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# ALEKSANDROV-TYPE ESTIMATES FOR A PARABOLIC MONGE-AMPÈRE EQUATION

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Abstract. A classical result of Aleksandrov allows us to estimate the size of a convex function $u$ at a point $x$ in a bounded domain $\Omega$ in terms of the distance from $x$ to the boundary of $\Omega$ if $\int_\Omega \det D^2 u\, dx < \infty$. This estimate plays a prominent role in the existence and regularity theory of the Monge-Ampère equation. Jerison proved an extension of Aleksandrov's result that provides a similar estimate, in some cases for which this integral is infinite. Gutiérrez and Huang proved a variant of the Aleksandrov estimate, relevant to solutions of a parabolic Monge-Ampère equation. In this paper, we prove Jerison-like extensions to this parabolic estimate.

## 1. Introduction

In studying the regularity and existence of weak solutions (in the sense of Aleksandrov) to the Dirichlet problem for the Monge-Ampère equation:

$$\begin{gathered} \det D^2 u = \mu \quad \text{in } \Omega, \\ u|_{\partial\Omega} = g, \end{gathered} \tag{1.1}$$

where $\mu$ is a Borel measure on the convex domain $\Omega$ and $g \in C(\partial\Omega)$, the following estimate of Aleksandrov plays a critical role. For its applications to this problem, see, for example, [12], [3], and [6]. A variant of this estimate appears in [2].

**Theorem 1.1** (Aleksandrov's estimate)**.** *Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$, and let $u \in C(\bar\Omega)$ be convex, with $u = 0$ on $\partial\Omega$. Then for all $x \in \Omega$,*

$$|u(x)|^n \le C_n (\operatorname{diam}\Omega)^{n-1} \operatorname{dist}(x, \partial\Omega)\, Mu(\Omega), \tag{1.2}$$

*where $C_n$ is a dimensional constant and $Mu$ is the Monge-Ampère measure associated to $u$.*

This estimate allows one to estimate the size of $u$ at a point $x$ in terms of the distance from $x$ to the boundary of the domain. However, if $u$ is such that $Mu(\Omega) = \infty$ (which can occur if $|Du| \to \infty$ at $\partial\Omega$), (1.2) does not give any information about the size of $u(x)$. Jerison, in [10], extended this inequality, using an affine-invariant normalized distance to the boundary, to an estimate (Theorem 2.6) that is useful even if $Mu(\Omega) = \infty$, provided $Mu$ does not blow up too quickly at the boundary.

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This result allows for a Caffarelli-style regularity theory for such problems, provided $Mu$ satisfies a technical requirement, weaker than the doubling condition, on the cross-sections of $u$; see [10] and [7].

The parabolic Monge-Ampère operator $u_t \det D_x^2 u$ was introduced in [11]. It is related to the problem of deformation of surfaces by Gauss curvature (see [13]). This operator is also considered in the following works: [16, 8, 5, 4, 14, 9, 15].

When studying entire solutions of the parabolic Monge-Ampère equation $-u_t \det D_x^2 u = 1$, Gutiérrez and Huang ([8]) extended the Aleksandrov estimate (Theorem 1.1) to parabolically convex functions on bounded bowl-shaped domains. This estimate again degenerates when the parabolic Monge-Ampère measure associated to $u$ of the entire domain is infinite. The purpose of this note is to extend the estimates of Jerison to the parabolic setting. These estimates are given below in Lemma 3.1 and Theorem 3.2. Because Jerison's estimates allow for a regularity theory for problem (1.1) when $\mu(\Omega) = \infty$, it is our hope that the estimates presented here will allow one to deduce regularity properties of parabolically convex solutions of the Dirichlet problem:

$$\begin{gathered} -u_t \, \det D^2 u = f \quad \text{in } E \\ u\big|_{\partial_p E} = g, \end{gathered}$$

where $f \geq 0$ may fail to be in $L^1(E)$, $E \subset \mathbb{R}^{n+1}$ is bowl-shaped, and $\partial_p E$ is the parabolic boundary of $E$. This would extend the regularity theory found in [5, 4, 14, 15], all of which assume that $f$ is bounded.

