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ASYMPTOTIC STABILITY RESULTS FOR CERTAIN INTEGRAL EQUATIONS

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Abstract. This paper shows the existence of asymptotically stable solutions to an integral equation. This is done by using a fixed point theorem, and without requiring that the solutions be bounded.

1. Introduction

Banás and Rzepka [3] study a very interesting property for the solutions of some functional equations. The same property was also studied by Burton and Zhang in [6], in a more general case. Let $F : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ be an operator, where $BC(\mathbb{R}_+)$ consists of bounded and continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, $\mathbb{R}_+ := [0, \infty)$, $d \geq 1$. Let $|\cdot|$ be a norm in $\mathbb{R}^d$.

The following definition is given in [3, 6], for solutions $x \in BC(\mathbb{R}_+)$ of the equation

$$x = Fx. \tag{1.1}$$

Definition 1.1. A function x is said to be an **asymptotically stable solution** of (1.1) if for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that for every $t \geq T$ and for every solution y of (1.1), we have

$$|x(t) - y(t)| \leq \varepsilon. \tag{1.2}$$

A sufficient condition for the existence of asymptotically stable solutions is given by the following proposition.

Proposition 1.2. *Assume that there exist a constant $k \in [0, 1)$ and a continuous function $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} a(t) = 0$, such that*

$$|(Fx)(t) - (Fy)(t)| \leq k|x(t) - y(t)| + a(t), \quad \forall t \in \mathbb{R}_+,\ \forall x, y \in BC(\mathbb{R}_+). \tag{1.3}$$

Then every solution of (1.1) is asymptotically stable.

The proof of this proposition is immediate. Let us remark that basically the property of the asymptotic stability is a property of the fixed points of the operator F. Actually, in [2, 6], the proof of the existence of an asymptotically stable solution in done by applying a fixed point theorem, i.e. Schauder's Theorem. It follows that

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it is sufficient to ask Definition 1.1 to be fulfilled only on the closed, bounded, and convex set on which Schauder's Theorem is applied.

Another remark concerning Proposition 1.2 is that if (1.3) is fulfilled then **every** solution of (1.1) is asymptotically stable. Moreover, by (1.3) we deduce that the result of Proposition 1.2 is appropriate for the case when $F = A + B$, where A is contraction and $\lim_{t\to\infty}(Bx)(t) = 0$, for every x belonging to the set on which the fixed point theorem is applied. On the other hand, the set of the fixed points of F should be "big" enough for Definition 1.1 to be consistent. In this direction, in the case when Schauder's fixed point Theorem is used an interesting result has been obtained by Zamfirescu in [8], stating that if B_ρ is the closed ball of radius $\rho > 0$ from a Banach space and $F : B_\rho \to B_\rho$ is a compact operator, then for most functions F, the set of solutions of (1.1) is homeomorphic to the Cantor set ("most" means "all" except those in a first category set).

Finally, let us remark that in order to fulfil Definition 1.1 it is not necessary that all the solutions of (1.1) to be bounded on $\mathbb{R}_+$. The result obtained by Burton and Zhang is contained in the following theorem.

Theorem 1.3. *Assume that*

(i) *$f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and there exist a continuous function $k : \mathbb{R}_+ \to [0, 1]$ with $0 \le k(t) < 1$ for $t > 0$ and a constant $x_0 \in \mathbb{R}^d$ such that $x_0 = f(0, x_0)$ and*
$$\lim_{t\to 0^+} (1 - k(t))^{-1}(f(t, x_0) - f(0, x_0)) = 0;$$
(ii) *for each $t \in \mathbb{R}_+$ and x, $y \in \mathbb{R}^d$,*
$$|f(t, x) - f(t, y)| \le k(t)|x - y|;$$
(iii) *$u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and there are continuous functions a, $b : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|u(t, s, x)| \le a(t)b(s)$, for all t, $s \in \mathbb{R}_+$ ($s \le t$) and all $x \in \mathbb{R}^d$ with*
$$\lim_{t\to 0^+} \frac{a(t)}{1 - k(t)} \int_0^t b(s)ds = 0$$
and
$$\lim_{t\to\infty} \frac{a(t)}{1 - k(t)} \int_0^t b(s)ds = 0.$$

