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VIABLE SOLUTIONS FOR SECOND ORDER NONCONVEX FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. We prove the existence of viable solutions for an autonomous second-order functional differential inclusions in the case when the multifunction that define the inclusion is upper semicontinuous compact valued and contained in the subdifferential of a proper lower semicontinuous convex function.

1. Introduction

Functional differential inclusions, well known as differential inclusions with memory, express the fact that the velocity of the system depends not only on the state of the system at given instant but depends upon the history of the trajectory until this instant. The class of functional differential inclusions contains a large variety of differential inclusions and control systems. In particular, this class covers the differential inclusions, the differential - difference inclusions and the Voltera inclusions. For a detailed discussion on this topic we refer to [2].

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$. Let σ be a positive number and $\mathcal{C}_\sigma := \mathcal{C}([-\sigma,0],\mathbb{R}^m)$ the Banach space of continuous functions from $[-\sigma,0]$ to $\mathbb{R}^m$ with the norm $\|x(.)\|_\sigma := \sup\{\|x(t)\|; t\in[-\sigma,0]\}$. For each $t\in[0,T]$, we define the operator $T(t)$ from $\mathcal{C}([-\sigma,T],\mathbb{R}^m)$ to $\mathcal{C}_\sigma$ as follows: $(T(t)x)(s) := x(t+s)$, $s\in[-\sigma,0]$. For a given nonempty subset K of $\mathbb{R}^m$ we introduce the set $\mathcal{K}_0 =: \{\varphi\in\mathcal{C}_\sigma; \varphi(0)\in K\}$.

The aim of this paper is to prove a viability result for the second order functional differential inclusion

$$x''\in F(T(t)x,x'),\quad (T(0)x,x'(0))=(\varphi_0,y_0)\in\mathcal{K}_0\times\Omega \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^m$ and $F:\mathcal{C}_\sigma\times\Omega\to 2^{\mathbb{R}^m}$ is an upper semicontinuous, compact valued multifunction such that $F(\psi,y)\subset\partial V(y)$ for every $(\psi,y)\in\mathcal{K}_0\times\Omega$ and V is a proper convex and lower semicontinuous function.

Bressan, Cellina and Colombo [4] prove the existence of local solutions to the Cauchy problem $x'\in F(x)$, $x(0)=x_0$, where F is upper semicontinuous, cyclically monotone and compact valued multifunction. While Rossi [22] prove a viability result for this problem. The first viability result for the first order functional

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differential inclusion was given by Haddad [13], [14] in the case when F is upper semicontinuous with convex compact values. Lupulescu [17] has been proved the local existence of solutions for nonconvex differential inclusion $x' \in F(T(t)x)$, $T(0)x = \varphi_0 \in \mathcal{C}_\sigma$, and the existence of viable solutions for this problem has been studied by Cernea and Lupulescu [9] in the case when F is upper semicontinuous compact values such that $F(\psi) \subset \partial V(\psi(0))$ for every $\psi \in \mathcal{K}_0$.

The first viability result for second order differential inclusions

$$x'' \in F(x, x'), \quad x(0) = x_0, x'(0) = y_0 \tag{1.2}$$

were given by Cornet and Haddad [10] in the case in which F is upper semicontinuous and with convex compact values. The nonconvex case has been studied by Lupulescu [16] and Cernea [7] in the finite dimensional case. The nonconvex case in Hilbert spaces has been studied by Ibrahim and Alkulaibi [15]. For other results, references and applications in this framework we refer to the papers: Casting [7], Auslender and Mechler [3], Aghezaaf and Sajid [1], Morchadi and Sajid [20], Marco and Murilio [18], Syam [23] and book of Motreanu and Pavel [19]. In the paper [12], Duc Ha and Monteiro Marques proved the several existence theorems for the nonconvex functional differential inclusions governed by the sweeping process.

2. Preliminaries and statement of the main result

For $x \in \mathbb{R}^m$ and $r > 0$ let $B(x, r) := \{y \in \mathbb{R}^m; \|y - x\| < r\}$ be the open ball centered at x with radius r, Ω and let $\overline{B}(x, r)$ be its closure. For $\varphi \in \mathcal{C}_\sigma$ let $B_\sigma(\varphi, r) := \{\psi \in \mathcal{C}_\sigma; \|\psi - \varphi\|_\sigma < r\}$ and $\overline{B}_\sigma(\varphi, r) := \{\psi \in \mathcal{C}_\sigma; \|\psi - \varphi\|_\sigma \leq r\}$. For $x \in \mathbb{R}^m$ and for a closed subset $A \subset \mathbb{R}^m$ we denote by $d(x, A)$ the distance from x to A given by $d(x, A) := \inf\{\|y - x\|; y \in A\}$.

Let $V : \mathbb{R}^m \to \mathbb{R}$ be a proper convex and lower semicontinuous function. The multifunction $\partial V : \mathbb{R}^m \to 2^{\mathbb{R}^m}$, defined by

$$\partial V(x) := \{\xi \in \mathbb{R}^m; V(y) - V(x) \geq \langle \xi, y - x\rangle, (\forall) y \in \mathbb{R}^m\},$$

is called subdifferential (in the sense of convex analysis) of the function V.