## 2. Preliminaries

We begin this section by reviewing the basic theory of weak or generalized solutions, in the Aleksandrov sense, to the (elliptic) Monge-Ampère equation. Proofs of these results and historical notes indicating their original sources can be found in the books [1] and [6].

Given $u : \Omega \to \mathbb{R}$ we recall that the normal mapping (or subgradient) of $u$ is defined by

$$\partial u(x_0) = \{p \in \mathbb{R}^n : u(x) \geq u(x_0) + p \cdot (x - x_0), \ \forall\, x \in \Omega\};$$

and if $E \subset \Omega$, then we set $\partial u(E) = \bigcup_{x \in E} \partial u(x)$. Note that the normal map of $u$ at a point $x_0$ is the set of points $p$ which are normal vectors for supporting hyperplanes to the graph of $u$ at $x_0$.

If $\Omega$ is open and $u \in C(\Omega)$ then the family of sets

$$S = \{E \subset \Omega : \partial u(E) \text{ is Lebesgue measurable}\}$$

is a Borel $\sigma$–algebra. The map $Mu : S \to \bar{\mathbb{R}}$ defined by $Mu(E) = |\partial u(E)|$ (where $|S|$ indicates the Lebesgue measure of the set $S$) is a measure, finite on compact subsets, called the Monge–Ampère measure associated with the function $u$. The convex function $u$ is a weak (Aleksandrov) solution of $\det D^2 u = \nu$ if the Monge–Ampère measure $Mu$ associated with $u$ equals the Borel measure $\nu$.

We use the notation $B_r(y)$ for the open Euclidean ball of radius $r$ with center $y$. The dimension of $B_r(y)$ should be clear from context.

**Definition 2.1.** A convex domain $\Omega \subset \mathbb{R}^n$ with center of mass at the origin is said to be normalized if $B_{\alpha_n}(0) \subset \Omega \subset B_1(0)$, where $\alpha_n = n^{-3/2}$.

The following lemma allows us to carry out our analysis in a normalized setting. It is a consequence of a result of John on ellipsoids of minimum volume. See Section 1.8 of [6] and its references for more detail.

**Lemma 2.2.** *If $\Omega$ is a bounded convex domain, there exists an affine transformation $T$ such that $T(\Omega)$ is normalized.*

We now introduce the normalized distance to the boundary used by Jerison in [10].

**Definition 2.3.** Let $\Omega \subset \mathbb{R}^n$ be bounded, open and convex. The normalized distance from $x \in \Omega$ to the boundary of $\Omega$ is

$$\delta(x,\Omega) = \min\Big\{\frac{|x-x_1|}{|x-x_2|} : x_1,\, x_2 \in \partial\Omega \text{ and } x, x_1, x_2 \text{ are collinear}\Big\}.$$

The most important properties of this distance for our purposes are summarized in the following lemma.

**Lemma 2.4.** *Let $\Omega$ be a bounded convex domain.*

(a) *If $T$ is an invertible affine transformation on $\mathbb{R}^n$, then*

$$\delta(x,\Omega) = \delta(Tx, T(\Omega)).$$

(b) *If $\Omega$ is normalized, $\delta(x,\Omega)$ is equivalent to* $\operatorname{dist}(x,\partial\Omega)$*, i.e. there exist constants $C_1$ and $C_2$ (depending only on the dimension) such that*

$$C_1\delta(x,\Omega) \le \operatorname{dist}(x,\partial\Omega) \le C_2\delta(x,\Omega)$$

*for all $x \in \Omega$, where* dist *is the Euclidean distance.*

(c) *For all $x \in \Omega$,* $\operatorname{dist}(x,\partial\Omega) \le \operatorname{diam}(\Omega)\delta(x,\Omega)$.

We now state Jerison's estimates. The first (Lemma 2.5) is [10, Lemma 7.2]. Estimate (2.1) is similar to Aleksandrov's estimate (1.2), with the normalized notion of distance replacing the standard one, and the Lebesgue measure of $\Omega$ replacing the diameter term.