Then (1.4) *below has at least one solution, and every solution of the equation*
$$x(t) = f(t, x(t)) + \int_0^t u(t, s, x(s))ds, \quad t \in \mathbb{R}_+ \tag{1.4}$$
is asymptotically stable and converges to the unique continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}^d$ satisfying
$$\psi(t) = f(t, \psi(t)), \quad t \ge 0.$$

Note that hypothesis (i) is not necessary; in our note [2] we prove a similar theorem without using this hypothesis. Let us remark that in (1.4) one has $F = A + B$, where A is a contraction in $BC(\mathbb{R}_+)$ and B is a compact operator which in the admitted hypotheses fulfills the property
$$\lim_{t\to\infty} (Bx)(t) = 0, \tag{1.5}$$

the limit being uniform with respect to $x \in BC(\mathbb{R}_+)$. The second result in our Note [2] is obtained in the absence of condition (1.5).

In the present paper we prove the existence of the asymptotically stable solutions to (1.1) when F is a sum of three operators, without requiring the boundedness of the solutions. The general result that we present needs a more sophisticated argument than the one used in [2]. To this aim, we consider the set of continuous functions as the fundamental space

$$X = C_c = C_c(\mathbb{R}_+, \mathbb{R}^d)$$

which equipped with the numerable family of seminorms

$$|x|_n := \sup_{t \in [0,n]} \{|x(t)|\}, \quad n \geq 1, \tag{1.6}$$

becomes a Fréchet space (i.e. a complete linear metrisable space). We will use in addition another family of seminorms,

$$\|x\|_n := |x|_{\gamma_n} + |x|_{h_n}, \quad n \geq 1, \tag{1.7}$$

where

$$|x|_{h_n} = \sup_{\gamma_n \leq t \leq n} \{e^{-h_n(t-\gamma_n)}|x(t)|\},$$

$\gamma_n \in (0, n)$ and $h_n > 0$ are arbitrary numbers.

Remark 1.4. The families (1.6) and (1.7) define the same topology on X, i.e. the topology of the uniform convergence on compact subsets of $\mathbb{R}_+$. Consequently, a family in X is relatively compact if and only if it is equicontinuous and uniformly bounded on compact subsets of $\mathbb{R}_+$.

Notations and general hypotheses. We consider the nonlinear integral equation

$$x(t) = q(t) + f(t, x(t)) + \int_0^t V(t,s)x(s)ds + \int_0^t G(t,s,x(s))ds, \; t \in \mathbb{R}_+, \tag{1.8}$$

where $q : \mathbb{R}_+ \to \mathbb{R}^d$, $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $V : \Delta \to \mathcal{M}_d(\mathbb{R})$, $G : \Delta \times \mathbb{R}^d \to \mathbb{R}^d$ are supposed to be continuous and $\Delta = \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad s \leq t\}$.

In what follows we denote by $|\cdot|$ a vector norm and also a matrix norm, such that for every vector $x \in \mathbb{R}^d$ and for every real quadratic $d \times d$ matrix $Z \in \mathcal{M}_d(\mathbb{R})$,

$$|Zx| \leq |Z||x|.$$

We will use the following general hypotheses:

(H1) There is a constant $L \in [0, 1)$ such that

$$|f(t,x) - f(t,y)| \leq L|x - y|, \; \forall x, y \in \mathbb{R}^d, \; \forall t \in \mathbb{R}_+;$$

(H2) There are two continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$|V(t,s)| \leq a(t)b(s), \quad \forall (t,s) \in \Delta;$$

(H3) There is a continuous function $\omega : \Delta \to \mathbb{R}_+$ such that

$$|G(t,s,x)| \leq \omega(t,s), \quad \forall (t,s) \in \Delta, \quad \forall x \in \mathbb{R}^d.$$

