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INITIAL BOUNDARY VALUE PROBLEM FOR A SYSTEM IN ELASTODYNAMICS WITH VISCOSITY

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Abstract. In this paper we prove existence of global solutions to boundary-value problems for two systems with a small viscosity coefficient and derive estimates uniform in the viscosity parameter. We do not assume any smallness conditions on the data.

1. Introduction

In this paper first we consider the boundary-value problem, for a system of nonlinear ordinary differential equations,

$$\begin{aligned} -\xi\frac{du}{d\xi}+u\frac{du}{d\xi}-\frac{d\sigma}{d\xi}&=\epsilon\frac{d^2u}{d\xi^2},\\ -\xi\frac{d\sigma}{d\xi}+u\frac{d\sigma}{d\xi}-k^2\frac{du}{d\xi}&=\epsilon\frac{d^2\sigma}{d\xi^2} \end{aligned}\tag{1.1}$$

for $\xi\in[0,\infty)$ with boundary conditions

$$\begin{aligned} u(0)=u_B, u(\infty)=u_R,\\ \sigma(0)=\sigma_B, \sigma(\infty)=\sigma_R. \end{aligned}\tag{1.2}$$

Next we consider the initial boundary value problem, for a system of parabolic equations in $x>0$ $t>0$,

$$\begin{aligned} u_t+uu_x-\sigma_x&=\epsilon u_{xx},\\ \sigma_t+u\sigma_x-k^2u_x&=\epsilon\sigma_{xx} \end{aligned}\tag{1.3}$$

in $\Omega=(x,t): x>0, t>0$, with the initial condition at $t=0$

$$u(x,0)=u_0(x), \sigma(x,0)=\sigma_0(x)\quad x>0,\tag{1.4}$$

and boundary condition, at $x=0$,

$$u(0,t)=u_B(t), \sigma(0,t)=\sigma_B(t)\quad t>0.\tag{1.5}$$

In both of these problems, $\epsilon>0$ is a small parameter. The system of equations (1.1) and (1.3) are approximations of initial boundary value problem for the system

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of equations which comes in elastodynamics:

$$\begin{aligned} u_t + uu_x - \sigma_x &= 0, \\ \sigma_t + u\sigma_x - k^2 u_x &= 0, \end{aligned} \tag{1.6}$$

where u is the velocity, σ is the stress and $k > 0$ is the speed of propagation of the elastic waves. This equation has been studied by many authors [1, 3, 4, 5] for the case when there is no boundary. The system (1.6) is nonconservative, strictly hyperbolic system with characteristic speeds

$$\lambda_1(u,\sigma) = u - k, \lambda_2(u,\sigma) = u + k \tag{1.7}$$

with Riemann invariants

$$r(u,\sigma) = \sigma + ku, s(u,\sigma) = \sigma - ku \tag{1.8}$$

respectively. The problem (1.1)-(1.2) is the vanishing self-similar approximations to study the boundary-Riemann problem for (1.6) and the problem (1.3)-(1.5) is the vanishing diffusion approximations for (1.6) with general initial-boundary data. Our aim is to show the existence of smooth solutions of these problems and derive estimates in the space of bounded variation, uniformly in $\epsilon > 0$. We do not give any restrictions on the size of the initial data.

In the study of $(u^\epsilon, \sigma^\epsilon)$ as ϵ tends to 0, there are two difficulties. The first is the nonconservative product which appear in the equation (1.6). For the self-similar case this difficulty can be overcome by the work of LeFloch and Tzavaras [7] on nonconservative products. The second is the study of the behaviour of $(u^\epsilon, \sigma^\epsilon)$ near the boundary $x = 0$. Since the characteristic speeds may change sign, the boundary may be characterestic at some points. This makes the study of the behaviour of $(u^\epsilon, \sigma^\epsilon)$ near $x = 0$, as ϵ goes to 0 difficult. This aspects are under investigation and will be taken up in a subsequent paper.

2. Self-similar vanishing diffusion approximation

In this section, we consider the system (1.1) and (1.2) and prove the existence of smooth solutions. Given the data $(u_B, \sigma_B), (u_R, \sigma_R)$, we define

$$r_B = \sigma_B + ku_B, r_R = \sigma_R + ku_R, s_B = \sigma_B - ku_B, s_R = \sigma_R - ku_R \tag{2.1}$$

The characteristic speeds (1.7) in terms of the Riemann invariants take the form

$$\lambda_1(r,s) = \frac{r-s}{2k} - k, \quad \lambda_2(r,s) = \frac{r-s}{2k} + k.$$

Consider the square

$$D = [\min(r_B, r_R), \max(r_B, r_R)] \times [\min(s_B, s_R), \max(s_B, s_R)],$$

and consider the minimum and maximum of the eigenvalues on this square

$$\lambda_j^m = \min_D \lambda_j(r,s), \lambda_j^M = \max_D \lambda_j(r,s), \quad j = 1, 2.$$

We shall prove the following result.