We say that a multifunction $F : \mathcal{K}_0 \times \Omega \to 2^{\mathbb{R}^m}$ is upper semicontinuous if for every $(\varphi, y) \in \mathcal{K}_0 \times \Omega$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(\psi, z) \subset F(\varphi, y) + B(0, \varepsilon),\ (\forall)(\psi, z) \in B_\sigma(\varphi, \delta) \times B(y, \delta).$$

This definition of the upper semicontinuous multifunction is less restrictive than the usual (see [2, Definition 1.1.1] or [11, Definition 1.1]). Actually such a property is called (ε, δ)-upper semicontinuity (see [11, Definition 1.2]) and it is only equivalent to the upper semicontinuity for compact-valued multifunctions (see [11, Proposition 1.1]).

For a multifunction $F : \mathcal{K}_0 \times \Omega \to 2^{\mathbb{R}^m}$ and for any $(\varphi, y) \in \mathcal{K}_0 \times \Omega$ we consider the functional differential inclusion

$$x'' \in F(T(t)x, x'), \quad T(0)x = \varphi_0, x'(0) = y_0 \tag{2.1}$$

under the following assumptions:

- (H1) K is a locally closed subset in $\mathbb{R}^m$, Ω is an open subset $\mathbb{R}^m$ and $F : \mathcal{K}_0 \times \Omega \to 2^{\mathbb{R}^m}$ is upper semicontinuous with compact values;
- (H2) There exists a proper convex and lower semicontinuous function $V : \mathbb{R}^m \to \mathbb{R}$ such that

$$F(\varphi, y) \subset \partial V(y) \text{ for every } (\varphi, y) \in \mathcal{K}_0 \times \Omega;$$

(H3) For every $(\varphi, y) \in \mathcal{K}_0 \times \Omega$ and for every $z \in F(\varphi, y)$ holds the following tangential condition:
$$\liminf_{h\downarrow 0} \frac{1}{h^2} d(\varphi(0) + hy + \frac{h^2}{2} z, K) = 0.$$

Remark 2.1. A convex function $V : \mathbb{R}^m \to \mathbb{R}$ is continuous in the whole space $\mathbb{R}^m$ [21, Corollary 10.1.1] and almost everywhere differentiable [21, Theorem 25.5]. Therefore, (H2) restricts strongly the multivaluedness of F.

Definition 2.2. By a viable solution of the functional differential inclusion (2.1) we mean any continuous function $x : [-\sigma, T] \to \mathbb{R}^m$, $T > 0$, that is absolutely continuous on $[0, T]$ with absolutely continuous derivative on $[0, T]$ such that $T(0)x = \varphi_0$ on $[-\sigma, T]$, $x'(0) = y_0$ and
$$\begin{gathered} x''(t) \in F(T(t)x, x'(t)), \quad \text{a.e. on } [0, T], \\ (x(t), x'(t)) \in K \times \Omega, \quad \text{for every } t \in [0, T]. \end{gathered}$$

Our main result is the following.

Theorem 2.3. *If Assumptions (H1)-(H3) are satisfied, then K is a viable domain for* (2.1).

3. Proof of the main result

We start this section with the following technical result, which will used to prove main result.

Lemma 3.1. *Assume that the hypotheses (H1) and (H3) are satisfied. Then, for each $(\varphi, y_0) \in \mathcal{K}_0 \times \Omega$ there exist $r > 0$ and $T > 0$ such that $K \cap B(\varphi(0), r)$ is closed and for each $k \in N^*$ there exist $m(k) \in N^*, t_k^p, x_k^p, y_k^p, z_k^p$ and a continuous function $x_k : [-\sigma, T] \to R^m$ such that for every $p \in \{0, 1, \dots, m(k) - 1\}$ we have:*

- (i) $h_k^p := t_k^{p+1} - t_k^p < \frac{1}{k}$ *and* $t_k^{m(k)-1} \le T < t_k^{m(k)}$,
- (ii) $x_k(t) = \varphi(t)$, *for every* $t \in [-\sigma, 0]$,
- (iii) $x_k(t) = x_k^p + (t - t_k^p) y_k^p + \frac{1}{2}(t - t_k^p)^2 z_k^p$, *for every* $t \in [t_k^p, t_k^{p+1}]$,
- (iv) $z_k^p \in F(T(t_k^p)x_k, y_k^p) + \frac{1}{k} B$,
- (v) $(x_k^p, y_k^p) \in Q_0$,
- (vi) $T(t_k^p)x_k \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$,

where $B := B(0, 1)$ and $Q_0 := (K \cap B(\varphi(0), r)) \times \overline{B}(y_0, r)$.

Remark 3.2. The following twp statements hold:

- (i) If $\alpha \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ then $\alpha(0) \in K \cap B(\varphi(0), r)$,
- (ii) If $K \cap B(\varphi(0), r)$ is closed in R^m then $\mathcal{K}_0 \cap B_\sigma(\varphi, r)$ is closed in C_σ.