2. Preliminary result

In X, we consider the equation

$$x(t) = q(t) + f(t, x(t)), \quad t \in \mathbb{R}_+. \tag{2.1}$$

Lemma 2.1. *Under assumptions (H1)-(H3), Equation* (2.1) *admits a unique solution.*

Proof. We define the operator $\Phi : X \to X$ through

$$(\Phi x)(t) = q(t) + f(t, x(t)), \quad x \in X, \quad t \in \mathbb{R}_+. \tag{2.2}$$

By hypothesis (H1) and (2.2) it follows that

$$|\Phi x - \Phi y|_n \le L|x - y|_n, \quad n \ge 1, \quad x, y \in X.$$

Let us define the sequence of iterates

$$\begin{gathered} x_0 \in X, \\ x_m = \Phi(x_{m-1}), \quad m \ge 1. \end{gathered}$$

Straightforward estimates lead us to

$$|x_{m+p} - x_m|_n \le \frac{L^m}{1-L}|x_1 - x_0|, \quad \forall m, p \ge 1.$$

Hence we obtain that for all $\varepsilon > 0$ and all n there exists $N = N(\varepsilon, n)$ such that

$$|x_{m+p} - x_m|_n < \varepsilon, \quad \forall p \ge 1, \ \forall m \ge N,$$

which means that $\{x_m\}_{m\ge 0}$ is a Cauchy sequence. Since X is complete, $\{x_m\}_{m\ge 0}$ is convergent. Then $\xi := \lim_{m\to\infty} x_m$ is a fixed point of Φ. The uniqueness of ξ is proved by contradiction. □

3. The associated equation

In (1.8), we make the transformation $x = y+\xi(t)$, where ξ is the function defined by Lemma 2.1. Then (1.8) becomes

$$y = Ay + By + Cy, \tag{3.1}$$

where

$$\begin{gathered} (Ay)(t) = q(t) + f(t, y(t) + \xi(t)) - \xi(t), \\ (By)(t) = \int_0^t V(t, s)[y(s) + \xi(s)]ds, \\ (Cy)(t) = \int_0^t G(t, s, y(s) + \xi(s))ds. \end{gathered}$$

Obviously, if y is a solution of (3.1), then $x = y + \xi(t)$ is a solution of (1.8), and conversely. The operators A, B, C satisfy the following properties

$$|(Ay_1)(t) - (Ay_2)(t)| \le L|y_1(t) - y_2(t)|, \quad A0 = 0, \tag{3.2}$$

$$|(By_1)(t) - (By_2)(t)| \le a(t)\int_0^t b(s)|y_1(s) - y_2(s)|ds, \tag{3.3}$$

$$|(Cy)(t)| \le \int_0^t \omega(t, s)ds. \tag{3.4}$$

We set $D = A + B$. Then we can state and prove the following useful lemma.

Lemma 3.1. *The operators C and D satisfy the following properties:*

(1) *$C : X \to X$ is compact operator;*
(2) *There exists a numerable set $\{\delta_n\}_n$ such that $\delta_n \in [0,1)$, for all $n \geq 1$ and for all $x, y \in X$ and for all $n \geq 1$,*

$$\|Dx - Dy\|_n \leq \delta_n \|x - y\|_n. \tag{3.5}$$

Proof. (1) First we prove that $C : X \to X$ is continuous. Let $y_m, y \in X$ be such that $y_m \to y$ in X, i.e. for all $\varepsilon > 0$ and all $n \geq 1$ there exists $N = N(\varepsilon, n)$ such that

$$|y_m - y|_n < \varepsilon, \quad \forall m \geq N.$$

Let $n \geq 1$ be fixed; we have

$$|(Cy_m)(t) - (Cy)(t)| \leq \int_0^t |G(t,s,y_m(s)+\xi(s)) - G(t,s,y(s)+\xi(s))|ds,$$

and so, for $t \in [0,n]$, we get

$$|(Cy_m)(t) - (Cy)(t)| \leq \int_0^n |G(t,s,y_m(s)+\xi(s)) - G(t,s,y(s)+\xi(s))|ds.$$