Theorem 2.1. *For each fixed $\epsilon > 0$ there exits a smooth solution $(u^\epsilon(\xi), \sigma^\epsilon(\xi))$ for (1.1) and (1.2) satisfying the estimates*

$$|u^\epsilon(\xi)| + |\sigma^\epsilon(\xi)| \le C, \int_0^\infty |\frac{du^\epsilon}{d\xi}| d\xi + \int_0^\infty |\frac{d\sigma^\epsilon}{d\xi}(\xi)| d\xi \le C, \tag{2.2}$$

If $\lambda_1^m > 0$, then

$$|u^\epsilon(\xi) - u_B| + |\sigma^\epsilon(\xi) - \sigma_B| \le \frac{C}{\delta} e^{\frac{-(\xi - \lambda_1^m)^2}{2\epsilon}}, \quad 0 \le \xi \le \lambda_1^m - \delta \tag{2.3}$$

If $\lambda_2^M > 0$, then

$$|u^\epsilon(\xi) - u_R| + |\sigma^\epsilon(\xi) - \sigma_R| \le \frac{C}{\delta} e^{\frac{-(\xi - \lambda_2^M)^2}{2\epsilon}}, \quad \xi \ge \lambda_2^M + \delta, \tag{2.4}$$

for some constant $C > 0$ independent of $\epsilon > 0$ and for $\delta > 0$, small.

Proof. To prove the theorem it is easier to work with Riemann invariants (1.8). The problem (1.1) and (1.2) takes the form

$$-\xi \frac{dr}{d\xi} + \lambda_1(r,s)\frac{dr}{d\xi} = \epsilon \frac{d^2 r}{d\xi^2}, \quad -\xi \frac{ds}{d\xi} + \lambda_2(r,s)\frac{ds}{d\xi} = \epsilon \frac{d^2 s}{d\xi^2} \tag{2.5}$$

on $[0, \infty)$ with boundary conditions

$$r(0) = r_B, \quad r(\infty) = r_R, \quad s(0) = s_B, \quad s(\infty) = s_R \tag{2.6}$$

where r_B, r_R, s_B and s_R are given by (2.1).

From the definition (1.8) of r, s, $u = \frac{r-s}{2k}, \sigma = \frac{r+s}{2}$. Then to prove (2.2)-(2.4), it is sufficient to prove the following estimates

$$\begin{gathered} r^\epsilon(\xi) \in [\min(r_B, \quad r_R), \max(r_B, r_R)], \quad \xi \in [0, \infty), \\ s^\epsilon(\xi) \in [\min(s_B, \quad s_R), \max(s_B, s_R)], \quad \xi \in [0, \infty); \end{gathered} \tag{2.7}$$

$$\begin{gathered} |r^\epsilon(\xi) - r_B| \le \frac{C}{\delta} e^{\frac{-(\xi - \lambda_1^m)^2}{2\epsilon}}, \quad \xi \le \lambda_1^m - \delta, \\ |s^\epsilon(\xi) - s_B| \le \frac{C}{\delta} e^{\frac{-(\xi - \lambda_2^m)^2}{2\epsilon}}, \quad \xi \le \lambda_2^m - \delta; \end{gathered} \tag{2.8}$$

$$\begin{gathered} |r^\epsilon(\xi) - r_R| \le \frac{C}{\delta} e^{\frac{-(\xi - \lambda_1^M)^2}{2\epsilon}}, \quad \xi \ge \lambda_1^M + \delta, \\ |s^\epsilon(\xi) - s_R| \le \frac{C}{\delta} e^{\frac{-(\xi - \lambda_2^M)^2}{2\epsilon}}, \quad \xi \ge \lambda_2^M + \delta; \end{gathered} \tag{2.9}$$

$$\int_0^\infty |\frac{dr^\epsilon}{d\xi}| d\xi \le |r_R - r_B|, \quad \int_0^\infty |\frac{ds^\epsilon}{d\xi}| d\xi \le |s_R - s_B|. \tag{2.10}$$

To prove these estimates we reduce (2.5) and (2.6) to an integral equation and use some ideas of Tzavaras [9] and Joseph and LeFloch [6]. Note that (2.1) can be written in the form

$$\begin{gathered} \frac{d^2 r}{d\xi^2} = (\frac{\lambda_1(r,s) - \xi}{\epsilon})\frac{dr}{d\xi}, \\ \frac{d^2 s}{d\xi^2} = (\frac{\lambda_2(r,s) - \xi}{\epsilon})\frac{ds}{d\xi}. \end{gathered} \tag{2.11}$$

For $j = 1, 2$, let

$$g^\epsilon{}_j(\xi) = \int_{\alpha_j}^\xi (y - \lambda_j(r,s)(y)) dy \tag{2.12}$$

Integrating the equation (2.11) once leads to

$$\begin{aligned} \frac{dr^\epsilon}{d\xi} &= (r_R - r_B)\frac{e^{\frac{-g_1(\xi)}{\epsilon}}}{\int_0^\infty e^{\frac{-g_1(y)}{\epsilon}}dy}, \\ \frac{ds^\epsilon}{d\xi} &= (s_R - s_B)\frac{e^{-g_2(\xi)}\epsilon}{\int_0^\infty e^{\frac{-g_2(y)}{\epsilon}}dy}. \end{aligned} \tag{2.13}$$

On integrating (2.13) using the boundary condition (2.6) we get,

$$\begin{aligned} r^\epsilon(\xi) &= r_B + (r_R - r_B)\frac{\int_0^\xi e^{\frac{-g_1(y)}{\epsilon}}dy}{\int_0^\infty e^{\frac{-g_1(y)}{\epsilon}}dy}, \\ s^\epsilon(\xi) &= s_B + (s_R - s_B)\frac{\int_0^\xi e^{\frac{-g_2(y)}{\epsilon}}dy}{\int_0^\infty e^{\frac{-g_2(y)}{\epsilon}}dy}. \end{aligned} \tag{2.14}$$

