3.2), $r \in \mathbb{R}$, and let $m \to +\infty$, we obtain the next particular order preserving property for A_1,

$$\int_\Omega (f_1 - f_2)\operatorname{sign}_0(u_1 - u_2) + \int_{\partial\Omega} (g_1 - g_2)\operatorname{sign}_0(\tau u_1 - \tau u_2) \ge 0 \tag{3.3}$$

If A_1 is m-completely accretive in X_1, we know from [5, proposition 3.7], (see also [2]), that the Yosida approximation $A_{1,\lambda} = \dfrac{I - J_\lambda^{A_1}}{\lambda} = A_1 J_\lambda^{A_1}$ satisfies, for every $U \in D(A_1)$,

$$A_{1,\lambda} \text{ is m-completely accretive, Lipschitz with coefficient } \frac{2}{\lambda} \text{ and } \lim_{\lambda\downarrow 0} A_{1,\lambda}U = A_1U. \tag{3.4}$$

The operator $\mathbf{a}$ of Leray-Lions type is defined as follows,

(H1) $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $(x, \xi) \mapsto \mathbf{a}(x, \xi)$ is a Carathéodory function in the sense that, $\mathbf{a}$ is continuous in ξ, for almost every $x \in \Omega$ and measurable in x for every $\xi \in \mathbb{R}^N$.

(H2) There exist p, $1 < p < N$, and $C_1 > 0$, so that,
$$\langle \mathbf{a}(x, \xi), \xi\rangle \ge C_1|\xi|^p, \text{ for a.e } x \in \mathbb{R}^N \text{ and every } \xi \in \mathbb{R}^N.$$

(H3) $\langle \mathbf{a}(x, \xi_1) - \mathbf{a}(x, \xi_2), \xi_1 - \xi_2\rangle > 0$, if $\xi_1 \ne \xi_2$, for a.e. $x \in \Omega$.

(H4) There exists some $h_0 \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$ and $C_2 > 0$, such that,
$$|\mathbf{a}(x, \xi)| \le C_2(h_0(x) + |\xi|^{p-1}), \text{ for a.e } x \in \Omega \text{ and every } \xi \in \mathbb{R}^N.$$

Definition 3.3. *If u is any measurable function on Ω, then u is a weak solution to the problem (1.4), if $u \in \mathcal{T}_{\mathrm{tr}}^{1,p}(\Omega)$, $\mathbf{a}(., \nabla u) \in L^1(\Omega)$ and for every $v \in \mathcal{C}_0^\infty(\mathbb{R}^N)$,*

$$\int_\Omega \langle \mathbf{a}(., \nabla u), Dv\rangle = \int_{\mathbb{R}^N} fv + \int_{\partial\Omega} g.\tau v,$$

It is well known, from [16], that uniqueness of weak solutions for degenerate elliptic equations, fails to be always true, then following [3], (see [13], for example, for another type of solutions, the renormalized solutions).

Definition 3.4. u is said to be an entropy solution to (1.4), if $u \in \mathcal{T}_{\mathrm{tr}}^{1,p}(\Omega)$ and u satisfies,

$$\int_\Omega \mathbf{a}(., Du)DT_k(u - \varphi) \le \int_\Omega fT_k(u - \varphi) + \int_{\partial\Omega} gT_k(\tau u - \varphi) \tag{3.5}$$

for every $\varphi \in \mathcal{T}_{\mathrm{tr}}^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We notice that if we set $K = k + \|\varphi\|_\infty$, then $\{|u-\varphi| \le k\} \subset \{|u| \le K\}$, thus $\mathbf{1}_{\{|u-\varphi|\le k\}}.\nabla u = \mathbf{1}_{\{|u-\varphi|\le k\}}.DT_K u = \mathbf{1}_{\{|u-\varphi|\le k\}}.Du$, since $T_K u = u$ on the set $\{|u-\varphi| \le k\}$.

We can prove easily as in [3] that if u is an entropy solution of (1.4), then u is a weak solution.

To discuss uniqueness, for the problem (1.4), the test functions φ must be taken in a class, larger than $\mathcal{C}_0^\infty(\overline{\Omega})$ and that contains $T_k u$. In [3], the class $\mathcal{T}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is well adapted to the problem with Dirichlet condition on $\partial\Omega$. In [18] this extension is obtained directly from [3], in the class $\mathcal{T}^{1,p}(\mathbb{R}^N) \cap \mathcal{L}_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, since we have the identity $\mathcal{T}_0^{1,p}(\mathbb{R}^N) \cap \mathcal{L}_0(\mathbb{R}^N) = \mathcal{T}^{1,p}(\mathbb{R}^N) \cap \mathcal{L}_0(\mathbb{R}^N)$, where $\mathcal{L}_0(\mathbb{R}^N) = \{u \in \mathcal{M}(\mathbb{R}^N) \text{ s.t. } |\{|u| > k\}| < +\infty, \text{ for every } k > 0\}$. In the present, we need the next lemma.