Indeed, the first statement is obvious. For the second statement, we assume that $K \cap B(\varphi(0), r)$ is closed in R^m and we consider a sequence $(\alpha_n)_n$ in $\mathcal{K}_0 \cap B_\sigma(\varphi, r)$ that is convergent(in the norm $\| \cdot \|_\sigma$) to $\alpha \in \mathcal{C}_\sigma$. Then follows that $\alpha \in B_\sigma(\varphi, r)$, $\alpha_n(0) \to \alpha(0)$ and $\alpha_n(0) \in K \cap B(\varphi(0), r)$; therefore, since $K \cap B(\varphi(0), r)$ is closed, we obtain that $\alpha(0) \in K$ and thus $\alpha \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$.

Proof of Lemma 2.3. Let (φ, y_0) be arbitrary and fixed in $\mathcal{K}_0 \times \Omega$. Since K is locally closed in $\mathbb{R}^m$, there exists $r > 0$ such that $K \cap B(\varphi(0), r)$ is closed. Moreover, since Ω is open set in $\mathbb{R}^m$, we can choose r such that $\overline{B}(y_0, r) \subset \Omega$. By [2, Proposition

1.1.3], F is locally bounded; therefore, we cam assume that there exists $M > 0$ such that

$$\sup\{\|v\|; v \in F(\psi), \psi \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)\} \le M. \tag{3.1}$$

Since φ is continuous on $[-\sigma, 0]$ we can choose $\eta > 0$ small enough such that

$$\|\varphi(t) - \varphi(s)\| < \frac{r}{4}, \text{ for all } t, s \in [-\sigma, 0] \text{ with } |t - s| < \eta. \tag{3.2}$$

Let

$$T := \min\{\eta, \frac{r}{4(M+1)}, \frac{r}{8(\|y_0\|+1)}, \frac{1}{2}\sqrt{\frac{r}{M+1}}\}. \tag{3.3}$$

Further on, for a fixed $k \in \mathbb{N}^*$, we put $x_k(t) = \varphi(t)$ for every $t \in [-\sigma, 0]$ and for $p = 0$ we take $t_k^0 = 0, x_k^0 = \varphi(0), y_k^0 = y_0$ and we choose an arbitrary element $z_k^0 \in F(\varphi, y_0) + \frac{1}{k}B$. Also, we can define $t_k^1, x_k^1, y_k^1, z_k^1$ and x_k on $[0, t_k^1]$ in the same way that in the next general case.

Suppose that, for a fixed $q \in \mathbb{N}^*$, we have constructed $t_k^p, x_k^p, y_k^p, z_k^p$ and x_k on $[0, t_k^p]$ such that the conditions $(i) - (vi)$ are satisfied for each $p \in \{1, 2, \dots, q-1\}$.

To define the next step h_k^q, we denote by H_k^q the set of all $h \in (0, \frac{1}{k})$ for which the following conditions are satisfied

(a) $h \in (0, T - t_k^q)$;
(b) there exists $u_k^q \in F(T(t_k^q)x_k, y_k^q)$ such that $d(u_k^q + hy_k^q + \frac{h^2}{2}u_k^q, K) \le \frac{h^2}{4k}$.

For a fixed $u \in F(T(t_k^q)x_k, y_k^q)$, since $(T(t_k^q)x_k)(0) = x(t_k^q) = x_k^q \in K$, using tangential condition (H3) applied in $(T(t_k^q)x_k, y_k^q) \in \mathcal{K}_0 \times \Omega$ we obtain that H_k^q is nonempty and that. Since $H_k^q \cap [\frac{d_k^q}{2}, d_k^q]$ is also nonempty, let us we chose $h_k^q \in H_k^q \cap [\frac{d_k^q}{2}, d_k^q]$. We define $t_k^{q+1} := t_k^q + h_k^q$ and so we have $t_k^q < t_k^{q+1}$ and $\sum_{i=0}^q t_k^i < T$. Since $h_k^q \in H_k^q$, it follows that the first condition in (i) is satisfies for $p = q$. Moreover, there exists $u_k^q \in F(T(t_k^q)x_k, y_k^q)$ such that

$$d(x_k^q + h_k^q y_k^q + \frac{(h_k^q)^2}{2}u_k^q, K) \le \frac{(h_k^q)^2}{4k}$$

and so there exists $v_k^q \in \mathbb{R}^m$ with $\|v_k^q\| \le \frac{1}{k}$ such that

$$x_k^{q+1} := x_k^q + h_k^q y_k^q + \frac{(h_k^q)^2}{2} z_k^q \in K \tag{3.4}$$

where

$$z_k^q := u_k^q + v_k^q \in F(T(t_k^q)x_k, y_k^q) + \frac{1}{k}B,$$

Also, we remark that by (3.1) we have

$$\|z_k^q\| \le M + 1 \tag{3.5}$$

Hence, if we put

$$x_k(t) = x_k^q + (t - t_k^q)y_k^q + \frac{1}{2}(t - t_k^q)^2 z_k^q, \quad \text{for every } t \in [t_k^q, t_k^{q+1}], \tag{3.6}$$

it follows that the conditions (iii) and (iv) are also satisfied for $p = q$. Let

$$y_k^{q+1} := y_k^q + h_k^q z_k^p. \tag{3.7}$$

By induction on p (which is left to the reader) one verifies that x_k^{q+1} and y_k^{q+1} defined above can be expressed as follows