But the convergence of $\{y_m\}_m$ and the continuity of ξ implies that there is a number $L_n > 0$ such that

$$|y_m(t) + \xi(t)| \leq L_n, \quad |y(t)+\xi(t)| \leq L_n, \quad \forall t \in [0,n], \quad n \geq 1.$$

Since the function G is uniformly continuous on the compact set

$$\big\{(t,s,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d, \quad t,s \in [0,n], \quad |x| \leq L_n\big\},$$

it follows that

$$|G(t,s,y_m(s)+\xi(s)) - G(t,s,y(s)+\xi(s))| \leq \frac{\varepsilon}{n}, \quad \forall m \geq N.$$

Then

$$|Cy_m - Cy|_n \leq \varepsilon, \quad \forall m \geq N,$$

and the continuity of C is proved.

It remains to show that C maps bounded sets into compact sets. Let $\mathcal{S} \subset C_c$ be bounded. We have to prove that for each $n \geq 1$ the family $\{Cy\big|_{[0,n]} : y \in \mathcal{S}\}$ is uniformly bounded and equicontinuous.

Recall that $\mathcal{S} \subset C_c$ is bounded if and only if for all n, there exists $p_n > 0$ such that for all $x \in \mathcal{S}$, $|x|_n \leq p_n$. Let $n \geq 1$ be arbitrary but fixed. For $t \in [0,n]$, $y \in \mathcal{S}$, we have

$$|(Cy)(t)| \leq \int_0^t |G(t,s,y(s)+\xi(s))|ds \leq \int_0^t \omega(t,s)ds \leq n\omega_n,$$

where

$$\omega_n := \sup_{(t,s)\in\Delta_n} \{\omega(t,s)\},$$

$$\Delta_n := \{(t,s) \in [0,n] \times [0,n], \ s \leq t\}. \tag{3.6}$$

Hence the family $\{Cy\big|_{[0,n]} : y \in \mathcal{S}\}$ is uniformly bounded.

Let $y \in \mathcal{S}$, $t \in [0,n]$; therefore $G(t,s,y(s)+\xi(s))$ is continuous and so $(Cy)(t)$ is a continuous function of t. Let $\xi_n := \sup_{t\in[0,n]}\{|\xi(t)|\}$. Now, $G(t,s,x)$ is uniformly continuous on

$$\Omega_n := \{(t,s,x), \quad 0 \leq s \leq t \leq n, \ |x| \leq p_n + \xi_n\}.$$

Hence, for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$, such that if $(t_i, s_i, x_i) \in \Omega_n$, $i = \overline{1,2}$, then

$$|(t_1, s_1, x_1) - (t_2, s_2, x_2)| < \delta$$

implies that

$$|G(t_1, s_1, x_1) - G(t_2, s_2, x_2)| < \varepsilon.$$

For $y \in \mathcal{S}$ and $t_1,\ t_2 \in [0, n]$ with $|t_1 - t_2| < \delta$, since δ can be chosen such that $\delta \le \varepsilon$, we have successively

$$\begin{aligned}|(Cy)(t_1) - (Cy)(t_2)| &\le \int_0^{t_1} \big|G(t_1, s, y(s) + \xi(s)) - G(t_2, s, y(s) + \xi(s))\big| ds \\ &\quad + \Big|\int_{t_2}^{t_1} G(t_2, s, y(s) + \xi(s)) ds\Big| \\ &\le \varepsilon n + \delta M_n \le \varepsilon(n + M_n),\end{aligned}$$

where $M_n := \sup_{(t,s,x)\in\Omega_n}\{|G(t, s, x)|\}$.

Hence the set $\{Cy\big|_{[0,n]} : y \in \mathcal{S}\}$ is equicontinuous. By Remark 1.4 we deduce that C is compact operator.