It follows that to solve (2.5) and (2.6) with estimates (2.7)–(2.10), it is enough to solve (2.14). To solve (2.14), we use the Schauder fixed point theorem applied to the function

$$F(r,s)(\xi) = (F_1(r,s)(\xi), F_2(r,s)(\xi))$$

where

$$\begin{aligned} F_1(r,s)(\xi) &= r_B + (r_R - r_B)\frac{\int_0^\xi e^{\frac{-g_1(y)}{\epsilon}}dy}{\int_0^\infty e^{\frac{-g_1(y)}{\epsilon}}dy}, \\ F_2(r,s)(\xi) &= s_B + (s_R - s_B)\frac{\int_0^\xi e^{\frac{-g_2(y)}{\epsilon}}dy}{\int_0^\infty e^{\frac{-g_2(y)}{\epsilon}}dy} \end{aligned} \tag{2.15}$$

and g_j, $j = 1, 2$ are given by (2.12). From (2.15) it is clear that $F_1(r,s)$ is a convex combination of r_B and r_R and $F_2(r,s)$ is a convex combination of s_B and s_R. So the estimate

$$\begin{aligned} F_1(r,s)(\xi) &\in [\min(r_B, r_R), \max(r_B, r_R)], \\ F_2(r,s)(\xi) &\in [\min(s_B, s_R), \max(s_B, s_R)] \end{aligned} \tag{2.16}$$

easily follows. Next we note that the expression on the right of (2.15) is independent of the choice of α_j because adding a constant to g_j does not change the value of the right hand side of (2.15). Take ρ_j as the point ξ where minimum of

$$\min \int_{\alpha_j}^{\xi} (y - \lambda_j(r,s)(y))dy$$

is achieved. This minimum is achieved because $\lambda_j(r,s)$ is bounded by the estimate (2.16) and so the term $\int_{\alpha_j}^{\xi} \lambda_j(r,s)(y)dy$ has at most linear growth as $\xi \to \infty$ where as the first term is $\xi^2/2 - \alpha_j^2/2$ has quadratic growth. Now take $\alpha_j = \rho_j$ in the definition of g_j, we have

$$g_j(\xi) \geq 0, \xi \in [0, \infty). \tag{2.17}$$

Suppose $\lambda_j^M > 0$, then because of the choice of ρ_j,

$$\begin{aligned} g_j(\xi) &= \int_{\rho_j}^{\xi} (y - \lambda_j(r,s)(y))dy \\ &\geq \int_{\lambda_j^M}^{\xi} (y - \lambda_j(r,s)(y))dy \\ &\geq \int_{\lambda_j^M}^{\xi} (y - \lambda_j^M)dy \\ &= \frac{(\xi - \lambda_j^M)^2}{2}, \quad \text{if } \xi \geq \lambda_j^M. \end{aligned}$$

So we have, for $\lambda_j^M > 0$,

$$g_j(\xi) \geq \frac{(\xi - \lambda_j^M)^2}{2}, \text{ if } \xi \geq \lambda_j^M. \tag{2.18}$$

Similarly, for $\lambda_j^m > 0$, we have

$$g_j(\xi) \geq \frac{(\xi - \lambda_j^m)^2}{2}, if \xi \leq \lambda_j^m. \tag{2.19}$$

Further,

$$\int_0^\infty e^{\frac{-g_j(\xi)\epsilon}{d}} \xi \geq \epsilon^{1/2} \int_0^\infty e^{\frac{-g_j(\rho_j + \epsilon^{1/2}\xi)\epsilon}{d}} \xi\,. \tag{2.20}$$

Now

$$\begin{aligned} g_j(\rho_j + \epsilon^{1/2}\xi) &= \int_{\rho_j}^{\rho_j + \epsilon^{1/2}\xi} (y - \lambda_j(y))dy \\ &= \int_0^{\epsilon^{1/2}\xi} (y + \rho_j - \lambda_j(\rho_j + y))dy \\ &\leq \epsilon\frac{\xi^2}{2} + (\lambda_j^M - \lambda_j^m)\epsilon^{1/2}\xi. \end{aligned} \tag{2.21}$$

From (2.20) and (2.21). we get for $j = 1, 2$

$$\begin{aligned} \int_0^\infty e^{\frac{-g_j(\xi)\epsilon}{d}} \xi &\geq \epsilon^{1/2} \int_0^\infty e^{\frac{-y^2}{2} - (\lambda_j^M - \lambda_j^m)\frac{y}{\epsilon^{1/2}}} dy \\ &= \epsilon \int_0^\infty e^{\frac{-\epsilon y^2}{2} - (\lambda_j^M - \lambda_j^m)y} dy \\ &\geq \epsilon \int_0^\infty e^{\frac{-y^2}{2} - (\lambda_j^M - \lambda_j^m)y} dy \end{aligned} \tag{2.22}$$