Lemma 3.5. *(i) If $\varphi \in \mathcal{T}_{\mathrm{tr}}^{1,p}(\Omega) \cap L^\infty(\Omega)$, then, there exists a sequence $(\varphi_n)_n$ in $\mathcal{C}_0^\infty(\overline{\Omega})$, $n \in \mathbb{N}$, such that $T_k(u-\varphi_n)$ converges a.e. on Ω to $T_k(u-\varphi)$, $\tau T_k(u-\varphi_n)$ converges a.e. on $\partial\Omega$ to $\tau T_k(u-\varphi)$ and $DT_k(u-\varphi_n)$ converges weakly in $(L^p(\Omega))^N$ to $DT_k(u-\varphi)$, for every $k > 0$.*
(ii)In particular, $u \in \mathcal{T}_{\mathrm{tr}}^{1,p}(\Omega)$ is an entropy solution to (1.4)*, if and only if u satisfies* (3.5)*, for every $\varphi \in \mathcal{T}_{\mathrm{tr}}^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Proof. (i) Let $(\theta_n)_n$ a regularizing sequence in $\mathbb{R}^N$, $\theta_n \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and $\phi_{n,m} = (\varphi.\mathbf{1}_{B(0,m)}) * \theta_n$, $m, n \in \mathbb{N}$. By the diagonal process, there exists a sequence φ_n that converges a.e. on Ω to φ. $T_k(u-\varphi)$ and $T_k(u-\varphi_n)$ are in $W^{1,1}(\Omega)$, we assume that $T_k(u-\varphi_n)$ converges a.e. on Ω to $T_k(u-\varphi)$ and the same type of convergence for the trace on $\partial\Omega$.

For the weak convergence of the sequence $DT_k(u-\varphi_n)$, it is sufficient to prove that
$$\int_\Omega \langle DT_k(u-\varphi_n), \psi\rangle \to \int_\Omega \langle DT_k(u-\varphi), \psi\rangle, \text{ for every } \psi \in [\mathcal{C}_0^\infty(\overline{\Omega})]^N.$$
By the divergence theorem, we have,
$$\int_\Omega \operatorname{div}[T_k(u-\varphi_n)\psi] = \int_{\partial\Omega} T_k(u-\varphi_n)(x)\langle \psi(x), \nu(x)\rangle d\sigma(x)$$
Where $\nu(x)$ is the outward normal vector in $x \in \partial\Omega$. Thus,
$$\begin{aligned}&\int_\Omega \langle DT_k(u-\varphi_n), \psi\rangle \\ &= -\int_\Omega [T_k(u-\varphi_n)\operatorname{div}\psi] + \int_{\partial\Omega} T_k(u-\varphi_n)(x)\langle \psi(x), \nu(x)\rangle d\sigma(x)\end{aligned}$$
The lemma is proved by applying dominated convergence in the last two integrals and then again the divergence theorem in the opposite sense. Part (ii) is an immediate consequence of part (i). □

For the sequel, the operators A and A_1 are given as follows,
$$\begin{aligned}&(u, \tau u) \in D(A), \text{ if } u \in \mathcal{T}_{\mathrm{tr}}^{1,p}(\Omega),\ \tau u \in L^1(\partial\Omega) \text{ and there exists} \\ &(f, g) \in X_1 \text{ such that, } u \text{ is an entropy solution for (1.4).} \\ &A_1 \text{is the restriction of } A \text{ to } X_1.\end{aligned} \tag{3.6}$$

Theorem 3.6. *(i) A_1 is m-completely accretive in X_1.*
(ii) A is one-to-one.

Proof. (i). In view of (H3) the inequality (3.2) is satisfied. Hence, A_1 is completely accretive in X_1. Now, we prove that $I + A_1$ is onto from X_1 to X_1.
Step 1: Construction of an approximate sequence. First, we consider the reflexive Banach space

$$E = W^{1,p}(\Omega) \times L^p(\partial\Omega), \quad \|U\|_E = (\|u\|^p_{W^{1,p}(\Omega)} + \|v\|^p_{L^p(\partial\Omega)})^{1/p}.$$

and we define a subspace X_0 and an operator A_0 as follows, $X_0 = \{(u, v) \in E : v = \tau u\}$, and $U = (u, \tau u) \in X_0$ is a solution to $A_0(u, \tau u) = (f, g) \in E'$ if,

$$\int_\Omega \langle \mathbf{a}(., Du), Dv\rangle = \int_\Omega fv + \int_{\partial\Omega} g.\tau v, \quad \text{for every } V = (v, \tau v) \in X_0.$$

Next, we consider, the convex functional

$$\Phi_n(u, v) = \frac{1}{2}\Big[\int_\Omega u^2(x)dx + \int_{\partial\Omega} v^2(x)d\sigma(x)\Big] + \frac{1}{np}\int_\Omega |u|^p dx; \, U = (u, v) \in X_0,$$

The real mapping $t \mapsto \langle A_0(u + tv), w\rangle$ is continuous, for all $u, v, w \in E$, then A_0 is monotone and Hemi-continuous (see [7], [11]), thus it is Pseudo-monotone. It is also coercive in the sense that

$$\lim_{\|U\|_E \to +\infty, U \in X_0} \frac{\langle A_0 U, U\rangle + \Phi_n(U)}{\|U\|_E} = +\infty.$$

Let $F = (f, g)$ be given in X_1 and $F_n = (f_n, g_n) = (T_n f, T_n g)$. Then $F_n \in X_1 \cap X_\infty \subset E'$, $\|f_n\|_1 \le \|f\|_1$ and $\|g_n\|_1 \le \|g\|_1$, f_n converges to f in $L^1(\Omega)$, and g_n converges to g in $L^1(\partial\Omega)$. By [7, corollary 30], there exists $U_n \in X_0$ that satisfies, for all $V \in X_0$,

$$\Phi_n(V) - \Phi_n(U_n) \ge \langle F_n - A_0 U_n, V - U_n\rangle_{E' \times E}$$