(2) Let $n \ge 1$ be arbitrary but fixed. Let $t \in [0, \gamma_n]$ be arbitrary. Then we have

$$\begin{aligned}|(Dx)(t) - (Dy)(t)| &\le L|x(t) - y(t)| + a(t)\int_0^t b(s)|x(s) - y(s)|ds \\ &\le (L + \gamma_n c_n)|x - y|_{\gamma_n},\end{aligned}$$

where $c_n := \sup_{(t,s)\in\Delta_n}\{a(t)b(s)\}$, and Δ_n is given by (3.6). Therefore,

$$|Dx - Dy|_{\gamma_n} \le (L + \gamma_n c_n)|x - y|_{\gamma_n}. \tag{3.7}$$

Let $t \in [\gamma_n, n]$ be arbitrary. Then we have

$$\begin{aligned}|(Dx)(t) - (Dy)(t)| &\le L|x(t) - y(t)| + a(t)\int_0^{\gamma_n} b(s)|x(s) - y(s)|ds \\ &\quad + a(t)\int_{\gamma_n}^t b(s)|x(s) - y(s)|ds.\end{aligned}$$

After easy computations, it follows that

$$\begin{aligned}&|(Dx)(t) - (Dy)(t)|e^{-h_n(t-\gamma_n)} \\ &< L|x(t) - y(t)|e^{-h_n(t-\gamma_n)} + \gamma_n c_n|x - y|_{\gamma_n} + \frac{c_n}{h_n}|x - y|_{h_n}\end{aligned}$$

and therefore

$$|Dx - Dy|_{h_n} \le (L + \frac{c_n}{h_n})|x - y|_{h_n} + \gamma_n c_n|x - y|_{\gamma_n}. \tag{3.8}$$

By (3.7) and (3.8) we obtain

$$\|Dx - Dy\|_n \le (L + 2\gamma_n c_n)|x - y|_{\gamma_n} + (L + \frac{c_n}{h_n})|x - y|_{h_n}. \tag{3.9}$$

Since $L < 1$, for $\gamma_n \in (0, \frac{1-L}{2c_n})$ we deduce that $L + 2\gamma_n c_n < 1$ and for $h_n > \frac{c_n}{1-L}$ we deduce that $L + \frac{c_n}{h_n} < 1$. Let $\delta_n := \max\{L + 2\gamma_n c_n, L + \frac{c_n}{h_n}\}$. It follows that $\delta_n \in [0, 1)$ and, since (3.9),

$$\|Dx - Dy\|_n \le \delta_n\|x - y\|_n, \quad \forall x, y \in X.$$

The proof of Lemma 3.1 is now complete. $\square$

Remark 3.2. Obviously, each operator D which fulfills (3.5) with $\delta_n > 0$, $\forall n \geq 1$ is continuous on X; if, in addition, $\delta_n < 1$, $\forall n \geq 1$, then $I - D$ is invertible and $(I-D)^{-1}$ is continuous (I denotes the identity operator). The proof of this assertion is immediate and it follows the classical model when X is a Banach space and D is a contraction.

4. Some remarks on Krasnoselskii's Theorem

A well known result in nonlinear analysis is Krasnoselskii's Theorem, which states as follows.

Theorem 4.1 (Krasnoselskii [7], [9])**.** *Let M be a non-empty bounded closed convex subset of a Banach space U. Suppose that $P : M \to U$ is a contraction and $Q : M \to U$ is a compact operator. If $H := P + Q$ has the property $H(M) \subset M$, then H admits fixed points in M.*

Burton [4] remarks that in practice it is difficult to check condition $H(M) \subset M$ and he proposes to replace it by the condition

$$(x = Px + Qy, \quad y \in M) \Longrightarrow (x \in M).$$

In another paper, [5], Burton and Kirk give another variant of Krasnoselskii's Theorem:

Theorem 4.2 (Burton and Kirk, [5])**.** *Let U be a Banach space, P, $Q : U \to U$, P a contraction with $\alpha < 1$ and Q a compact operator. Then either*

(a) $x = \lambda P(\frac{x}{\lambda}) + \lambda Qx$ *has a solution for* $\lambda = 1$ *or*
(b) *the set* $\{x \in U : x = \lambda P(\frac{x}{\lambda}) + \lambda Qx,\ \lambda \in (0, 1)\}$ *is unbounded.*

This result has been generalized in [1], obtaining the following proposition.

