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EIGENVALUES AND SYMMETRIC POSITIVE SOLUTIONS FOR A THREE-POINT BOUNDARY-VALUE PROBLEM

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ABSTRACT. In this paper, we consider the second-order three-point boundary-value problem

$$u''(t) + f(t, u, u', u'') = 0, \quad 0 \le t \le 1,$$
$$u(0) = u(1) = \alpha u(\eta).$$

Under suitable conditions and using Schauder fixed point theorem, we prove the existence of at least one symmetric positive solution. We also study the existence of positive eigenvalues for this problem. We emphasis the highest-order derivative occurs nonlinearly in our problem.

1. INTRODUCTION

In this paper, we discuss the existence of symmetric positive solution for the second-order three-point boundary-value problem (BVP)

$$u''(t) + f(t, u, u', u'') = 0, \quad 0 \le t \le 1, \tag{1.1}$$
$$u(0) = u(1) = \alpha u(\eta), \tag{1.2}$$

where $\alpha \in (0, 1)$, $\eta \in (0, \frac{1}{2}]$; $f : [0, 1] \times [0, \infty) \times (-\infty, \infty) \times (-\infty, 0] \to [0, \infty)$ is continuous, $f(t, 0, 0, 0) \not\equiv 0$, $t \in [0, 1]$, $f(\cdot, u, v, w)$ is symmetric on $[0, 1]$ for all $(u, v, w) \in [0, \infty) \times (-\infty, \infty) \times (-\infty, 0]$, in which the second-order derivative may occur nonlinearly. We also establish new result for second order differential equations of the form

$$u''(t) + \lambda f(t, u, u', u'') = 0, \quad 0 \le t \le 1,$$

subject to condition (1.2).

The three-point boundary-value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. In the past few years, there has been much attention focused on questions of positive solutions of three-point boundary-value problems for nonlinear ordinary differential equations. The main approach is reformulate the problem to an operator equation of the form $u = Au$, where A is a suitable operator and then applying a fixed point

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theorem, such as Krasnoselskii's fixed point theorem or Leggett-Williams fixed point theorem and so on; see for example [2]-[16] and references therein. But for the existence of symmetric positive solutions has received very little attention in the literature. Thus the aim of the present paper is to establish a simple criterion for the existence of positive solution to the BVP (1.1)-(1.2) by Schauder fixed point theorem. The emphasis in this work is the highest-order derivative occurs nonlinearly in our problem and we study the existence of the symmetric positive solution.

The organization of this paper is as follows. In Section 2, we present some preliminary results that will be used to prove our main results. In Section 3, we discuss the existence of symmetric positive solution for the BVP (1.1)-(1.2), the existence of positive eigenvalues for this problem and we give two examples to illustrate our results.

2. Preliminaries

Consider the three-point boundary-value problem

$$u'' + h(t) = 0, \quad 0 \le t \le 1, \tag{2.1}$$

$$u(0) = u(1) = \alpha u(\eta), \tag{2.2}$$

where $\eta \in (0,1)$.

Lemma 2.1. *Let $\alpha \neq 1$, $h \in C[0,1]$. Then the three-point BVP (2.1)-(2.2) has a unique solution*

$$u(t) = \int_0^1 G(t,s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)h(s)ds, \tag{2.3}$$

where

$$G(x,y) = \begin{cases} x(1-y), & 0 \le x \le y \le 1, \\ y(1-x), & 0 \le y \le x \le 1. \end{cases} \tag{2.4}$$

Proof. From (2.1) we have $u''(t) = -h(t)$. For $t \in [0,1]$, integrating from 0 to t, we get

$$u'(t) = -\int_0^t h(s)ds + B.$$

For $t \in [0,1]$, integrate from 0 to t yields

$$u(t) = -\int_0^t \Big(\int_0^x h(s)ds\Big)dx + Bt + A,$$

i.e.

$$u(t) = -\int_0^t (t-s)h(s)ds + Bt + A.$$

So,

$$\begin{gathered} u(0) = A, \\ u(1) = -\int_0^1 (1-s)h(s)ds + B + A, \\ u(\eta) = -\int_0^\eta (\eta-s)h(s)ds + \eta B + A. \end{gathered}$$

Combining this with (2.2), we have

$$B = \int_0^1 (1-s)h(s)ds,$$
$$A = \frac{\alpha\eta}{1-\alpha}\int_0^1 (1-s)h(s)ds - \frac{\alpha}{1-\alpha}\int_0^\eta (\eta - s)h(s)ds.$$

Therefore, the three-point BVP (2.1)-(2.2) has a unique solution

$$\begin{aligned} u(t) =& -\int_0^t (t-s)h(s)ds + t\int_0^1 (1-s)h(s)ds \\ &+ \frac{\alpha\eta}{1-\alpha}\int_0^1 (1-s)h(s)ds - \frac{\alpha}{1-\alpha}\int_0^\eta (\eta-s)h(s)ds \\ =& \int_0^1 G(t,s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)h(s)ds\,. \end{aligned}$$

This completes the proof. □

Remark 2.2. For any $x, y \in [0,1]$, we have

$$G(x,y) \le G(x,x) \text{ and } G(x,y) = G(1-x, 1-y).$$

From (2.3) we obtain the following lemma.