**Lemma 2.5.** *Let $\Omega$ be an open convex set and suppose $u \in C(\overline{\Omega})$ is convex and zero on $\partial\Omega$. Then, for all $x \in \Omega$,*

$$|u(x)|^n \le C\delta(x,\Omega)|\Omega| Mu(\Omega) \tag{2.1}$$

*where $C$ is a constant depending only on the dimension.*

Note that the estimate (2.1) gives no information when $Mu(\Omega) = \infty$. If this is the case, $Mu$ must blow up near $\partial\Omega$, but this is precisely where $\delta(\cdot,\Omega)$ is small. As a consequence, the estimate in the next result ([10, Lemma 7.3]) may be meaningful.

**Theorem 2.6.** *Let $\Omega$ be bounded, open, convex and normalized, and suppose $u \in C(\overline{\Omega})$ is convex and zero on $\partial\Omega$. For each $\epsilon \in (0,1]$, there exists a constant $C(n,\epsilon)$ such that*

$$|u(x_0)|^n \le C(n,\epsilon)\delta(x_0,\Omega)^\epsilon \int_\Omega \delta(x,\Omega)^{1-\epsilon}\, dMu(x) \tag{2.2}$$

*for all $x_0 \in \Omega$.*

We now introduce some terminology and notation for the parabolic problem. Let $D \subset \mathbb{R}^{n+1}$ and let $t \in \mathbb{R}$. Then define

$$D(t) = \{x \in \mathbb{R}^n : (x,t) \in D\}.$$

**Definition 2.7.** The domain $D$ is said to be bowl-shaped if $D(t)$ is convex for every $t$ and $D(t_1) \subset D(t_2)$ whenever $t_1 \leq t_2$. If $D$ is bounded, let $t_0 = \inf\{t : D(t) \neq \emptyset\}$. Then the parabolic boundary of $D$ is defined to be

$$\partial_p D = (\bar{D}(t_0) \times \{t_0\}) \cup \Big( \bigcup_{t \in \mathbb{R}} (\partial D(t) \times \{t\}) \Big).$$

For a bowl-shaped domain $D$ we define the set $D_{t_0}$ to be $D_{t_0} = D \cap \{(x,t) : t \leq t_0\}$.

**Definition 2.8.** A function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $u = u(x,t)$, is called parabolically convex (or convex-monotone) if it is continuous, convex in $x$ and non-increasing in $t$.

We now define the parabolic normal map and parabolic Monge-Ampère measure. As in the elliptic case, this will lead to the notion of weak solution for this operator. Let $D \subset \mathbb{R}^{n+1}$ be an open, bounded bowl-shaped domain, and $u$ be a continuous real-valued function on $D$. The parabolic normal mapping of $u$ at a point $(x_0, t_0)$ is the set-valued function $P_u(x_0, t_0)$ given by

$$\begin{gathered}\{(p,h) : u(x,t) \geq u(x_0,t_0) + p \cdot (x - x_0) \text{ for all } t \leq t_0 \text{ and } x \in D(t), \\ h = p \cdot x_0 - u(x_0,t_0)\}.\end{gathered}$$

As before, the parabolic normal mapping of a set $E \subset D$ is defined to be the union of the parabolic normal maps of each point in the set. The family of subsets $E$ of $D$ for which $P_u(E)$ is Lebesgue measurable is a Borel $\sigma$-algebra and the map $M_p(E) = |P_u(E)|$ is a measure, called the parabolic Monge-Ampère measure associated to the function $u$. These results are proved in [16]. We remark that, because of the translation invariance of the Lebesgue measure, the parabolic Monge-Ampère measure of a function $u$ is identical to the parabolic Monge-Ampère measure of $u - \lambda$ for any constant $\lambda$.

We conclude this section with a parabolic analog of Aleksandrov's estimate (Theorem 1.1) due to Gutiérrez and Huang ([8]).