$$x_k^{q+1} = \varphi(0) + (\sum_{j=0}^{q} h_k^j)y_0 + \frac{1}{2}\sum_{j=0}^{q}(h_k^j)^2 z_k^i + \sum_{i=0}^{q-1}\sum_{j=i+1}^{q} h_k^i h_k^j z_k^i \tag{3.8}$$

and

$$y_k^{q+1} = y_0 + \sum_{j=0}^{q-1} h_k^j z_k^j. \tag{3.9}$$

Moreover, by (3.8) and (3.9), from (3.6) we obtain

$$x_k(t) = \varphi(0) + (t - t_k^p)(y_0 + \sum_{j=0}^{q-1} h_k^j z_k^j) + \frac{1}{2}(t - t_k^p)^2 z_k^p + w_q, \quad t \in [0, t_k^{q+1}] \tag{3.10}$$

where

$$w_q = (\sum_{j=0}^{q-1} h_k^j)y_0 + \frac{1}{2}\sum_{j=0}^{q-1}(h_k^j)^2 z_k^i + \sum_{i=0}^{q-2}\sum_{j=i+1}^{q-1} h_k^i h_k^j z_k^i.$$

It easy to see that, using $\sum_{j=0}^{q-1} h_k^j < T$, we obtain

$$\|w_q\| \le T\|y_0\| + \frac{M+1}{2}T^2 < \frac{r}{8} + \frac{r}{8} = \frac{r}{4}. \tag{3.11}$$

Now, we check (v) and (vi) for $p = q$. To check condition (v), we observe that by (3.3), (3.5) and (3.8) we have

$$\begin{aligned} &\|x_k^{q+1} - \varphi(0)\| \\ &\le \big(\sum_{j=0}^{q-1} h_k^j\big)\|y_0\| + \frac{1}{2}\sum_{j=0}^{q-1}\big(h_k^j\big)^2(M+1) + \sum_{i=0}^{q-2}\sum_{j=i+1}^{q-1} h_k^i h_k^j (M+1) \\ &\le T\|y_0\| + \frac{M+1}{2}T^2 < \frac{r}{8} + \frac{r}{8} = \frac{r}{4} \end{aligned} \tag{3.12}$$

and by (3.5) and (3.9) we have

$$\|y_k^{q+1} - y_0\| \le T(M+1) < \frac{r}{4}. \tag{3.13}$$

Therefore, by (3.4) (3.12) and (3.13) we have

$$(x_k^{q+1}, y_k^{q+1}) \in (K \cap B(\varphi(0), r/4)) \times B(y_0, r/4) \subset Q_0 \tag{3.14}$$

and so the condition (v) is checked for $p = q$. Also, we observe that by (3.3), (3.5), (3.10) and (3.11) we have

$$\begin{aligned} \|x_k(t) - \varphi(0)\| &\le T[\|y_0\| + T(M+1)] + \frac{M+1}{2}T^2 + \|w_q\| \\ &\le \frac{r}{8} + \frac{r}{8} + \frac{2r}{8} = \frac{r}{4}, \end{aligned} \tag{3.15}$$

which means that $x_k(t) \in B(\varphi(0), r/4)$ for every $t \in [0, t_k^{q+1}]$.

Furthermore, if $-\sigma \le s \le -t_k^q$, then, by the fact that $0 < t_k^{q+1} < T$, $t_k^{q+1} + s \in [-\sigma, 0]$ and by (3.2), we have

$$\|(T(t_k^{q+1})x_k)(s) - \varphi(s)\| = \|x_k(t_k^{q+1} + s) - \varphi(s)\| = \|\varphi(t_k^{q+1} + s) - \varphi(s)\| < r/4.$$

If $-t_k^{q+1} \leq s \leq 0$, then $t_k^{q+1} + s \in [0, t_k^{q+1}]$. Hence, there exists $t_k^p < t_k^{q+1}$ such that $t_k^{q+1} + s \in [t_k^p, t_k^{p+1}]$ and so, because $|s| < t_k^{q+1} < T$, by (3.2), (3.3) and (3.15) we have

$$\begin{aligned}\|(T(t_k^{q+1})x_k)(s) - \varphi(s)\| &= \|x_k(t_k^{q+1} + s) - \varphi(s)\|\\ &\leq \|x_k(t_k^{q+1} + s) - \varphi(0)\| + \|\varphi(s) - \varphi(0)\|\\ &< \frac{r}{4} + \frac{r}{4} = \frac{r}{2}.\end{aligned}$$

Hence we have that $T(t_k^{q+1})x_k \in B_\sigma(\varphi, r)$ and since $(T(t_k^{q+1})x_k)(0) = x_k(t_k^{q+1}) = x_k^{q+1} \in K$ we deduce that $T(t_k^{q+1})x_k \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ and so the condition (vi) is checked for $p = q$.

Thus the conditions $(i) - (vi)$ are satisfied for every $p \in \{1, 2 \dots, q\}$ and so the inductive procedure can be continued further.

In the following, we show that this iterative process if finite, i.e., there exists $m(k) \in \mathbb{N}^*$ such that the second condition in (i) is satisfied.

For this, we assume by contradiction that $t_k^p < T$ for every $p \in \mathbb{N}$. Then the bounded increasing sequence $(t_k^p)_p$ converges to $t_k^* \leq T$.