Lemma 2.3. *Let $\alpha \in (0,1)$, $h \in C^+[0,1]$. Then the unique solution $u(t)$ of the BVP (2.1)-(2.2) is nonnegative on $[0,1]$, and if $h(t) \not\equiv 0$, then $u(t) > 0$, for all $t \in [0,1]$.*

Proof. Let $y \in C^+[0,1]$. From the fact that $u''(t) = -y(t) \le 0$, for all $t \in [0,1]$, we know that the graph of $u(t)$ is concave on $[0,1]$. From (2.2) and (2.3) we have that

$$\begin{aligned} u(1) = u(0) =& \frac{\alpha\eta}{1-\alpha}\int_0^1 (1-s)y(s)ds - \frac{\alpha}{1-\alpha}\int_0^\eta (\eta-s)y(s)ds \\ =& \frac{\alpha\eta}{1-\alpha}\int_\eta^1 (1-s)y(s)ds + \frac{\alpha}{1-\alpha}\int_0^\eta [\eta(1-s) - (\eta-s)]y(s)ds \\ =& \frac{\alpha\eta}{1-\alpha}\int_\eta^1 (1-s)y(s)ds + \frac{\alpha(1-\eta)}{1-\alpha}\int_0^\eta sy(s)ds \ge 0. \end{aligned}$$

It follows that $u(t) \ge 0$, for all $t \in [0,1]$, and if $h(t) \not\equiv 0$, then $u(t) > 0$, for all $t \in [0,1]$. □

Lemma 2.4. *Let $\alpha, \eta \in (0,1)$, $h(t)$ be symmetric on $[0,1]$. Then the unique solution $u(t)$ of the BVP (2.1)-(2.2) is symmetric on $[0,1]$.*

Proof. From (2.3), we have

$$u(t) = \int_0^1 G(t,s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)h(s)ds.$$

Therefore,

$$\begin{aligned}
u(1-t) &= \int_0^1 G(1-t,s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)h(s)ds \\
&= \int_1^0 G(1-t,1-s)h(1-s)d(1-s) + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)h(s)ds \\
&= \int_0^1 G(t,s)h(s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)h(s)ds \\
&= u(t).
\end{aligned}$$

i.e., $u(t)$ is symmetric on $[0,1]$. □

Define an integral operator $T : E \to E$ by

$$Tu(t) = \int_0^1 G(t,s)f(s,u,u',u'')ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)f(s,u,u',u'')ds. \tag{2.5}$$

for $t \in [0,1]$. It is easy to see that (1.1)-(1.2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator T. The main tool in our approach is the following Schauder fixed point theorem (See [1]).

Theorem 2.5. *Let E be Banach space and $B \subset E$ be a bounded closed convex subset, $T : E \to E$ be a completely continuous operator such that $T(B) \subset B$. Then T has a fixed point in B.*

3. Main results

In this section, we study the existence of positive solution for the BVP (1.1)-(1.2). We obtain the following existence results.

Theorem 3.1. *Assume that $\alpha, \eta \in (0,1)$, $f : [0,1]\times[0,\infty)\times(-\infty,\infty)\times(-\infty,0] \to [0,\infty)$ is continuous, $f(t,0,0,0) \not\equiv 0$, $t \in [0,1]$, $f(\cdot,u,v,w)$ is symmetric on $[0,1]$ for all $(u,v,w) \in [0,\infty)\times(-\infty,\infty)\times(-\infty,0]$. If there exist constant M such that*

$$\max\Big\{1, \frac{1-\alpha+4\alpha\eta(1-\eta)}{4(1-\alpha)}\Big\}L \le 2M,$$

where

$$L = \max\{f(t,u,v,w) : 0 \le t \le 1,\ 0 \le u \le M,\ -M \le v \le M,\ -2M \le w \le 0\}.$$

Then $(1.1)-(1.2)$ has at least one symmetric positive solution.

Proof. Let $E = C^2[0,1]$ be a Banach space with norm $\|u\| = \max\{|u|_0, |u'|_0, |u''|_0\}$, where $|u|_0 = \max_{0\le t\le 1}|u(t)|$, $B = \{u(t) : u(t) \in C^2[0,1]$ is symmetric on $[0,1]$, $0 \le u(t) \le M$, $-M \le u'(t) \le M$, $-2M \le u''(t) \le 0\}$, then B is a bounded closed convex subset of E. Now we prove

$$T(B) \subset B. \tag{3.1}$$

In fact, for any $u \in B$, from (2.3), (2.4) and Remark 2.2 we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)f(s,u,u',u'')ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)f(s,u,u',u'')ds \\
&\le \Big(\int_0^1 G(t,s)ds + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)ds\Big)L \\
&\le \left(\int_0^1 t(1-t)ds + \frac{\alpha}{1-\alpha}\int_0^\eta (1-\eta)sds + \frac{\alpha}{1-\alpha}\int_\eta^1 \eta(1-s)ds\right)L \\
&= \Big(\frac{1}{2}t(1-t) + \frac{\alpha\eta(1-\eta)}{2(1-\alpha)}\Big)L \\
&\le \frac{1-\alpha+4\alpha\eta(1-\eta)}{8(1-\alpha)}L \le M,
\end{aligned}$$