**Theorem 2.9.** *Let $D \subset \mathbb{R}^{n+1}$ be an open bounded bowl-shaped domain, and let $u \in C(\bar{D})$ be a parabolically convex function with $u = 0$ on $\partial_p D$. If $(x_0, t_0) \in D$, then*

$$|u(x_0,t_0)|^{n+1} \leq C_n \operatorname{dist}(x_0, \partial D(t_0)) \operatorname{diam}(D(t_0))^{n-1} M_p(D_{t_0})$$

*where $C_n$ is a dimensional constant, and $M_p$ is the parabolic Monge-Ampère measure associated to $u$.*

## 3. Parabolic estimates

In this section, we prove parabolic versions of Jerison's estimates. We adapt the arguments given in [10] to our situation. The first is the analog of Lemma 2.5.

**Lemma 3.1.** *Let $D$ be a bounded, open bowl-shaped domain in $\mathbb{R}^{n+1}$. Suppose $u \in C(\bar{D})$ is parabolically convex and $u|_{\partial_p D} = 0$. Then there exists a dimensional constant $C_n$ such that*

$$|u(x_0,t_0)|^{n+1} \leq C_n \delta(x_0, D(t_0)) |D(t_0)| |P_u(D_{t_0})|$$

*for all $(x_0, t_0) \in D$, where $\delta(x_0, D(t_0))$ is the normalized distance from $x_0$ to the boundary of the $n$-dimensional convex set $D(t_0)$, and $|P_u(D_{t_0})| = M_p(D_{t_0})$ is the Lebesgue measure of the set $P_u(D_{t_0}) \subset \mathbb{R}^{n+1}$.*

*Proof.* $D(t_0)$ is a bounded convex subset of $\mathbb{R}^n$. By Lemma 2.2, we may choose an affine transformation $T$ of $\mathbb{R}^n$ that normalizes $D(t_0)$. Define $\tilde{T}:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ by $\tilde{T}(x,t)=(Tx,t)$. Then $\tilde{T}(D_{t_0})\subset B_1(0)\times(-\infty,t_0]$. Let $v(z)=u(\tilde{T}^{-1}z)$ for $z\in\tilde{T}(D)$. Then $\tilde{T}(D)$ is a bowl-shaped domain, $v$ is continuous on the closure of $\tilde{T}(D)$, is parabolically convex, and is zero on $\partial_p\tilde{T}(D)$.

Now apply the parabolic Aleksandrov estimate (Theorem 2.9) to $v$ in $\tilde{T}(D)$ to obtain

$$\begin{aligned}|u(x_0,t_0)|^{n+1}&=|v(\tilde{T}(x_0,t_0))|^{n+1}\\&\le C_n\operatorname{dist}(Tx_0,\partial\tilde{T}(D(t_0)))[\operatorname{diam}(\tilde{T}(D(t_0)))]^{n-1}|P_v(\tilde{T}(D_{t_0}))|.\end{aligned}\tag{3.1}$$

Next, we establish the change-of-variable formula

$$|P_v(\tilde{T}(D_{t_0}))|=|\det T^{-1}|\,|P_u(D_{t_0})|.\tag{3.2}$$

For simplicity, we make the following abuse of notation: when we write $u$ or $v$ as functions of $x$ only, we mean the restrictions of $u$ and $v$ to $D(t_0)$. Let $p\in\partial u(x_0)$. Then

$$u(x,t_0)\ge u(x_0,t_0)+p\cdot(x-x_0)$$

for all $x\in D(t_0)$. Since $u$ is non-increasing in $t$,

$$u(x,t)\ge u(x,t_0)\ge u(x_0,t_0)+p\cdot(x-x_0)$$

for all $t\le t_0$ and $x\in D(t)$, so $(p,h)\in P_u(x_0,t_0)$ where $h=p\cdot x_0-u(x_0,t_0)$. If $p\notin\partial u(x_0)$, then $(p,h)\notin P_u(x_0,t_0)$; therefore, $p\in\partial u(x_0)$ if and only if $(p,h)\in P_u(x_0,t_0)$. It is not hard to see that $p\in\partial u(x_0)$ if and only if $(T^{-1})^tp\in\partial v(Tx_0)$. Then as above, for $t\le t_0$ and $y\in\tilde{T}(D)(t)$,