Now, we show that $(x_k^p)_p$ and $(y_k^p)_p$ is a Cauchy sequence. Indeed, for $q < p$, by (3.5), (3.8) and (3.9), we have

$$\begin{gathered}\|x_k^p - x_k^q\| \leq \|y_0\|(t_k^p - t_k^q) + \frac{1}{2}(M+1)(t_k^p - t_k^q)^2,\\ \|y_k^p - y_k^q\| \leq (M+1)(t_k^p - t_k^q).\end{gathered}$$

Since $(t_k^p)_p$ is a Cauchy sequence, the sequences $(x_k^p)_p$ and $(y_k^p)_p$ are also Cauchy. Therefore, there exists $x_k^* = \lim_{p\to\infty} x_k^p$ and $y_k^* = \lim_{p\to\infty} y_k^p$. Since $K \cap B(\varphi(0), r)$ is closed and $(x_k^p, y_k^p) \in (K \cap B(\varphi(0), r)) \times B(y_0, r)$ for every $p \in \mathbb{N}$, we have that $(x_k^*,\, y_k^*) \in (K \cap B(\varphi(0), r)) \times \overline{B}(y_0, r)$.

Now, if we put $x_k(t_k^*) := x_k^*$ then, for any sequence $(s_k^p)_p$ with $t_k^p \leq s_k^p \leq t_k^{p+1}$ for every $p \in \mathbb{N}$, the inequality

$$\begin{aligned}\|x_k(s_k^p) - x_k^*\| &\leq \|x_k(s_k^p) - x_k^p\| + \|x_k^p - x_k^*\|\\ &\leq (s_k^p - t_k^p)\|y_0\| + \frac{1}{2}(M+1)(s_k^p - t_k^p)^2 + \|x_k^p - x_k^*\|\end{aligned}$$

implies $\|x_k(s_k^p) - x_k^*\| \to 0$ as $p \to \infty$.

Therefore, there exists $\lim_{t\to t_k^*} x_k(t) = x_k^* = x_k(t_k^*)$. Accordingly, x_k is continuous on $[-\sigma, t_k^*]$ and hence $T(t_k^p)x_k \to T(t_k^*)x_k$ as $p \to \infty$. Thus, since $T(t_k^p)x_k \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ for every $p \in \mathbb{N}$ and $\mathcal{K}_0 \cap B_\sigma(\varphi, r)$ is closed, we have that $T(t_k^*)x_k \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$.

Furthermore, by (3.5) we deduce that there exists a subsequence (again denote by) $(z_k^p)_p$ such that $z_k^p \to z_k^*$ as $p \to \infty$.

Also, since $u_k^p \in F(T(t_k^p)x_k, y_k^p)$, by (3.1) we have $\|u_k^p\| \leq M$ for every $p \in \mathbb{N}$ and so there exists a subsequence (again denote by) $(u_k^p)_p$ such that $u_k^p \to u_k^*$ as $p \to \infty$. Therefore, since $(T(t_k^p)x_k, y_k^p, u_k^p) \in \operatorname{graph}(F), y_k^p \to y_k^*,\ u_k^p \to u_k^*,\ T(t_k^p)x_k \to T(t_k^*)x_k$ as $p \to \infty$ and $\operatorname{graph}(F)$ is closed we have that $u_k^* \in F(T(t_k^*)x_k, y_k^*)$.

Since $(T(t_k^p)x_k)(0) = x_k(t_k^*) = x_k^* \in K$, we can apply tangential condition (H3) in $(T(t_k^*)x_k, y_k^*)$. Therefore, we can chose $h \in (0, T - t_k^*)$ such that

$$d(x_k^* + hy_k^* + \frac{h^2}{2}u_k^*, K) \leq \frac{h^2}{4k}. \tag{3.16}$$

We would like to prove that h as defined above belongs to H_k^p for every p sufficiently large.

Since $h \in (0, T - t_k^*)$ implies that $h \in (0, T - t_k^p)$ then the condition (a) of the definition H_k^p is checked for every $p \in \mathbb{N}$. Since $(t_k^p)_p$ is increasing to t_k^*, there exists $p_1 \in \mathbb{N}$ such that for every $p \geq p_1$ we have $t_k^* - t_k^p < h$, and so $t_k^p < t_k^* < t_k^p + h < t_k^* + h$ for every $p \geq p_1$. Also, there exists $p_2 \geq p_1$ such that $p \geq p_2$ implies that

$$\|x_k^p - x_k^*\| < \frac{h^2}{12k}, \quad \|y_k^p - y_k^*\| < \frac{h}{12k}, \quad \|u_k^p - u_k^*\| < \frac{1}{6k}.$$

Therefore, for every $p \geq p_2$ we have

$$\begin{aligned} \Delta_k &:= \|(x_k^p + hy_k^p + \frac{h^2}{2}u_k^p) - (x_k^* + hy_k^* + \frac{h^2}{2}u_k^*)\| \\ &\leq \|y_k^p - x_k^*\| + h\|y_k^p - y_k^*\| + \frac{h^2}{2}\|u_k^p - u_k^*\| \leq \frac{h^2}{4k}. \end{aligned} \tag{3.17}$$

Using the inequality

$$d(x_k^p + hy_k^p + \frac{h^2}{2}u_k^p, K) \leq d(x_k^* + hy_k^* + \frac{h^2}{2}u_k^*, K) + \Delta_k,$$

by (3.16) and (3.17), we obtain that h and u_k^p satisfy the second condition of the definition of H_k^p, for every $p \geq p_2$.