Thus, $F_n - A_0 U_n = \partial\Phi_n(U_n) \in E'$, $\partial\Phi_n$ is the subdifferential of Φ_n, $\partial\Phi_n$ is univalued here. In other words, $U_n = (u_n, \tau u_n) \in X_0$ and U_n satisfies,

$$\int_\Omega u_n v + \int_{\partial\Omega} \tau u_n \tau v + \int_\Omega \langle \mathbf{a}(., Du_n), Dv\rangle + \frac{1}{n}\int_\Omega |u_n|^{p-2} u_n v = \int_\Omega f_n.v + \int_{\partial\Omega} g_n \tau v, \tag{3.7}$$

for every $V = (v, \tau v) \in X_0$.
Step 2: we claim that the sequence $(\frac{1}{n}|u_n|^{p-1})_n$converges to 0 in $L^1(\Omega)$. If we take $v = T_k(u_n)$ in (3.7), then $v \in W^{1,p}(\Omega)$, and we obtain,

$$\frac{C_1}{k}\int_{\{|u_n|<k\}} |Du_n|^p \le \|F_n\|_1 \le \|F\|_1 \tag{3.8}$$

We deduce from (2.4), that $\big(|u_n|^{p-1}\big)_n$ is uniformly bounded in the Marcinkiewicz space $M^{\frac{N}{N-1}}(\Omega)$ and then in $L^1(\Omega)$. After passing to a subsequence, we assume that $\frac{1}{n}|u_n|^{p-1}$ converges in $L^1(\Omega)$ and a.e. on Ω to 0.
Step 3: Convergence in measure of the sequence $(u_n)_n$. We consider the decomposition,

$$\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|u_n| \le k, |u_m| \le k, |u_n - u_m| > t\}$$

Since $(u_n)_n$ is uniformly bounded in the Marcinkiewicz space M^{p_1}, then, for every $\varepsilon > 0$, there exists k_0 such that $|\{|u_n| > k\}| < \varepsilon$ and $|\{|u_m| > k\}| < \varepsilon$, if $k > k_0$. Next, if we select some $k > k_0$, since $(T_k u_n)_n$, is bounded in $W^{1,p}(\Omega)$, we assume then, up to a subsequence, that $(T_k u_n)_n$ is a Cauchy sequence in $L^1(\Omega)$ and in measure. Then we have $|\{|u_n| \le k, |u_m| \le k, |u_n - u_m| > t\}| < \varepsilon$, if m, n are sufficiently large.

Hence, up to a subsequence, $(u_n)_n$ converges in measure and a.e. to some u and $u \in M^{p_1}(\Omega)$.

Step 4: Convergence of the sequence $(Du_n)_n$. If k, l, t, ε are positive real numbers, we have the inclusions,

$$\begin{aligned}\{|Du_n - Du_m| > t\} \subset &\{|u_n - u_m| > k\} \cup \{|Du_n| > l\} \cup \{|Du_m| > l\} \\ &\cup \{|Du_n| \le l, |Du_m| \le l, |u_n - u_m| \le k, |Du_n - Du_m| > t\}.\end{aligned}$$

We proceed, first, with the last term in the previous inclusion

Let a compact $\mathbf{K}$ and a function μ be given as follows,

$$\mathbf{K} = \{(\xi, \zeta) \in \mathbb{R}^N \times \mathbb{R}^N : |\xi| \le l, |\zeta| \le l, |\xi - \zeta| \ge t\},$$
$$\mu(x) = \min_{(\xi,\zeta)\in\mathbf{K}} \langle \mathbf{a}(x, \xi) - \mathbf{a}(x, \zeta), \xi - \zeta\rangle$$

We derive from (H1) and (H3), that μ is defined for a.e. $x \in \Omega$ and is positive. Thus $|\{\mu = 0\}| = 0$, and there exists $\eta > 0$, such that for every measurable subset S of Ω, if $\int_S \mu < \eta$, then $|S| < \varepsilon$. By applying this last statement to,

$$S = \{|Du_n| \le l, |Du_m| \le l, |u_n - u_m| \le k, |Du_n - Du_m| > t\},$$

Since,

$$\int_S \mu \le \int_S \langle \mathbf{a}(x, Du_n) - \mathbf{a}(x, Du_m), Du_n - Du_m\rangle \le k\|F\|_1,$$

then $|S| < \varepsilon$, if k is small enough.

Next, if k is fixed small enough, from the step 3, we have, $|\{|u_n - u_m| > k\}| < \varepsilon$, if m, n are sufficiently large.

According to (3.8) and (2.5), $(Du_n)_n$ is uniformly bounded in the Marcinkiewicz space M^{p_2}, $p_2 = \frac{N(p-1)}{N-1}$. Hence $|\{|Du_n| > l\}| < \varepsilon$ and $|\{|Du_m| > l\}| < \varepsilon$, for l sufficiently large.

Then, we may assume that, Du_n converges in measure and a.e. on Ω, if $n \to +\infty$ to some V and $|V| \in M^{p_2}(\mathbb{R}^N)$.

We claim that $u \in \mathcal{T}^{1,p}(\Omega)$ and $\nabla u = V$. For a fixed $k > 0$, on one hand $T_k u_n$ is converging to $T_k u$ by dominated convergence, and therefore $DT_k u_n$ is converging to $DT_k u$ in $\mathcal{D}'(\mathbb{R}^N)$, on the other hand, $(DT_k u_n)_n$ is bounded in $L^p(\Omega)$, thus, $DT_k u_n$ is converges weakly to a V_k in L^p, therefore also in $\mathcal{D}'(\mathbb{R}^N)$. By uniqueness, $DT_k u = V_k \in L^p(\Omega)$ and $DT_k u_n$ converges weakly in $L^p(\Omega)$ to $DT_k u$. Consequently, $u \in \mathcal{T}^{1,p}(\Omega)$.