$$v(y,t)\ge v(y,t_0)+(T^{-1})^tp\cdot(y-Tx_0).$$

Hence, $(T^{-1})^tp\in\partial v(Tx_0)$ if and only if $((T^{-1})^tp,\tilde{h})\in P_v(Tx_0,t_0)$, where $\tilde{h}=(T^{-1})^tp\cdot Tx_0-v(Tx_0,t)=p\cdot x_0-u(x_0)=h$. In other words, $(p,h)\in P_u(x_0,t_0)$ if and only if $((T^{-1})^tp,h)\in P_v(Tx_0,t_0)$. We also have $((T^{-1})^tp,h)=(\tilde{T}^{-1})^t(p,h)$ which implies that

$$(\tilde{T}^{-1})^tP_u(E)=P_v(\tilde{T}(E))$$

for any Borel set $E\subset D$. In particular, $(\tilde{T}^{-1})^tP_u(D_{t_0})=P_v(\tilde{T}(D_{t_0}))$. This implies that

$$|\det\tilde{T}^{-1}|\,|P_u(D_{t_0})|=|P_v(\tilde{T}(D_{t_0}))|,$$

but $\det\tilde{T}^{-1}=\det T^{-1}$, showing (3.2). Then using equation (3.2), Lemma 2.4, inequality (3.1), and the fact that $|\det T^{-1}|\le C(n)|D(t_0)|_n$, we prove the claimed estimate:

$$\begin{aligned}|u(x_0,t_0)|^{n+1}&\le C_n\delta(Tx_0,T(D(t_0)))|P_v(\tilde{T}(D_{t_0}))|\\&=C_n\delta(x_0,D(t_0))|P_v(\tilde{T}(D_{t_0}))|\\&=C_n\delta(x_0,D(t_0))|\det T^{-1}||P_u(D_{t_0})|\\&\le C_n\delta(x_0,D(t_0))|D(t_0)||P_u(D_{t_0})|.\end{aligned}$$

□

The next result extends Theorem 2.6 to the parabolic setting.

**Theorem 3.2.** *Let $0 < \epsilon \le 1$. Let $E$ be a bounded open bowl-shaped domain in $\mathbb{R}^{n+1}$, such that $E \subset B_1(0) \times (-\infty, \infty)$. Suppose $u \in C(\bar{E})$ is parabolically convex and zero on $\partial_p E$. Let $M_p$ be the parabolic Monge-Ampère measure associated to $u$. Then there exists $C = C(\epsilon, n)$ such that*

$$|u(x_0,t_0)|^{n+1} \le C\delta(x_0, E(t_0))^\epsilon \int_{E_{t_0}} \delta(x, E(t_0))^{1-\epsilon}\, dM_p(x,t).$$

*for all $(x_0, t_0) \in E$.*

*Proof.* Without loss of generality, we may assume that $u(x_0,t_0) = -1$ (if this is not the case, multiply $u$ by a suitably chosen positive constant). Let $s_k = s2^{-k\beta}$ where $s$ and $\beta$ are positive and chosen to satisfy $\beta(n+1) \le \epsilon$ and $\sum_{k=1}^\infty s_k \le 1/2$.

$$A := \delta(x_0, E(t_0))^\epsilon \int_{E_{t_0}} \delta(x, E(t_0))^{1-\epsilon}\, dM_p(x,t).$$

It suffices to show that $A \ge C(s)$, a constant depending on $s$ and $\epsilon$.

For $k = 1, 2, \dots$, let $E_k = \{(x,t) \in E : u(x,t) \le \lambda_k = -1 + s_1 + \dots + s_k\}$. Define $E_0 = \{(x,t) \in E : u(x,t) \le -1\}$. Note that $E_k \subset E_{k+1}$ for $k = 1, 2, \dots$, and that $E_0 \ne \emptyset$. Each of the sets $E_k$ is bowl-shaped and $u|_{\partial_p E_k} = \lambda_k$ (taking $\lambda_0 = -1$). Fix $t$ and let $\delta_k(t) = \operatorname{dist}(\partial E_k(t), \partial E(t))$.