Therefore, for $p \geq p_2$ we have that $h \in H_k^p$ and hence $d_k^p := \sup H_k^p \geq h$ for every $p \geq p_2$. But $h_k^p \in [\frac{d_k^p}{2}, d_k^p]$, and so $h_k^p \geq \frac{h}{2} > 0$ for every $p \geq p_1$, which is in contradiction with $h_k^p = t_k^{p+1} - t_k^p \to 0$ as $p \to \infty$. This contradiction can be eliminated only of the iterative process is finite, i.e.,if there exists $m(k) \in \mathbb{N}^*$ such that $t_k^{m(k)-1} \leq T < t_k^{m(k)}$ and the conditions $(i) - (vi)$ are satisfied for every $p \in \{0, 1, \dots, m(k) - 1\}$. $\square$

Proof of Theorem 2.3. Assume that hypotheses (H1)-(H3) are satisfied. Also, since the multifunction $y \to \partial V(y)$ is locally bounded we can choose $r > 0$ and $M > 0$ such that V is Lipschitz continuous with the constant M in $\overline{B}(\varphi(0, r))$.

We prove that the sequence $x_k(\cdot)$, constructed by Lemma 3.1, has a subsequence that converges to a solution of (2.1). First, for every $k \geq 1$ we define the function $\theta_k : [0, T] \to [0, T]$ by $\theta_k(t) = t_k^p$ for every $t \in [t_k^p, t_k^{p+1}]$.

Since $|\theta_k(t) - t| \leq \frac{1}{k}$ for every $t \in [0, T]$, then $\theta_k(t) \to t$ uniformly on $[0, T]$. By the fact that $x_k^p = x_k(\theta_k(t))$ for every $t \in [t_k^p, t_k^{p+1}]$ and for every $k \geq 1$ and by (v) and (vi) we have

$$x_k(\theta_k(t)) \in K \cap B(\varphi(0), r), \text{ for every } t \in [0, T] \text{ and for every } k \geq 1. \tag{3.18}$$

and

$$T(\theta_k(t))x_k \in \mathcal{K}_0 \cap B_\sigma(\varphi, r), \text{ for every } t \in [0, T] \text{ and for every } k \geq 1. \tag{3.19}$$

Also, by (iii) and (iv) we have

$$x_k''(t) \in F(T(\theta_k(t))x_k, x_k'(\theta_k(t))) + \frac{1}{k}B, \text{ a.e. on } [0, T] \text{ and for every } k \geq 1. \tag{3.20}$$

Moreover, by (iii) and (iv) we have

$$x_k'(t) = y_k^p + (t - t_k^p)z_k^p \text{ for every } t \in [t_k^p, t_k^{p+1}]$$

and

$$x_k''(t) = z_k^p \in F(T(t_k^p)x_k, y_k^p) + \frac{1}{k}B \text{ for every } t \in [t_k^p, t_k^{p+1}].$$

Hence, by (3.3), (3.5), and (3.15) we obtain

$$\|x_k''(t)\| = \|z_k^p\| \le M+1,$$
$$\|x_k'(t)\| \le \|y_k^p\| + (t-{}_k^p)\|z_k^p\| \le \|y_k^p - y_0\| + \|y_0\| + T(M+1) \le \|y_0\| + 2r,$$
$$\|x_k(t)\| \le \|x_k(t) - \varphi(0)\| + \|\varphi(0)\| \le \|\varphi(0)\| + r\,.$$

Therefore, $x_k''(\cdot)$ is bounded in $L^2([0,T],\mathbb{R}^m)$, $x_k(\cdot)$ $x_k'(\cdot)$ are bounded in the space $C([0,T],\mathbb{R}^m)$. Moreover, for all $t',t'' \in [0,T]$, we have

$$\|x_k(t') - x_k(t'')\| = \|\int_{t'}^{t''} x_k'(t)dt\| \le \int_{t'}^{t''} \|x_k'(t)\|dt \le (\|y_0\| + 2r)|t'-t''|,$$
$$\|x_k'(t') - x_k'(t'')\| = \|\int_{t'}^{t''} x_k''(t)dt\| \le \int_{t'}^{t''} \|x_k'(t)\|dt \le (\|\varphi(0)\| + r)|t'-t''|,$$

i. e. the sequence $x_k(\cdot)$, is equi-lipschitzian and the sequence $x_k'(\cdot)$ is equi-uniformly continuous.

Hence, by [2, Theorem 0.3.4], there exists a subsequence (again denoted by) $x_k(\cdot)$ and an absolutely continuous function $x:[0,T]\to\mathbb{R}^m$ such that

(a) $x_k(\cdot)$ converges uniformly to $x(\cdot)$,
(b) $x_k'(\cdot)$ converges uniformly to $x'(\cdot)$,
(c) $x_k''(\cdot)$ converges weakly in $L^2([0,T],\mathbb{R}^m)$ to $x''(\cdot)$.