From (2.15) and (2.22) we get for $j = 1, 2$

$$|\frac{dF_j(r,s)}{d\xi}(\xi)| \leq \frac{C}{\epsilon}. \tag{2.23}$$

Further, from (2.15), (2.18), (2.19) and (2.22), we get: For $\lambda_1^m > 0$,

$$|F_1(r,s)(\xi) - r_B| \leq \frac{C}{\epsilon} \int_0^\xi e^{\frac{-(s-\lambda_1^m)^2}{2\epsilon}} ds = \frac{C\sqrt{2\epsilon}}{\epsilon} \int_{\frac{-\lambda_1^m}{\sqrt{2\epsilon}}}^{\frac{(\xi - \lambda_1^m)}{\sqrt{2\epsilon}}} e^{-s^2} ds, 0 \leq \xi \leq \lambda_1^m.$$

For the case $\lambda_2^m > 0$,

$$|F_2(r,s)(\xi) - s_B| \leq \frac{C}{\epsilon}\int_0^\xi e^{\frac{-(s-\lambda_2^m)^2}{2\epsilon}} ds = \frac{C\sqrt{2\epsilon}}{\epsilon}\int_{\frac{-\lambda_2^m}{\sqrt{2\epsilon}}}^{\frac{(\xi-\lambda_2^m)}{\sqrt{2\epsilon}}} e^{-s^2} ds, 0 \leq \xi \leq \lambda_2^m.$$

From (2.15), (2.18), (2.19) and (2.22), we have for the case $\lambda_1^M > 0$,

$$|F_1(r,s)(\xi) - r_R| \leq \frac{C}{\epsilon}\int_\xi^\infty e^{\frac{-(s-\lambda_k^M)^2}{2\epsilon}} ds = \frac{C\sqrt{2\epsilon}}{\epsilon}\int_{\frac{(\xi-\lambda_1^M)}{\sqrt{2\epsilon}}}^\infty e^{-s^2} ds, \xi \geq \lambda_1^M.$$

For the case $\lambda_2^M > 0$

$$|F_2(r,s)(\xi) - s_R| \leq \frac{C}{\epsilon}\int_\xi^\infty e^{\frac{-(s-\lambda_k^M)^2}{2\epsilon}} ds = \frac{C\sqrt{2\epsilon}}{\epsilon}\int_{\frac{(\xi-\lambda_2^M)}{\sqrt{2\epsilon}}}^\infty e^{-s^2} ds, \xi > \lambda_2^M$$

Now using the asymptotic expansion

$$\int_y^\infty e^{-y^2} dy = (\frac{1}{2y} - O(\frac{1}{y^2}))e^{-y^2}, \quad y \to \infty$$

in the above two inequalities, we get

$$\begin{aligned} |F_1(r,s)(\xi) - r_B| &\leq \frac{C}{\delta} e^{\frac{-(\xi-\lambda_1^m)^2}{2\epsilon}}, \quad \xi \leq \lambda_1^m - \delta, \\ |F_2(r,s)(\xi) - s_B| &\leq \frac{C}{\delta} e^{\frac{-(\xi-\lambda_2^m)^2}{2\epsilon}}, \quad \xi \leq \lambda_2^m - \delta; \end{aligned} \tag{2.24}$$

$$\begin{aligned} |F_1(r,s)(\xi) - r_R| &\leq \frac{C}{\delta} e^{\frac{-(\xi-\lambda_1^M)^2}{2\epsilon}}, \quad \xi \geq \lambda_1^M + \delta, \\ |F_2(r,s)(\xi) - s_R| &\leq \frac{C}{\delta} e^{\frac{-(\xi-\lambda_2^M)^2}{2\epsilon}}, \quad \xi \geq \lambda_2^M + \delta. \end{aligned} \tag{2.25}$$

If $\lambda_j^M < 0$, it can be easily seen that $g_j(\xi) \geq \frac{\xi^2}{2}$ and an analysis similar to the one given earlier gives

$$|F_1(r,s)(x) - r_R| \leq \frac{C}{\delta} e^{\frac{-\xi^2}{2\epsilon}}, \quad \xi > 0 \tag{2.26}$$

$$|F_2(r,s)(x) - s_R| \leq \frac{C}{\delta} e^{\frac{-\xi^2}{2\epsilon}}, \quad \xi > 0. \tag{2.27}$$

The estimates (2.16), (2.23)–(2.27) show that F is compact and maps the convex set

$$[\min(r_B, r_R), \max(r_B, r_R)] \times [\min(s_B, s_R), \max(s_B, s_R)]$$

into itself. So by Schauder fixed point theorem F has a fixed point and hence (2.10) has a solution. Further it satisfies the estimates (2.2)-(2.4). The proof of the theorem is complete. □

3. Vanishing diffusion approximation

In this section we consider (1.3) in the domain $\Omega_T = [x > 0, 0 \leq t \leq T]$, for $T > 0$, with initial condition (1.4) and boundary condition (1.5) and prove the following result.