Since $\delta_k(t) \not\to 0$ as $k \to \infty$ (if $\delta_k(t) \to 0$, then $u$ would be smaller than $-\frac{1}{2}$ somewhere on $\partial_p E$), we may choose $k$ to be the smallest nonnegative integer for which $\delta_{k+1}(t) > \frac{1}{2}\delta_k(t)$.

Let $x_k \in \partial E_k(t)$ be a point closest to $\partial E(t)$. Then we have that

$$\operatorname{dist}(x_k, \partial E_{k+1}(t)) < \frac{1}{2}\delta_k(t) < \delta_{k+1}(t). \tag{3.3}$$

The second of these inequalities holds because of the choice of $k$. The first inequality requires the following geometric argument. Let $L$ be a line segment of length $\delta_k$ from $x_k$ to $\partial E(t)$. The segment $L$ meets $\partial E_{k+1}(t)$ at a point, $x_{k+1}$. Let $\ell$ represent the length of the part of $L$ that connects $\partial E_{k+1}(t)$ to $\partial E(t)$. Then

$$\begin{aligned}
\delta_k &= |x_k - x_{k+1}| + \ell \\
&\ge |x_k - x_{k+1}| + \operatorname{dist}(\partial E_{k+1}(t), \partial E(t)) \\
&= |x_k - x_{k+1}| + \delta_{k+1} \\
&> |x_k - x_{k+1}| + \frac{1}{2}\delta_k.
\end{aligned}$$

Therefore, $\frac{1}{2}\delta_k > |x_k - x_{k+1}| \ge \operatorname{dist}(x_k, \partial E_{k+1}(t))$. Now we apply Lemma 3.1 to the function $u(x,t) - \lambda_{k+1}$ on the set $E_{k+1}$ to get

$$|u(x_k,t) - \lambda_{k+1}|^{n+1} \le C_n \delta(x_k, E_{k+1}(t)) |E_{k+1}(t)|\, M_p((E_{k+1})_t).$$

The point $x_k \in \partial E_k(t)$, so $u(x_k,t) = \lambda_k$ and $|u(x_k,t) - \lambda_{k+1}| = |\lambda_k - \lambda_{k+1}| = s_{k+1}$. Thus,

$$s_{k+1}^{n+1} \le C_n \delta(x_k, E_{k+1}(t)) |E_{k+1}(t)|\, M_p((E_{k+1})_t). \tag{3.4}$$

Let $L_t$ be a shortest segment from $x_k$ to $\partial E_{k+1}(t)$ and let $z \in \partial E_{k+1}(t)$ be the other endpoint of $L_t$. Let $\rho$ denote

$$\rho = |L_t| = |x_k - z| = \operatorname{dist}(x_k, \partial E_{k+1}(t)). \tag{3.5}$$

Since the set $E_{k+1}(t)$ is convex, the hyperplane $\Pi$ (of dimension $n-1$) normal to $L_t$ through $z$ is a support plane for $E_{k+1}(t)$. Let $\Pi'$ be the support plane parallel to $\Pi$ on the opposite side of $E_{k+1}(t)$, so that $E_{k+1}(t)$ is contained between the two planes, and let $r = \operatorname{dist}(\Pi, \Pi')$. Then since $E_{k+1}(t) \subset B_1(0)$, there exists a constant $C = C(n)$ such that

$$|E_{k+1}(t)| \le Cr. \tag{3.6}$$

We remark that the $C$ in (3.6) can be chosen to be the volume of the unit ball in $\mathbb{R}^{n-1}$.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation normalizing $E_{k+1}(t)$. Then $\operatorname{dist}(T(\Pi), T(\Pi'))$ is bounded between two dimensional constants $C_1$ and $C_2$, with $C_1 < C_2$, and $C_1 \frac{\rho}{r} \le \operatorname{dist}(Tx_k, T(\Pi)) \le C_2 \frac{\rho}{r}$. By Lemma 2.4, we have