Moreover, since all functions x_k agree with φ on $[-\sigma,0]$, we can obviously say that $x_k \to x$ on $[-\sigma,T]$, if we extend x in such a way that $x_k = \varphi$ on $[0,T]$. By (a), (b) and the uniformly converges of $\theta_k(\cdot)$ to t on $[0,T]$ we deduce that $x_k(\theta_k(t)) \to x(t)$ uniformly on $[0,T]$ and $x_k'(\theta_k(t)) \to x'(t)$ uniformly on $[0,T]$. Also, it is clearly that $T(0)x = \varphi$ on $[-\sigma,0]$.

Let us denote the modulus continuity of a function ψ on interval I of $\mathbb{R}$ by

$$\omega(\psi,I,\varepsilon) := \sup\{\|\psi(t)-\psi(s)\|; s,t\in I, |s-t|<\varepsilon\}, \varepsilon>0.$$

Then we have

$$\begin{aligned}\|T(\theta_k(t))x_k - T(t)x_k\|_\sigma &= \sup_{-\sigma\le s\le 0} \|x_k(\theta_k(t)+s) - x_k(t+s)\|\\ &\le \omega(x_k,[-\sigma,T],\frac{1}{k}) \le \omega(\varphi,[-\sigma,0],\frac{1}{k}) + \omega(x_k,[0,T],\frac{1}{k})\\ &\le \omega(\varphi,[-\sigma,0],\frac{1}{k}) + \frac{(\|y_0\|+2r)T}{k},\end{aligned}$$

hence

$$\|T(\theta_k(t)x_k - T(t)x_k\|_\sigma \le \delta_k \tag{3.21}$$

for every $k\ge 1$, where $\delta_k := \omega(\varphi,[-\sigma,0],\frac{1}{k}) + \frac{(\|y_0\|+2r)T}{k}$. Thus, by continuity of φ, we have $\delta_k \to 0$ as $k\to\infty$, hence

$$\|T(\theta_k(t)x_k - T(t)x_k\|_\infty \to 0 \quad \text{as } k\to\infty,$$

and so, since the uniform convergence of $x_k(\cdot)$ to $x(\cdot)$ on $[-\sigma,T]$ implies

$$T(t)x_k \to T(t)x \quad \text{uniformly on } [0,T], \tag{3.22}$$

we deduce that

$$T(\theta_k(t)x_k \to T(t)x \quad \text{in } \mathcal{C}_\sigma. \tag{3.23}$$

Since $T(\theta_k(t))x_k \in \mathcal{K}_0 \cap B_\sigma(\varphi,r)$ for every $t\in[0,T]$ and for every $k\ge 1$, thus by (3.19), (3.23) and by Remark 3.2 we have $T(t)x \in \mathcal{K}_0\cap B_\sigma(\varphi,r)$.

Since $\|x'_k(t) - x'_k(\theta_k(t))\| \le \frac{(M+1)T}{k}$, by (3.20) and (3.21), we have
$$d(T(t)x_k, x'_k(t), x''_k(t)), \operatorname{graph}(F)) \le \delta_k + \frac{(M+1)T+1}{k} \tag{3.24}$$
for every $k \ge 1$. By (H2), (b), (c), (3.22) and [2, Theorem 1.4.1], we obtain
$$x''(t) \in \operatorname{co} F(T(t)x, x'(t)) \subset \partial V(x'(t)) \text{ a.e. on } [0,T], \tag{3.25}$$
where co stands for the closed convex hull. Since the functions $t \to x(t)$ and $t \to V(x'(t))$ are absolutely continuous, we obtain from [5, Lemma 3.3] and (3.25) that
$$\frac{d}{dt}V(x'(t)) = \|x''(t)\|^2 \quad \text{a.e. on } [0,T]$$
hence
$$V(x'(t)) - V(x'(0)) = \int_0^T \|x''(t)\|^2 dt \tag{3.26}$$
On the other hand, since $x''_k(t) = z_k^p$ for every $t \in [t_k^p, t_k^{p+1}]$, by (iv), there exists $w_k^p \in \frac{1}{k}B$ such that
$$z_k^p - w_k^p \in F(T(t_k^p)x_k, y_k^p) \subset \partial V(x'_k(t_k^p)), \quad \forall k \in \mathbb{N}^*$$
and so the properties of subdifferential of a convex function imply that, for every $p < m(k) - 2$, and for every $k \in \mathbb{N}^*$ we have
$$\begin{aligned} V(x'_k(t_k^{p+1})) - V(x'_k(t_k^p)) &\ge \langle z_k^p - w_k^p, x'_k(t_k^{p+1}) - x'_k(t_k^p)\rangle \\ &= \langle z_k^p, \int_{t_k^p}^{t_k^{p+1}} x''_k(t)dt\rangle - \langle w_k^p, \int_{t_k^p}^{t_k^{p+1}} x''_k(t)dt\rangle \\ &= \int_{t_k^p}^{t_k^{p+1}} \|x''_k(t)\|^2 dt - \langle w_k^p, \int_{t_k^p}^{t_k^{p+1}} x''_k(t)dt\rangle; \end{aligned}$$
hence
$$V(x'_k(t_k^{p+1}) - V(x'_k(t_k^p)) \ge \int_{t_k^p}^{t_k^{p+1}} \|x''(t)\|^2 dt - \langle w_k^p, \int_{t_k^p}^{t_k^{p+1}} x''_k(t)dt\rangle. \tag{3.27}$$
Analogously, if $t \in [t_k^{m(k)-1}, T]$, then by (i) we have
$$\begin{aligned} &V(x'_k(T)) - V(x'_k(t_k^{m(k)-1})) \\ &\ge \langle z_k^{m(k)-1} - w_k^{m(k)-1}, \int_{t_k^{m(k)-1}}^T x''_k(t)dt\rangle \\ &= \int_{t_k^{m(k)-1}}^T \|x''_k(t)\|^2 dt - \langle w_k^{m(k)-1}, \int_{t_k^{m(k)-1}}^T x''_k(t)dt\rangle. \end{aligned} \tag{3.28}$$
By adding the $m(k) - 1$ inequalities from (3.27) and the inequality from (3.28), we get
$$V(x'_k(T)) - V(x'(0)) \ge \int_0^T \|x''_k(t)\|^2 dt + \alpha(k), \tag{3.29}$$
where
$$\alpha(k) = -\sum_{p=0}^{m(k)-2} \langle w_k^p, \int_{t_k^p}^{t_k^{p+1}} x''_k(t)dt\rangle - \langle w_k^{m(k)-1}, \int_{t^{m(k)-1}}^T x''_k(t)dt\rangle.$$