Next, we prove that

$$DT_k u_n \text{ converges in measure and a.e. on } \Omega \text{ to } \nabla u, \text{ as } n \to +\infty \text{ and } k \to +\infty. \tag{3.9}$$

Since $T_{k+\alpha} \circ T_k = T_k$, for every $k > 0$ and $\alpha > 0$, then, by the same arguments as for $(Du_n)_n$, we obtain from (3.9) that $(DT_k u_n)_n$ converges in measure to some v_k, then claim that $(DT_k u_n)_n$ converges weakly to v_k, since that leads to $v_k = DT_k u$.

Indeed, if $\varepsilon > 0$, and $\varphi \in L^{p'}(\Omega)$, then for every $k > 0$, we can select two positive constants c_k and $\eta > 0$ such that, for every $n \in \mathbb{N}$ and every measurable subset $S \subset \Omega$, we have

$$\|DT_k u_n\|_p \le c_k, \quad \text{and} \quad \|\varphi\|_{L^{p'}(S)} \le \frac{\varepsilon}{4c_k}, \quad \text{if } |S| \le \eta$$

By Fatou lemma, we have $\|v_k\|_p \le c_k$. Next, if we set,
$$\lambda = \frac{\varepsilon}{2|\Omega|^{\frac{1}{p}}\|\varphi\|_{L^{p'}(\Omega)}}, \quad S_\lambda = \{|DT_k u_n - v_k| > \lambda\}$$
and n_0 such that $|S_\lambda| \le \eta$ for $n \ge n_0$, then
$$\begin{aligned} |\int_\Omega (DT_k u_n - v_k)\varphi| &\le \int_{S_\lambda} |DT_k u_n - v_k||\varphi| + \int_{\Omega\setminus S_\lambda} |DT_k u_n - v_k||\varphi| \\ &\le 2c_k\|\varphi\|_{L^{p'}(S_\lambda)} + \lambda|\Omega|^{\frac{1}{p}}\|\varphi\|_{L^{p'}(\Omega)} \le \varepsilon\,. \end{aligned}$$
Thus, $DT_k u_n$ converges in measure and a.e to $DT_k u$, as $n \to +\infty$. Consequently, for every $k > 0$, there exists some $n_k \in \mathbb{N}$, such that, $d(DT_k u_{n_k}, DT_k u) \le \frac{1}{k}$, where d is the metric on $\mathcal{M}$. On the other hand,
$$d(DT_k u, \nabla u) = \int_\Omega \frac{|DT_k u - \nabla u|}{1 + |DT_k u - \nabla u|} \le |\{|u| > k\}| \to 0, \quad \text{as } k \to +\infty. \tag{3.10}$$
Therefore, we assume the subsequence $(DT_k u_{n_k})_k$ converges in measure and a.e. to ∇u, as $k \to +\infty$.

But for a.e. $x \in \Omega$, if $k > |u(x)|$ and $k' > |u(x)|$, we have
$$\begin{aligned} &|DT_k u_n(x) - DT_{k'} u_m(x)| \\ &\le |DT_k u_n(x) - Du(x)| + |Du(x) - DT_{k'} u_m(x)| \\ &= |DT_k u_n(x) - DT_k u(x)| + |DT_{k'} u(x) - DT_{k'} u_m(x)| \le \varepsilon \end{aligned}$$
if m and n are sufficiently large,

At last by the same argument as in (3.10), we conclude that, up to a subsequence $DT_k u_n$ converges in measure and a.e. to Du_n, if $k \to +\infty$. Since $(Du_n)_n$ converges in measure and a.e. to ∇u, we conclude that $DT_k u_n$ converges in measure and a.e. to ∇u, as $n, k \to +\infty$. This completes the proof of (3.9).

Applying classical arguments for Carathéodory functions, we assume that the sequence $(\mathbf{a}(., Du_n))_n$ converges in measure to $\mathbf{a}(., Du)$. ¿From (H4) and the fact that $|Du_n|^{p-1}$ a.e. uniformly bounded in the Marcinkiewicz space $M^{\frac{N}{N-1}}(\Omega)$, we deduce that $(\mathbf{a}(., Du_n))_n$ is equi-integrable on Ω. Hence, $\mathbf{a}(., Du_n)$ converges in $L^1(\Omega)$ to $\mathbf{a}(., Du)$.

Step 5: Convergence of the trace. We prove that $(\tau u_n)_n$ converges to some $w \in \mathcal{M}(\partial\Omega)$, that $u \in \mathcal{T}^{1,p}_{\rm tr}(\Omega)$ and $w = \tau u$, $d\sigma$ a.e.

Since the trace operator is completely continuous from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$, we assume that $T_k \tau u_n = \tau T_k u_n \to \tau T_k u$, a.e. on $S_k = \{x \in \partial\Omega;\ |T_k u| < k\}$, for every $k > 0$. Thus τu_n converges a.e. to w on $\partial\Omega$, $w = \tau T_k u$, a.e. on S_k, for every $k > 0$.

On the other hand, $DT_k u_n$ converges weakly in L^p and in measure to $DT_k u$, we deduce also that $DT_k u_n$ converges to $DT_k u$ in $L^1(\Omega)$.