$$\begin{aligned}
\delta(x_k, E_{k+1}(t)) &= \delta(Tx_k, T(E_{k+1}(t))) \\
&\le C \operatorname{dist}(Tx_k, \partial T(E_{k+1}(t))) \\
&\le C \operatorname{dist}(Tx_k, T(\Pi)) \\
&\le C \frac{\rho}{r}.
\end{aligned}$$

Inserting this inequality into (3.4) and using (3.6), we get

$$s_{k+1}^{n+1} \le C\frac{\rho}{r}|E_{k+1}(t)|\, M_p((E_{k+1})_t) \le C\rho\, M_p((E_{k+1})_t) < C\delta_{k+1}(t)\, M_p((E_{k+1})_t),$$

where the last inequality holds since $\rho < \delta_{k+1}(t)$ (see (3.3) and (3.5)). Therefore,

$$s_{k+1}^{n+1} < C\delta_{k+1}(t)\, M_p((E_{k+1})_t). \tag{3.7}$$

Since $u$ is non-increasing in $t$ and $E_0(t_0) \neq \emptyset$, $\delta_0(t)$ is defined for any $t \ge t_0$. On the other hand, for some values of $t$, $\delta_0(t)$ might not be defined; for instance, this is the case when $u > -1$ on $E(t)$. Then for any $t \ge t_0$, by the choice of $k$, we have $\delta_{k+1}(t) < \delta_k(t) \le 2^{-k}\delta_0(t)$.

Since $\delta_0(t_0) \le \operatorname{dist}(x_0, \partial E(t_0))$ and $\operatorname{diam}(E(t_0)) \le 2$, we may conclude by Lemma 2.4*(c)* that $2^{-k}\delta_0(t_0) \le C2^{-k}\delta(x_0, E(t_0))$ for a dimensional constant $C$. Therefore,

$$\begin{aligned}
\delta_{k+1}(t_0)\, M_p((E_{k+1})_{t_0}) &= \delta_{k+1}(t_0)^{\epsilon} \int_{(E_{k+1})_{t_0}} \delta_{k+1}(t_0)^{1-\epsilon}\, dM_p(y,s) \\
&\le C2^{-k\epsilon}\delta(x_0, E(t_0))^{\epsilon} \int_{(E_{k+1})_{t_0}} \delta_{k+1}(t_0)^{1-\epsilon}\, dM_p(y,s) \\
&\le C2^{-k\epsilon}\delta(x_0, E(t_0))^{\epsilon} \int_{(E_{k+1})_{t_0}} \delta(y, E(t_0))^{1-\epsilon} dM_p(y,s).
\end{aligned} \tag{3.8}$$

The last inequality holds since

$$\delta_{k+1}(t_0) = \operatorname{dist}(\partial E_{k+1}(t_0), \partial E(t_0)) \le \operatorname{dist}(y, \partial E(t_0)) \le C\delta(y, E(t_0))$$

for all $y \in E_{k+1}(t_0)$. Then from (3.7) and (3.8) we obtain that

$$s_{k+1}^{n+1} \le C2^{-k\epsilon}\delta(x_0, E(t_0))^{\epsilon} \int_{(E_{k+1})_{t_0}} \delta(y, E(t_0))^{1-\epsilon}\, dM_p(y,s) \le C2^{-k\epsilon}A.$$

Recall that

$$s_{k+1}^{n+1} = s^{n+1}2^{-(n+1)(k+1)\beta} \ge s^{n+1}2^{-\epsilon(k+1)}$$

since $\beta(n+1) \le \epsilon$. Hence

$$s^{n+1}2^{-\epsilon(k+1)} \le C2^{-k\epsilon}A \Rightarrow s^{n+1} \le CA,$$

where $C$ depends on $\epsilon$, so $A \geq C(s)$ as desired. □

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