Since

$$\begin{aligned}|\alpha(k)| &\le \sum_{p=0}^{m(k)-2} |\langle w_k^p, \int_{t_k^p}^{t_k^{p+1}} x_k''(t)dt\rangle| + |\langle w_k^{m(k)-1}, \int_{t^{m(k)-1}}^{T} x_k'(t)dt\rangle| \\ &\le \sum_{p=0}^{m(k)-2} \|w_k^p\| \cdot \| \int_{t_k^p}^{t_k^{p+1}} x_k''(t)dt\| + \|w_k^{m(k)-1}\| \cdot \| \int_{t^{m(k)-1}}^{T} x_k''(t)dt\| \\ &\le \frac{(M+1)(2m(k)-1)}{k}\end{aligned}$$

it follows that $\alpha(k) \to 0$ as $k \to \infty$; hence, by (3.29), passing to the limit as $k \to \infty$, we obtain

$$V(x'(t)) - V(y_0) \ge \limsup_{k\to\infty} \int_0^T \|x_k''(t)\|^2 dt. \tag{3.30}$$

Therefore, by (3.26) and (3.30) we have

$$\int_0^T \|x''(t)\|^2 dt \ge \limsup_{k\to\infty} \int_0^T \|x_k''(t)\|^2 dt$$

and, since $x_k''(\cdot)$ converges weakly in $L^2([0,T],\mathbb{R}^m)$ to $x''(\cdot)$, by the lower semi-continuity of the norm in $L^2([0,T],\mathbb{R}^m)$ (e.g. [6, Proposition III 30]) we obtain that

$$\lim_{k\to\infty} \int_0^T \|x_k''(t)\|^2 dt = \int_0^T \|x''(t)\|^2 dt,$$

i. e. $x_k''(\cdot)$ converges strongly in $L^2([0,T],\mathbb{R}^m)$ to $x''(\cdot)$, hence a subsequence (again denote by) $x_k''(\cdot)$ converges pointwise a.e. to $x''(\cdot)$.

Since, by (3.24)

$$\lim_{k\to\infty} d((T(t)x_k, x_k'(t),_k''(t)), \mathrm{graph}(F)) = 0$$

and since, by (H1), the graph of F is closed ([2, Proposition 1.1.2]), we have

$$x''(t) \in F(T(t)x, x'(t)) \quad \text{a.e. on } [0,T].$$

It remains to prove that $(x(t), x'(t)) \in K \times \Omega$ for every $t \in [0,T]$. Indeed, since $\|x_k(t) - x_k^p\| \le \frac{2\|y_0\|+3(M+1)T}{k}$, $\|x_k'(t) - y_k^p\| \le \frac{(M+1)T}{k}$ we have

$$\lim_{k\to\infty} d((x_k(t), x_k'(t)), (x_k^p, y_k^p)) = 0.$$

Since $(x_k^p, y_k^p) \in Q_0 := (K \cap B(\varphi(0), r)) \times \overline{B}(y_0, r) \subset K \times \Omega$ for every $k \in \mathbb{N}$, by (a) and (b) we have

$$\lim_{k\to\infty} d((x(t), x'(t)), (x_k(t), x_k'(t))) = 0.$$

On the other hand,

$$\begin{aligned}&d((x(t), x'(t)), Q_0) \\ &\le d((x(t), x'(t)), (x_k(t), x_k'(t))) + d((x_k(t), x_k'(t)), (x_k^p, y_k^p)) + d((x_k^p, y_k^p), Q_0);\end{aligned}$$

hence, by passing to the limit as $k \to \infty$ we obtain

$$d((x(t), x'(t)), Q_0) = 0, \quad \text{for every } t \in [0,T].$$

Since Q_0 is closed, we obtain that $(x(t), x'(t)) \in Q_0 \subset K \times \Omega$ for all $t \in [0,T]$, which completes the proof. □

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