which implies

$$0 \le Tu(t) \le M, \quad \forall\, t \in [0,1]. \tag{3.2}$$

In addition, from

$$(Tu)'(t) = -\int_0^t sf(s,u,u',u'')ds + \int_t^1 (1-s)f(s,u,u',u'')ds$$

we have

$$\begin{aligned}
(Tu)'(t) &\le \int_0^1 (1-s)f(s,u,u',u'')ds \\
&\le L\int_0^1 (1-s)ds \\
&= \frac{1}{2}L \le M
\end{aligned}$$

and

$$\begin{aligned}
(Tu)'(t) &\ge -\int_0^t sf(s,u,u',u'')ds \\
&\ge -\int_0^1 sf(s,u,u',u'')ds \\
&\ge -\frac{1}{2}L \ge -M
\end{aligned}$$

which implies

$$-M \le (Tu)'(t) \le M, \quad \forall\, t \in [0,1]. \tag{3.3}$$

Finally, from

$$(Tu)''(t) = -f(t,u,u',u''), \quad \forall\, t \in [0,1].$$

we know

$$0 \ge (Tu)''(t) \ge -L \ge -2M, \quad \forall\, t \in [0,1]. \tag{3.4}$$

Therefore, from (3.2), (3.3), (3.4) and Lemma 2.1 we have $Tu \in B$ which implies (3.1). Thus, by Schauder fixed theorem, T has a fixed point $u^* \in B$. From $f(t,0,0,0) \not\equiv 0$, $t \in [0,1]$. We know u^* is a symmetric positive solution of (1.1)-(1.2). $\square$

Example 3.2. Consider the three-point boundary-value problem

$$u''(t) + 4t(1-t)(1+\sqrt{u}) + \frac{1}{8}\min\{t, 1-t\}(u')^2 + \sqrt{-2u''} = 0, \quad 0 \le t \le 1, \tag{3.5}$$

$$u(0) = u(1) = \frac{1}{2}u(\frac{1}{4}). \tag{3.6}$$

Set $f(t, u, v, w) = 4t(1-t)(1+\sqrt{u}) + \frac{1}{8}\min\{t, 1-t\}v^2 + \sqrt{-2w}$, $M = 16$. A direct computation shows $L = 29$ and $\max\{1, \frac{1-\alpha+4\alpha\eta(1-\eta)}{4(1-\alpha)}\}L = 29 < 32 = 2M$, by Theorem 3.1, the BVP (3.5)-(3.6) has at least one symmetric positive solution.

As an application of Theorem 3.1, we consider the eigenvalue problem

$$u''(t) + \lambda f(t, u, u', u'') = 0, \quad 0 \le t \le 1, \tag{3.7}$$

$$u(0) = u(1) = \alpha u(\eta), \tag{3.8}$$

We have the following existence result.

Theorem 3.3. *Assume that* α, $\eta \in (0, 1)$ *are constants,* $\lambda > 0$ *is parameter,* $f : [0,1]\times[0,\infty)\times(-\infty,\infty)\times(-\infty,0] \to [0,\infty)$ *is continuous,* $f(t, 0, 0, 0) \not\equiv 0$, $t \in [0, 1]$, $f(\cdot, u, v, w)$ *is symmetric on* $[0, 1]$ *for all* $(u, v, w) \in [0,\infty) \times (-\infty,\infty) \times (-\infty, 0]$. *Then for each*

$$\lambda \in \Big(0, \frac{2M}{\max\{1, \frac{1-\alpha+4\alpha\eta(1-\eta)}{4(1-\alpha)}\}L}\Big],$$

where $M > 0$ *and* $L = \max\{f(t, u, v, w)|\ 0 \le t \le 1,\ 0 \le u \le M,\ -M \le v \le M,\ -2M \le w \le 0\}$, *the eigenvalue problem* (3.7)-(3.8) *has at least one symmetric positive solution.*

Example 3.4. Consider the eigenvalue problem

$$u''(t) + \lambda f(t, u, u', u'') = 0, \quad 0 \le t \le 1, \tag{3.9}$$

$$u(0) = u(1) = \frac{15}{16}u(\frac{1}{2}), \tag{3.10}$$

where $f(t, u, v, w) = 1 + \frac{4}{13}t(1-t)(1+3\sqrt{u}) + \frac{1}{4}t(1-t)|v| - \frac{1}{16}\min\{t, 1-t\}w$. Set $M = 16$. A direct computation shows $L = 4$ and $\max\{1, \frac{1-\alpha+4\alpha\eta(1-\eta)}{4(1-\alpha)}\}L = 16$, by Theorem 3.3, the eigenvalue problem (3.9)-(3.10) has at least one symmetric positive solution.

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