We summarize, $DT_k u_n$ $DT_k u$, we deduce that $L^1(\Omega)$.
$$\begin{gathered} (u_n)_n \text{ converges in measure to some } u, \\ DT_k u_n \text{ converges to } DT_k u \text{ in } L^1(\Omega), \\ \tau u_n \text{ converges a.e. to } w \text{ on } \partial\Omega. \end{gathered}$$
We conclude, as defined in (2.2), that $u \in \mathcal{T}^{1,p}_{\rm tr}(\Omega)$ and $\tau u = w$.

Step 6: It remains to prove that u is an entropy solution to
$$u - \operatorname{div}[\mathbf{a}(., Du)] = f \quad \text{in } \Omega$$

$$\tau u+\frac{\partial u}{\partial \nu_{\mathbf{a}}}=g \quad \text{on } \partial\Omega$$

If $\varphi\in\mathcal{C}_0^\infty(\overline{\Omega})$, and $v=T_k(u_n-\varphi)$ in (3.7), then we have

$$\begin{aligned}
&\int_\Omega\langle\mathbf{a}(.,Du_n),DT_k(u_n-\varphi)\rangle\\
&=\int_\Omega(f_n-u_n-\frac{1}{n}|u_n|^{p-2}u_n)T_k(u_n-\varphi)+\int_{\partial\Omega}(g_n-\tau u_n)\tau T_k(u_n-\varphi),
\end{aligned}$$

$u_n\to u$ a.e. on Ω and $\tau u_n\to\tau u$ a.e. on $\partial\Omega$.

For the left member, we notice first that $D(u-\varphi)=0$, a.e. on the set $\{|u-\varphi|=k\}$. Then $DT_k(u_n-\varphi)=D(u_n-\varphi)\mathbf{1}_{\{|u_n-\varphi|<k\}}$ a.e. , and $\langle\mathbf{a}(.,Du_n),DT_k(u_n-\varphi)\rangle$ converges a.e on Ω, to $\langle\mathbf{a}(.,Du),Du-D\varphi)\rangle\mathbf{1}_{\{|u-\varphi|<k\}}=\langle\mathbf{a}(.,Du),DT_k(u-\varphi))\rangle$. Next

$$\lim_n\int_{\Omega\cap\{|u-\varphi|<k\}}\langle\mathbf{a}(.,Du_n),D\varphi\rangle\mathbf{1}_{\{|u_n-\varphi|<k\}}=\int_{\Omega\cap\{|u-\varphi|<k\}}\langle\mathbf{a}(.,Du),D\varphi\rangle$$

On the other hand, $\langle\mathbf{a}(.,Du_n),DT_ku_n\rangle\geq 0$, a.e. and $\mathbf{a}(.,Du_n)$ converges in $L^1(\Omega)$. Therefore,

$$\begin{aligned}
&\int_\Omega\langle\mathbf{a}(.,Du),DT_k(u-\varphi)\rangle\\
&=\int_\Omega\liminf_n\langle\mathbf{a}(.,Du_n),DT_k(u_n-\varphi)\rangle\\
&=\int_{\Omega\cap\{|u_n-\varphi|<k\}}\liminf_n\langle\mathbf{a}(.,Du_n),DT_ku_n\rangle-\int_{\Omega\cap\{|u_n-\varphi|<k\}}\lim_n\langle\mathbf{a}(.,Du_n),D\varphi\rangle\\
&\leq\liminf_n\int_{\Omega\cap\{|u_n-\varphi|<k\}}\langle\mathbf{a}(.,Du_n),DT_ku_n\rangle-\lim_n\int_{\Omega\cap\{|u_n-\varphi|<k\}}\langle\mathbf{a}(.,Du_n),D\varphi\rangle\\
&=\liminf_n\int_\Omega\langle\mathbf{a}(.,Du_n),DT_k(u_n-\varphi)\rangle\\
&\leq\liminf_n\int_\Omega(f_n-u_n-\frac{1}{n}|u_n|^{p-2}u_n)T_k(u_n-\varphi)\\
&\quad+\liminf_n\int_{\partial\Omega}(g_n-\tau u_n)T_k(\tau u_n-\tau\varphi).
\end{aligned}$$

By the Lebesgue theorem, we have

$$\lim_n\int_\Omega f_nT_k(u_n-\varphi)+\lim_n\int_{\partial\Omega}g_n\tau T_k(u_n-\varphi)=\int_\Omega fT_k(u-\varphi)+\int_{\partial\Omega}g\tau T_k(u-\varphi).$$

Next we prove that

$$\liminf_n\int_\Omega\big(-u_n-\frac{1}{n}|u_n|^{p-2}u_n\big)T_k(u_n-\varphi)\leq\int_\Omega(-u)T_k(u-\varphi).$$

In view of the fact that $(\frac{1}{n}|u_n|^{p-2}u_n)$ converges to 0 in $L^1(\Omega)$, we have

$$\begin{aligned}
&\liminf_n\int_\Omega -u_nT_k(u_n-\varphi)\\
&\leq\limsup_n\int_\Omega -u_nT_k(u_n-\varphi)\\
&\leq\limsup_n\int_\Omega -(u_n-\varphi)T_k(u_n-\varphi)-\lim_n\int_\Omega\varphi T_k(u_n-\varphi))
\end{aligned}$$

$$\begin{aligned}&\leq \int_\Omega \limsup_n \big[-(u_n-\varphi)T_k(u_n-\varphi)\big] - \int_\Omega \varphi T_k(u-\varphi)\\ &= \int_\Omega -uT_k(u-\varphi).\end{aligned}$$

In the same way, we have,

$$\liminf_n \int_{\partial\Omega} -\tau u_n \tau T_k(u_n-\varphi) \leq \int_{\partial\Omega} -\tau u \tau T_k(u-\varphi).$$

This completes the proof of (i).