Proposition 4.3. *Let X be a Fréchet space, $C, D : X \to X$ two operators. Admit that:*

(a) *C is compact operator on X;*
(b) *D fulfills condition* (3.5) *for a family of seminorms $|\cdot|_n$, $n \geq 1$;*
(c) *The following set is bounded*

$$\{x \in X, \quad x = \lambda D(\frac{x}{\lambda}) + \lambda Cx, \quad \lambda \in (0, 1)\}\,. \tag{4.1}$$

Then the operator $C + D$ admits fixed points.

The proof of this proposition is a consequence of Schaefer's Theorem ([9]).

5. Existence result

One can state and prove now an existence theorem for (3.1) (and so for (1.8)).

Theorem 5.1. *If hypotheses (H1)–(H3) are fulfilled, then* (3.1) *admits solutions.*

Proof. We will use Proposition 4.3. Taking into account Lemma 3.1, it will be sufficient to show that the set (4.1) is bounded. We recall a general result stating that if a set is bounded with respect to a family of seminorms, then it will be bounded with respect to every other equivalent family of seminorms. So, let $y \in X$,

$y = \lambda D(\frac{y}{\lambda}) + \lambda C y$, $\lambda \in (0,1)$. Then, since $\lambda < 1$, from (3.2) and hypotheses (H2) and (H3), we deduce successively

$$\begin{aligned}
|y(t)| &= \Big|\lambda A(\frac{y}{\lambda})(t) + \int_0^t V(t,s)y(s)ds \\
&\quad + \lambda \int_0^t V(t,s)\xi(s)ds + \lambda \int_0^t G(t,s,y(s)+\xi(s))ds\Big| \\
&\le L|y(t)| + a(t)\int_0^t b(s)|y(s)|ds + a(t)\int_0^t b(s)|\xi(s)|ds + \int_0^t \omega(t,s)ds
\end{aligned}$$

and so

$$|y(t)| \le \frac{a(t)}{1-L}\int_0^t b(s)|y(s)|ds + \frac{a(t)}{1-L}\int_0^t b(s)|\xi(s)|ds + \frac{1}{1-L}\int_0^t \omega(t,s)ds. \tag{5.1}$$

Let us denote

$$c(t) := \frac{a(t)}{1-L}\int_0^t b(s)|\xi(s)|ds + \frac{1}{1-L}\int_0^t \omega(t,s)ds. \tag{5.2}$$

Then (5.1) becomes

$$|y(t)| \le \frac{a(t)}{1-L}\int_0^t b(s)|y(s)|ds + c(t). \tag{5.3}$$

We set

$$w(t) = \int_0^t b(s)|y(s)|ds$$

and, since (5.3), we obtain

$$w(0) = 0, \quad w'(t) = b(t)|y(t)| \le \frac{a(t)b(t)}{1-L}w(t) + b(t)c(t). \tag{5.4}$$

By (5.4), classical estimates lead us to conclude

$$\begin{aligned}
|y(t)| &\le \frac{a(t)}{1-L}e^{\frac{1}{1-L}\int_0^t a(s)b(s)ds} \cdot \int_0^t e^{-\frac{1}{1-L}\int_0^s a(u)b(u)du} b(s)c(s)ds + c(t) \\
&=: h(t), \quad \forall t \in \mathbb{R}_+.
\end{aligned} \tag{5.5}$$