Theorem 3.1. *Assume that* $u_0^\epsilon(x), \sigma_0^\epsilon(x) \in W^{1,1}(0,\infty)$ *and* $u_B^\epsilon, \sigma_B^\epsilon \in W^{1,1}(0,T)$ *for every* $T > 0$. *Further assume that* $(u_0^\epsilon(0), \sigma_0^\epsilon(0)) = (u_B^\epsilon(0), \sigma_B^\epsilon(0))$. *Then there exists a classical solution* $(u^\epsilon, \sigma^\epsilon)$ *of the problem* (1.3)–(1.5) *in* Ω_T *with the following estimates:*

$$\begin{gathered}
\|u^\epsilon\|_{L^\infty(\Omega_T)} \le \frac{1}{k} \max\left[\|\sigma_0^\epsilon\|_{L^\infty} + k\|u_0^\epsilon\|_{L^\infty}, \|\sigma_B^\epsilon\|_{L^\infty(0,T)} + k\|u_B^\epsilon\|_{L^\infty(0,T)}\right] \\
\|\sigma^\epsilon\|_{L^\infty(\Omega_T)} \le \max\left[\|\sigma_0^\epsilon\|_{L^\infty} + k\|u_0^\epsilon\|_{L^\infty}, \|\sigma_B^\epsilon\|_{L^\infty(0,T)} + k\|u_B^\epsilon\|_{L^\infty(0,T)}\right]
\end{gathered} \tag{3.1}$$

$$\begin{aligned}
\int_0^\infty (|\partial_x u^\epsilon(x,t)\| dx &\le \frac{1}{k}\int_0^\infty (|\partial_x u_0^\epsilon(x)| + k|\partial_x \sigma_0^\epsilon(x)|)\,dx \\
&\quad + \frac{1}{k}\int_0^T (|\partial_t u_B^\epsilon(t)| + k|\partial_t \sigma_B^\epsilon(t)|)\,dt, \\
\int_0^\infty |\partial_x \sigma^\epsilon(x,t)|)\,dx &\le \int_0^\infty (\partial_x|u_0^\epsilon(x)| + k|\partial_x \sigma_0^\epsilon(x)|)\,dx \\
&\quad + \int_0^T (|\partial_t u_B^\epsilon(t)| + k|\partial_t \sigma_B^\epsilon(t)|)\,dt.
\end{aligned} \tag{3.2}$$

We prove this theorem in several steps. Since we are dealing with the case $\epsilon > 0$ fixed in this theorem we suppress the dependence of ϵ and write u, σ, r, s for$u^\epsilon, \sigma^\epsilon, r^\epsilon, s^\epsilon$. We rewrite the problem (1.1) - (1.3) in terms of the Riemann invariants (r,s) as

$$\begin{gathered}
r_t + (\frac{s-r}{2k} - k) r_x = \epsilon r_{xx}, \\
s_t + (\frac{s-r}{2k} + k) s_x = \epsilon s_{xx}.
\end{gathered} \tag{3.3}$$

with initial conditions

$$r(x,0) = r_0(x) = \sigma_0(x) + k u_0(x), s(x,0) = s_0(x) = \sigma_0(x) - k u_0(x) \tag{3.4}$$

and the boundary conditions

$$r(0,t) = r_B(t) = \sigma_B(t) + k u_B(t), s(0,t) = s_B(t) = \sigma_B(t) - k u_B(t). \tag{3.5}$$

First we assume that r_0 and s_0 are C^∞ functions on$[0,\infty)$ which are in $W^{1,1}(0,\infty)$ and boundary data r_B and s_B are C^∞ which are in $W^{1,1}(0,T)$. The general result then follows from a simple density arguments. To prove the theorem we define a sequence of functions $(r_n(x,t), s_n(x,t))$,$n = 0, 1, 2, \dots$, iteratively,

$$(r_0(x,t), s_0(x,t)) = (r_0(x), s_0(x)),$$

and for $n = 1, 2, \dots, (r_n(x,t), s_n(x,t))$ is defined by the solution of linear problems

$$\begin{gathered}
(r_n)_t + (\frac{s_{n-1} - r_{n-1}}{2k} - k)(r_n)_x = \epsilon (r_n)_{xx}, \\
(s_n)_t + (\frac{s_{n-1} - r_{n-1}}{2k} + k)(s_n)_x = \epsilon (s_n)_{xx}.
\end{gathered} \tag{3.6}$$

with initial conditions

$$r_n(x,0) = r_0(x), s_n(x,0) = s_0(x) \tag{3.7}$$

and the boundary conditions

$$r_n(0,t) = r_B(t), s_n(0,t) = s_B(t). \tag{3.8}$$

Fix $T > 0$, then by linear theory of parabolic equations, see Friedman [2], there exists a unique C^∞ solution (r_1, s_1) to (3.6)–(3.8). Further, the solution decay to 0 as x tends to ∞ and by maximum principle

$$\begin{aligned} \|r_1(x,t)\|_{L^\infty(\Omega_T)} &= \max\big[\|r_0\|_{L^\infty[0,\infty)}, \|r_B\|_{L^\infty[0,T]}\big], \\ \|s_1(x,t)\|_{L^\infty(\Omega_T)} &= \max\big[\|s_0\|_{L^\infty[0,\infty)}, \|s_B\|_{L^\infty[0,T]}\big]. \end{aligned} \tag{3.9}$$

Iteratively we get unique solution (r_n, s_n) of the problem (3.6)–(3.8) in $C^\infty(\Omega_T)$ and

$$\begin{aligned} \|r_n(x,t)\|_{L^\infty(\Omega_T)} &= \max\big[\|r_0\|_{L^\infty[0,\infty)}, \|r_B\|_{L^\infty[0,T]}\big], \\ \|s_n(x,t)\|_{L^\infty(\Omega_T)} &= \max\big[\|s_0\|_{L^\infty[0,\infty)}, \|s_B\|_{L^\infty[0,T]}\big]. \end{aligned} \tag{3.10}$$