(ii). If u is an entropy solution to (1.4) with data $F=(f,g)\in X_1$, and $h,k>0$, then by taking $T_{k+h}(u-T_ku)$ as test functions in (3.7), and applying (H2), we obtain

$$C_1\int_{\{h\leq|u|\leq k+h\}}|Du|^p \leq k\int_{\{h\leq|u|\leq k+h\}}|f| + k\int_{\{h\leq|\tau u|\leq k+h\}}|g| \leq k\|F\|_1.$$

In particular (2.3) while taking $h=0$.

We deduce, then from (2.5) and the condition $\tau u\in L^1(\partial\Omega)$ in (3.6) that,

$$\lim_{h\to+\infty}\int_{\{h\leq|u|\leq k+h\}}|Du|^p = 0. \tag{3.11}$$

Next, if $u_1,u_2\in \mathcal{T}^{1,p}_{\mathrm{tr}}(\Omega)$ are two entropy solutions to (1.4) with the same data (f,g), by taking the same decomposition as in [3], for a fixed k,

$$\begin{aligned}S_1(h) &= \{|u_1-u_2|\leq k\}\cap[\{|u_1|<h\}\cup\{|u_2|<h\}]\\ S_2(h) &= \{|u_1-u_2|\leq k\}\cap[\{|u_1|\geq h\}\cup\{|u_2|<h\}]\\ S_2'(h) &= \{|u_1-u_2|\leq k\}\cap[\{|u_2|\geq h\}\cup\{|u_1|<h\}],\end{aligned}$$

and selecting $\varphi=T_hu_2$ as test function in the equation related to u_1, we have

$$\begin{aligned}&\int_{\{|u_1-T_hu_2|\leq k\}\cap\{|u_2|<h\}}\langle \mathbf{a}(.,Du_1),Du_1-Du_2\rangle\\ &+\int_{\{|u_1-T_hu_2|\leq k\}\cap\{|u_2|\geq h\}}\langle \mathbf{a}(.,Du_1),Du_1\rangle\\ &\leq \int_\Omega fT_k(u_1-T_hu_2)+\int_{\partial\Omega} gT_k(\tau u_1-\tau T_hu_2)\end{aligned}$$

Then, taking into account that

$$\begin{gathered}\int_\Omega \langle \mathbf{a}(.,Du_1),Du_1\rangle \mathbf{1}_{\{|u_1-T_hu_2|\leq k\}}\mathbf{1}_{\{|u_2|\geq h\}}\geq 0,\\ \int_{S_2}\langle \mathbf{a}(.,Du_1),-Du_2\rangle \leq \int_{S_2}\langle \mathbf{a}(.,Du_1),Du_1-Du_2\rangle,\end{gathered}$$

we have

$$\begin{aligned}&\int_{S_2}\langle \mathbf{a}(.,Du_1),Du_2\rangle + \int_{S_1}\langle \mathbf{a}(.,Du_1),Du_1-Du_2\rangle\\ &\leq \int_\Omega fT_k(u_1-T_hu_2)+\int_{\partial\Omega} gT_k(\tau u_1-\tau T_hu_2)\end{aligned}$$

On the other hand, if $S_3=\{h-k\leq|u_2|<h\}$ and $S_4=\{h\leq|u_1|\leq h+k\}$, then

$$\Big|\int_{S_2}\langle \mathbf{a}(.,Du_1),-Du_2\rangle\Big| \leq C_3\|Du_2\|_{L^p(S_3)}\Big(\|h_0\|_{L^{p'}(S_4)}+\||Du_1|^{p-1}\|_{L^{p'}(S_4)}\Big)$$

With the help of (2.4) and (2.5) we conclude that
$$\lim_{h\to+\infty}\int_{S_2(h)}\langle \mathbf{a}(.,Du_1),-Du_2\rangle = 0. \tag{3.12}$$
Next, we do the same for the equation related to u_2, with test function $\varphi = T_h u_1$ an add the two inequalities:
$$\begin{aligned}
&\liminf_{h\to+\infty}\int_\Omega \langle \mathbf{a}(.,Du_1)-\mathbf{a}(.,Du_2),Du_1-Du_2\rangle \mathbf{1}_{S_1(h)}\\
&+\lim_{h\to+\infty}\int_{S_2(h)}\langle \mathbf{a}(.,Du_1),-Du_2\rangle + \lim_{h\to+\infty}\int_{S_2'(h)}\langle \mathbf{a}(.,Du_2),-Du_1\rangle\\
&\le \lim_{h\to+\infty}\int_\Omega f[T_k(u_1-T_hu_2)+T_k(u_2-T_hu_1)]\\
&\quad+\lim_{h\to+\infty}\int_{\partial\Omega} g[T_k(\tau u_1-\tau T_hu_2)+T_k(\tau u_2-\tau T_hu_1)].
\end{aligned}$$
Then, by applying the Lebesgue dominated convergence on the right, and Fatou lemma on the left, taking into account (3.12), we obtain,
$$\int_{\{|u_1-u_2|<k\}}\langle \mathbf{a}(.,Du_1)-\mathbf{a}(.,Du_2),Du_1-Du_2\rangle = 0,\ k>0\,.$$
It arises from (H3), that $Du_1 = Du_2$, a.e. in Ω. □