Since h is a continuous function, by (5.5) it follows that

$$|y|_n \le \sup_{t\in[0,n]}\{h(t)\},$$

which allows us to conclude that the set (4.1) is bounded and so the proof of Theorem 5.1 is complete. □

6. Main result

Theorem 6.1. *Assume hypotheses (H1)–(H3). If*

$$\lim_{t\to\infty} h(t) = 0 \tag{6.1}$$

then every solution $x(t)$ to (1.8) is asymptotically stable and

$$\lim_{t\to\infty} |x(t) - \xi(t)| = 0.$$

Proof. Let x_1, x_2 be two solutions to (1.8). Then $y_i = x_i + \xi$, $i \in \overline{1,2}$ are solutions to (3.1). Similar estimates as in the proof of the boundedness of the set (4.1) in Theorem 5.1, allow us to conclude that

$$|y_i(t)| \leq h(t), \quad \forall t \in \mathbb{R}_+,\ \forall i \in \overline{1,2}.$$

Then, from (5.5), for every $t \in \mathbb{R}_+$ we have

$$|x_1(t) - x_2(t)| = |y_1(t) - y_2(t)| \leq 2h(t).$$

Finally, by (6.1), the conclusion follows. □

Next, we present an example when condition (6.1) holds.

Remark 6.2. Let the following assumptions be fulfilled:

(1) $\lim_{t\to\infty} a(t) = 0$;
(2) $\int_0^\infty b(t)dt < \infty$;
(3) $\int_0^\infty a(t)b(t)dt < \infty$;
(4) $\int_0^t b(s)|\xi(s)|ds < \infty$;
(5) $\lim_{t\to\infty} \int_0^t \omega(t,s)ds = 0$.

Then (6.1) holds. Indeed, since (2)–(5) and

$$\begin{aligned}
&\exp\Big(-\frac{1}{1-L}\int_0^t a(u)b(u)du\Big)b(t)c(t)\\
&\leq \frac{a(t)b(t)}{1-L}\int_0^t b(s)|\xi(s)|ds + \frac{b(t)}{1-L}\int_0^t \omega(t,s)ds, \quad \forall t \in \mathbb{R}_+,
\end{aligned}$$

it follows that

$$\int_0^\infty \exp\Big(-\frac{1}{1-L}\int_0^s a(u)b(u)du\Big)b(s)c(s)ds < \infty. \tag{6.2}$$

Then, from (1), (3), (4), (5), and (6.2), we deduce that

$$\lim_{t\to\infty} h(t) = 0.$$

Remark 6.3. Unlike [3], under assumptions (1)–(5), the mapping ξ is not necessarily bounded (see also Remark 4 in [6]).

Remark 6.4. If the mapping a is decreasing, then hypothesis (3) follows from hypothesis (2).

References

[1] C. Avramescu, Some remarks on a fixed point theorem of Krasnoselskii, *Electronic Journal of Qualitative Theory of Differential Equations*, **5**, 1-15 (2003).

[2] C. Avramescu, C. Vladimirescu, On the existence of asymptotically stable solutions of certain integral equations, *in preparation*.

[3] J. Banaś and B. Rzepka, An application of a measure of noncompactness in the study of asymptotic stability, *Applied Mathematics Letters*, **16**, 1-6 (2003).

[4] T.A. Burton, A fixed-point theorem of Krasnoselskii, *Applied Mathematics Letters*, **11**(1), 85-88 (1998).

[5] T.A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii type, *Mathematische Nachrichten*, **189**, 23-31 (1998).

[6] T.A. Burton and Bo Zhang, Fixed points and stability of an integral equation: nonuniqueness, *Applied Mathematics Letters*, **17**, 839-846 (2004).

[7] M.A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Cambridge University Press, New York, 1964.

[8] T. Zamfirescu, A generic view on the theorems of Brouwer and Schauder, *Mathematische Zeitschrift*, **213**, 387-392 (1993).
[9] E. Zeidler, *Nonlinear Functional Analysis and its Applications. I. Fixed-Point Theorems*, Springer-Verlag, New York, 1986.

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