Note that

$$\begin{aligned} \lambda_{1n}(x,t) &= \frac{s_n(x,t) - r_n(x,t)}{2k} - k, \\ \lambda_{2n}(x,t) &= \frac{s_n(x,t) - r_n(x,t)}{2k} + k. \end{aligned} \tag{3.11}$$

By (3.9) and (3.10), we have there exists a constant $\lambda \geq 1$ such that

$$\sup_{\Omega_T} |\lambda_{in}(x,t)| \leq \lambda, \quad \text{for } i = 1,2,\ n = 0,1,2,\dots \tag{3.12}$$

For future use we write (3.6)–(3.8) in the integral formulation. For this we introduce the standard boundary heat kernels

$$\begin{aligned} p_\epsilon(x,y,t) &= \frac{1}{\sqrt{4\pi t\epsilon}}[e^{\frac{-(x-y)^2}{4t\epsilon}} - e^{\frac{-(x+y)^2}{4t\epsilon}}], \\ q_\epsilon(x,t,s) &= \frac{-2}{\sqrt{\pi}}\partial_s[\int_{\frac{x}{2\sqrt{\epsilon(t-s)}}}^\infty e^{-y^2}\, dy]. \end{aligned}$$

Then (3.6)–(3.8) is equivalent to

$$\begin{aligned} r_n(x,t) &= \int_0^\infty r_0(y)p_\epsilon(x,y,t)\, dy + \int_0^t r_B(s)q_\epsilon(x,t,s)\, ds \\ &\quad - \int_0^t \int_0^\infty p_\epsilon(x,y,t-s)\lambda_{1n-1}(y,s)\partial_y r_n(y,s)\, dy\, ds \\ s_n(x,t) &= \int_0^\infty s_0(y)p_\epsilon(x,y,t)\, dy + \int_0^t s_B(s)q_\epsilon(x,t,s)\, ds \\ &\quad - \int_0^t \int_0^\infty p_\epsilon(x,y,t-s)\lambda_{2,n-1}(y,s)\partial_y s_n(y,s)\, dy\, ds. \end{aligned} \tag{3.13}$$

With these preliminaries we start the proof of the theorem. First we show that the map $(r_{n-1}, s_{n-1}) \to (r_n, s_n)$ is a contraction in $L^\infty(\Omega_{T_0})$, where T_0 is given by

$$T_0 = \frac{1}{9C_0^2} \tag{3.14}$$

where

$$C_0 = \frac{1}{(\pi\epsilon)^{1/2}}[2\lambda + \frac{1}{2k}(\int_0^\infty (|v_0'(x)| + |w_0'(x)|)\, dx + \int_0^T (|v_B'(t)| + |w_B'(t)|)\, dt)]$$

With this notation we shall prove the following lemma.

Lemma 3.2. *(a) Let $T > 0$ be fixed. Then for $n = 1, 2, \dots$ and $0 \le t \le T$,*

$$\begin{aligned}\int_0^\infty |\partial_x r_n(x,t)|\,dx &\le \int_0^\infty |r_0'|\,dx + \int_0^T |r_B'(t)|\,dt,\\ \int_0^\infty |\partial_x s_n(x,t)|\,dx &\le \int_0^\infty |s_0'|\,dx + \int_0^T |s_B'(t)|\,dt.\end{aligned} \tag{3.15}$$

(b) For $n = 2, 3, \dots$,

$$\|(v_n - v_{n-1}, w_n - w_{n-1})\|_{L^\infty(\Omega_{T_0})} \le \frac{1}{2}\|(v_{n-1} - v_{n-2}, w_{n-1} - w_{n-2})\|_{L^\infty(\Omega_{T_0})} \tag{3.16}$$

Proof. First we prove the estimate (3.15) for r_n, the estimate for s_n is similar. For a fixed $t > 0$, let $y_0(t) = 0$ and $y_i(t)$, $i = 1, 2, \dots$ are the points where $\partial_x r_n(x,t)$ changes sign and let $k = 0$ if $\partial_x r_n(x,t) \ge 0$ and $k = 1$ if $\partial_x r_n(x,t) \le 0$. Following Oleinik [8], we write,

$$\int_0^\infty |\partial_x r_n(x,t)|\,dx = \sum_{i=0}^\infty (-1)^{i+k} \int_{y_i(t)}^{y_{i+1}(t)} \partial_x r_n(x,t)\,dx \tag{3.17}$$

Let us take the case $k = 0$, the other case is similar. Differentiating (3.17), we get

$$\frac{d}{dt}\int_0^\infty |\partial_x r_n(x,t)|\,dx = \sum_{i=0}^\infty (-1)^i \int_{y_i(t)}^{y_{i+1}(t)} \partial_t(\partial_x r_n(x,t))\,dx \tag{3.18}$$

where we have used $\frac{d}{dt}(y_0(t)) = 0$ and $\partial_x r_n(y_i(t), t) = 0$ if $i = 1, 2, \dots$. Now differentiating the first equation of (3.6) with respect to x, multiplying the resulting equation by $(-1)^i$ and then integrating from $y_i(t)$ to $y_{i+1}(t)$, we get for $i = 1, 2, \dots$

$$\begin{aligned}&(-1)^i \int_{y_i(t)}^{y_{i+1}(t)} \partial_t[\partial_x r_n](x,t)\,dx\\ &= \epsilon[(-1)^i \partial_x(\partial_x r_n)(y_{i+1}(t), t) + (-1)^{i+1}\partial_x(\partial_x r_n)(y_i(t), t).\end{aligned} \tag{3.19}$$