4. An order preserving property

Theorem 4.1. *(i) If $u_1, u_2 \in \mathcal{T}^{1,p}_{\rm tr}(\Omega)$ are entropy solutions for*
$$\begin{gathered}
-\operatorname{div}[\mathbf{a}(.,Du_i)] = f_i, \quad \text{in } \Omega\\
\frac{\partial u_i}{\partial \nu_a} = g_i \quad \text{on } \partial\Omega,
\end{gathered} \tag{4.1}$$
$i=1,2$, and $\varphi = \operatorname{sign}_0(u_1-u_2)$, $\psi = \operatorname{sign}_0(\tau u_1-\tau u_2)$, then we have the following order preserving property:
$$\int_{\Omega\cap\{u_1-u_2\}}|f_1-f_2| + \int_{\partial\Omega\cap\{\tau u_1-\tau u_2\}}|g_1-g_2| + \int_\Omega (f_1-f_2)\varphi + \int_{\partial\Omega}(g_1-g_2)\psi \ge 0 \tag{4.2}$$
(ii) If furthermore, $U_i = (u_i,\tau u_i) \in Dom(A_1)$, then for every $\varphi \in \operatorname{sign}(u_1-u_2)$ and $\psi \in \operatorname{sign}(\tau u_1-\tau u_2)$, we have
$$\int_\Omega (f_1-f_2)\varphi + \int_{\partial\Omega}(g_1-g_2)\psi \ge 0\,. \tag{4.3}$$

Proof. (i) If $U_i = (u_i,\tau u_i) \in D(A)$, is an entropy solution for the problem (4.1) with data $F_i = (f_i,g_i) \in X_1$, $i=1,2$, then from Theorem 3.6, there exist $V_{i,n} = (v_{i,n},\tau v_{i,n}) \in X_1$ such that $V_{i,n}$ is an entropy solution for,
$$\begin{gathered}
\frac{1}{n}v_{n,i} - \operatorname{div}[\mathbf{a}(.,Dv_{n,i})] = f_i \quad \text{in } \Omega\\
\frac{1}{n}\tau v_{n,i} + \frac{\partial v_{n,i}}{\partial \nu_{\mathbf{a}}} = g_i \quad \text{on } \partial\Omega\,, \quad n\in\mathbb{N}.
\end{gathered}$$
By taking $\varphi = 0$ in the entropy condition and applying (H2), we have
$$\frac{C_1}{k}\int_{\{|u_{n,i}|<k\}}|Dv_{n,i}|^p \le \big\|F_{n,i}-\frac{1}{n}V_{n,i}\big\|_1 \le 2\|F_i\|_1$$

Then applying the same proof as in the previous section, we assume then that $(v_{n,i})_n$ converges in measure and a.e. to some $w_i \in M^{p_1}(\Omega)$, $\tau v_{n,i}$ converges $d\sigma$ a.e. to τw_i. Thus $(\frac{1}{n} v_{n,i})_n$ and $(\frac{1}{n}\tau v_{n,i})_n$ converge a.e. to 0 and $\mathbf{a}(., Dv_{n,i})$ converges in $L^1(\Omega)$ to $\mathbf{a}(., Dw_i)$, where w_i is an entropy solution to the problem

$$\begin{gathered} -\operatorname{div}[\mathbf{a}(., Dw_i)] = f_i \quad \text{in } \Omega \\ \frac{\partial w_i}{\partial \nu_{\mathbf{a}}} = g_i \quad \text{on } \partial\Omega \end{gathered}$$

Applying again Theorem 3.6, we have $Dw_i = Du_i$ a.e. Hence, there exist some constants c_1, $c_2 \in \mathbb{R}$, such that $w_i = u_i + c_i$. Consider, then the sequences

$$\begin{gathered} u_{n,i} = v_{n,i} - c_i, \quad f_{n,i} = f_i - \frac{1}{n} u_{n,i} - \frac{1}{n} c_i, \\ g_{n,i} = g_i - \frac{1}{n}\tau u_{n,i} - \frac{1}{n} c_i, \quad \varphi_n = \operatorname{sign}_0(u_{n,1} - u_{n,2}), \\ \psi_n = \operatorname{sign}_0(\tau u_{n,1} - \tau u_{n,2}), \quad i = 1, 2,\ n \in \mathbb{N}\,. \end{gathered}$$

Then $u_{n,i}$ is an entropy solution to

$$\begin{gathered} -\operatorname{div}[\mathbf{a}(., Du_{n,i})] = f_{n,i} \quad \text{in } \Omega \\ \frac{\partial u_{n,i}}{\partial \nu_{\mathbf{a}}} = g_{n,i} \quad \text{on } \partial\Omega, \quad n \in \mathbb{N}. \end{gathered}$$

Since $(u_{n,i}, \tau u_{n,i}) \in X_1$, from (3.3), we have

$$\int_\Omega (f_{n,1} - f_{n,2})\varphi_n + \int_{\partial\Omega} (g_{n,1} - g_{n,2})\psi_n \geq 0.$$

After suppressing a negative part on the left, this leads to,

$$\int_\Omega (f_1 - f_2)\varphi_n + \int_{\partial\Omega} (g_1 - g_2)\psi_n - \frac{1}{n}(c_1 - c_2)\big[\int_\Omega \varphi_n + \int_{\partial\Omega} \psi_n\big] \geq 0.$$

In particular,

$$\begin{aligned} &\int_{\{u_1 = u_2\}} |f_1 - f_2| + \int_{\{\tau u_1 = \tau u_2\}} |g_1 - g_2| \\ &+ \lim_{n\to+\infty} \int_{\{u_1 \neq u_2\}} (f_1 - f_2)\varphi_n + \lim_{n\to+\infty} \int_{\{\tau u_1 \neq \tau u_2\}} (g_1 - g_2)\psi_n \geq 0 \end{aligned} \tag{4.4}$$

Since $(u_{n,i})_n$ converges in measure and a.e. to u_i and $\tau u_{n,i}$ converges $d\sigma$ -a.e. to τu_i, then we have

$$\begin{gathered} \lim_{n\to+\infty} \varphi_n = \operatorname{sign}_0(u_1 - u_2), \quad \text{a.e. on the set } \{u_1 - u_2 \neq 0\}, \\ \lim_{n\to+\infty} \psi_n = \operatorname{sign}_0(\tau u_1 - \tau u_2), \quad \text{a.e. on the set} \{\tau u_1 - \tau u_2 \neq 0\}\,. \end{gathered}$$

Then passing to the limit $n \to +\infty$ in (4.4), we obtain (4.2).
(ii) This is exactly the same as in [18, theorem 4.1.(ii)], while changing the integrals on $\mathbb{R}^N$ by integrals on Ω. $\square$

5. Existence and uniqueness

Theorem 5.1. *If β, γ are non decreasing continuous functions on $\mathbb{R}$ such that $\beta(0) = \gamma(0) = 0$ and $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$, then there exists an entropy solution $u \in T^{1,p}_{\rm tr}(\Omega)$ to the problem*

$$\begin{gathered} -\operatorname{div}[\mathbf{a}(., Du)] + \beta(u) = f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu_{\mathbf{a}}} + \gamma(\tau u) = g \quad \text{on } \partial\Omega \end{gathered} \tag{5.1}$$

with, $(\beta(u), \gamma(\tau u)) \in X_1$ and $\|(\beta(u), \gamma(\tau u))\|_1 \leq \|(f, g)\|_1$ and u is unique, up to an additive constant. Furthermore, if β or γ is one-to-one, then the entropy solution is unique.

Proof. Existence: Let E, and X_0, be the same spaces, and A_0 the same operator on X_0 as in the proof of the Theorem 3.6. Then we define the sequence $(\Phi_n)_n$ of convex and lower semi-continuous functions in X_0, as follows:

$$j_\beta(r) = \int_0^r \beta(s)ds, \quad j_\gamma(r) = \int_0^r \gamma(s)ds$$

and for $U = (u, v) \in X_0$,

$$\Phi_n(u, v) = \begin{cases} \frac{1}{2}\big[\int_\Omega j_\beta(u) + \int_{\partial\Omega} j_\gamma(\tau u)\big] + \frac{1}{np}\int_\Omega |u|^p dx, \\ \qquad \text{if } j_\beta(u) \in L^1(\Omega) \text{ and } j_\gamma(\tau u) \in L^1(\partial\Omega) \\ +\infty \quad \text{otherwise.} \end{cases}$$

Let $F_n = (f_n, g_n) = (T_n f, T_n g) \in E' \cap X_1$. Applying again, [7, Corollary 30], there exits $U_n = (u_n, \tau u_n) \in X_0$, a solution for

$$\begin{aligned} &\int_\Omega \langle a(., Du_n), Dv\rangle + \int_\Omega \beta(u_n)v + \int_{\partial\Omega} \gamma(\tau u_n).\tau v \\ &+ \frac{1}{n}\int_\Omega |u_n|^{p-2}u_n.v + \frac{1}{n}\int_\Omega |u_n|^{p-2}u_n.v \\ &= \int_\Omega f_n v + \int_{\partial\Omega} g_n v, \quad \text{for all } V = (v, \tau v) \in X_0. \end{aligned} \tag{5.2}$$

If $\tilde{F}_n = (f_n - \beta(u_n) - \frac{1}{n}|u_n|^{p-2}u_n, g_n - \gamma(\tau u_n))$, then $\|\tilde{F}_n\|_1 \leq 3\|F\|_1$. Thus we obtain, as previously,

$$C_1 \int_{\{h \leq |u| \leq k+h\}} |Du_n|^p \leq 3k\|F\|_1.$$

We assume that $(u_n)_n$ converges in measure to some u, that $\frac{1}{n}|u_n|^{p-2}u_n$ converges to 0 in $L^1(\Omega)$. Then applying (4.2), we have

$$\begin{aligned} &\int_\Omega |\beta(u_n) - \beta(u_m)| + \int_{\partial\Omega} |\gamma(\tau u_n) - \gamma(\tau u_n)| \\ &\leq \frac{1}{n}\int_\Omega |u_n|^{p-1} + \frac{1}{m}\int_\Omega |u_m|^{p-1} + \int_\Omega |f_n - f_m| + \int_{\partial\Omega} |g_n - g_m| \end{aligned}$$

Hence, $(\beta(u_n), \gamma(\tau u_n))_n$ is a Cauchy sequence in X_1.

The rest of the proof of existence and uniqueness up to a constant of a solution for (1.1) and the entropy condition is the same as for Theorem 3.6, and finally, by Fatou lemma, we have, $\|(\beta(u), \gamma(\tau u))\|_1 \leq \|(f, g)\|_1$.

Uniqueness: Applying again (4.2) we have uniqueness for the nonlinear perturbation $(\beta(u), \gamma(\tau u))$. Thus we have uniqueness up to a constant for the solution u. If u_1, u_2 are two entropy solutions and $u_2 = u_1 + c$, then, $\tau u_2 = \tau u_1 + c$. Thus $c = 0$, if β or γ is one-to-one. $\square$

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