For $i = 0$,

$$\int_{y_0(t)}^{y_1(t)} \partial_t[\partial_x r_n](x,t)\,dx = \epsilon[\partial_x(\partial_x v_n)(y_1(t), t) - \partial_x(\partial_x r_n)(0,t)] + (\lambda_{1,n-1}\partial_x r_n)(0,t), \tag{3.20}$$

where we have used $(\partial_x r_n)(y_i(t), t) = 0$, for $i = 1, 2, \dots$. From (3.6) and the boundary condition (3.8), we have

$$\epsilon\partial_{xx} r_n(0,t) - \lambda_{1,n-1}(0,t)\partial_x(0,t) = r_B'(t) \tag{3.21}$$

Also in the present case $\partial_x r_n(x,t)$ changes from positive to negative at $x = y_i(t)$ when i is odd and negative to positive when i is even and hence $\partial_{xx} v_n(y_i(t), t) \le 0$ when i is odd and $\partial_{xx} v_n(y_i(t), t) \ge 0$ when i is even. Using these facts in (3.18)–(3.21) we get,

$$\frac{d}{dt}\int_0^\infty |\partial_x r_n(x,t)|\,dx \le |r_B'(t)|$$

Integrating this from 0 to t and using initial conditions (3.7), we get,

$$\int_0^\infty |\partial_x r_n(x,t)|\,dx \le \int_0^\infty |r_0'(x)|\,dx + \int_0^t |r_B'(t)|\,dt$$

Thus for any $T > 0$ fixed,we have

$$\int_0^\infty |\partial_x r_n(x,t)|\,dx \le \int_0^\infty |r_0'(x)|\,dx + \int_0^T |r_B'(s)|\,ds, \quad \text{if } 0 \le t \le T \tag{3.22}$$

The estimate for s_n is similar. To prove the second part we use the integral representation (3.13) to get

$$\begin{aligned} r_n(x,t) - r_{n-1}(x,t) = &-\int_0^t \int_0^\infty p_\epsilon(x,y,t-s)\big[\lambda_{1n-1}(y,s)\partial_y r_n(y,s) \\ &- \lambda_{1,n-2}(y,s)\partial_y r_{n-1}(y,s)\big]\,dy\,ds \end{aligned}$$

This can be written as

$$r_n(x,t) - r_{n-1}(x,t) = a_n(x,t) + b_n(x,t) \tag{3.23}$$

where

$$a_n(x,t) = -\int_0^t \int_0^\infty p_\epsilon(x,y,t-s)\big(\frac{r_{n-1} - s_{n-1}}{2k} - k\big)\partial_y(r_n - r_{n-1})\,dy\,ds \tag{3.24}$$

and

$$b_n(x,t) = -\int_0^t \int_0^\infty p_\epsilon(x,y,t-s)\big(\frac{(s_{n-1} - s_{n-2})}{2k} - \frac{(r_{n-1} - r_{n-2})}{2k}\big)\partial_y r_{n-1}\,dy\,ds \tag{3.25}$$

Integrating by parts and changing variables we get

$$\begin{aligned} &a_n(x,t) \\ &= \frac{1}{(\pi\epsilon)^{1/2}} \int_0^t \frac{1}{(t-s)^{1/2}} \int_{-\infty}^{x/(4(t-s)\epsilon)^{1/2}} z e^{-z^2}\big(\frac{(s_{n-1} - r_{n-1})}{2k} + k\big)(r_n - r_{n-1})\,dz \\ &\quad - \frac{1}{(\pi\epsilon)^{1/2}} \int_0^t \frac{1}{(t-s)}^{1/2} \int_{x/(4(t-s)\epsilon)^{1/2}}^\infty z e^{-z^2}\big(\frac{(s_{n-1} - r_{n-1})}{2k} + k\big)(r_n - r_{n-1})\,dz \\ &\quad + \int_0^t \int_0^\infty p_\epsilon(x,y,t-s)\partial_y \frac{(s_{n-1} - r_{n-1})}{2k}(r_n - r_{n-1})\,dy\,ds. \end{aligned} \tag{3.26}$$

So we get for $0 \le t \le t_0 \le T$,

$$\begin{aligned} |a_n(x,t)| \le{}& \frac{t_0^{1/2}}{(\pi\epsilon)^{1/2}} \|r_n - r_{n-1}\|_{L^\infty(\Omega_{t_0})}\Big[2\lambda \\ &+ \frac{1}{2k}\big(\int_0^\infty (|r_0'(x)| + |s_0'(x)|)\,dx + \int_0^T (|r_B'(t)| + |s_B'(t)|)\,dt\big)\Big]. \end{aligned} \tag{3.27}$$

Similarly, for $0 \le t \le t_0 \le T$,

$$\begin{aligned} |b_n(x,t)| \le{}& \frac{t_0^{1/2}}{(\pi\epsilon)^{1/2}} \frac{(\|r_{n-1} - r_{n-2}\|_{L^\infty(\Omega_{t_0})} + \|s_{n-1} - s_{n-2}\|_{L^\infty(\Omega_{t_0})})}{2k} \\ &\times \Big[\int_0^\infty |r_0'(x)|\,dx + \int_0^T |r_B'(t)|\,dt\Big] \end{aligned} \tag{3.28}$$

From (3.23)–(3.28), we get for $0 \le t \le t_0 \le T$,

$$\begin{aligned}
&|r_n(x,t) - r_{n-1}(x,t)| \\
&\le \frac{t_0^{1/2}}{(\pi\epsilon)^{1/2}}\Big[2\lambda + \frac{1}{2k}\Big(\int_0^\infty (|r_0'(x)| + |s_0'(x)|)\,dx + \int_0^T (|r_B'(t)| + |s_B'(t)|)\,dt\Big)\Big] \\
&\quad \times \|r_n - s_{n-1}\|_{L^\infty(\Omega_{t_0})} + \frac{t_0^{1/2}}{(\pi\epsilon)^{1/2}}\frac{1}{2k}\Big(\int_0^\infty |r_0'(x)|\,dx + \int_0^T |r_B'(t)|\,dt\Big) \\
&\quad \times (\|r_{n-1} - r_{n-2}\|_{L^\infty(\Omega_{t_0})} + \|s_{n-1} - s_{n-2}\|_{L^\infty(\Omega_{t_0})})
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
|s_n(x,t) - s_{n-1}(x,t)| &\le \frac{t_0^{1/2}}{(\pi\epsilon)^{1/2}}[2\lambda + \frac{1}{2k}(\int_0^\infty (|r_0'(x)| + |s_0'(x)|)\,dx \\
&\quad + \int_0^T (|r_B'(t)| + |s_B'(t)|)\,dt)] \times \|s_n - s_{n-1}\|_{L^\infty(\Omega_{t_0})} \\
&\quad + \frac{(t_0^{1/2}}{(\pi\epsilon)^{1/2}}\frac{1}{2k}\Big(\int_0^\infty |s_0'(x)|\,dx + \int_0^T |s_B'(t)|\,dt\Big) \\
&\quad \times (\|r_{n-1} - r_{n-2}\|_{L^\infty(\Omega_{t_0})} + \|s_{n-1} - s_{n-2}\|_{L^\infty(\Omega_{t_0})}).
\end{aligned} \tag{3.30}$$

From (3.29) and (3.30), we get

$$\begin{aligned}
&\|r_n - r_{n-1}\|_{L^\infty(\Omega_{t_0})} + \|s_n - s_{n-1}\|_{L^\infty(\Omega_{t_0})} \\
&\le C(t_0)^{1/2}[\|r_n - r_{n-1}\|_{L^\infty(\Omega_{t_0})} + \|s_n - s_{n-1}\|_{L^\infty(\Omega_{t_0})}] \\
&\quad + C(t_0)^{1/2}[\|r_{n-1} - r_{n-2}\|_{L^\infty(\Omega_{t_0})} + \|s_{n-1} - s_{n-2}\|_{L^\infty(\Omega_{t_0})}]
\end{aligned} \tag{3.31}$$

where C_0 is given by (3.14). Now take $t_0 = T_0 = \frac{1}{9C_0^2}$ in (3.14) and the estimate (3.16) follows. The proof of Lemma is complete. □

Proof of Theorem 3.1. First we shall prove that there exists a continuous function (r,s) such that the sequence (r_n, s_n) converges uniformly to to (r,s) on Ω_T. Estimate (3.16) shows that (r_n, s_n) converges uniformly to a continuous function (r_{T_0}, s_{T_0}) on Ω_{T_0}. Now we consider the region

$$\Omega_{T_0,2T_0} = [(x,t) : x \ge 0, T_0 \le t \le 2T_0].$$

Consider problem (3.6) in $\Omega_{T_0,2T_0}$ with initial data at T_0 as $(r_n(x,T_0), s_n(x,T_0))$. Now use the estimates (3.10) and (3.15) and using the same argument to get the estimate (3.16) to get

$$\begin{aligned}
&\|(r_n - r_{n-1}, s_n - s_{n-1})\|_{L^\infty(\Omega_{T_0,2T_0})} \\
&\le \frac{1}{2}\|(r_{n-1} - r_{n-2}, s_{n-1} - s_{n-2})\|_{L^\infty(\Omega_{T_0,2T_0})} \\
&\quad + \frac{3}{2}\|(r_n(x,T_0) - r_{n-1}(x,T_0), s_n(x,T_0) - s_{n-1}(x,T_0)\|_{L^\infty[0,\infty)}.
\end{aligned}$$

Iterating this inequality leads to

$$\begin{aligned}
&\|(r_n - r_{n-1}, s_n - s_{n-1})\|_{L^\infty(\Omega_{T_0,2T_0})} \\
&\le (\frac{1}{2})^{(n-2)} \|(r_2 - r_1, s_2 - s_1)\|_{L^\infty(\Omega_{T_0,2T_0})} \\
&\quad + 3(n-1)(\frac{1}{2})^{(n-2)} \|(r_n(x,T_0) - r_{n-1}(x,T_0), s_n(x,T_0 - s_{n-1}(x,T_0)\|_{L^\infty[0,\infty)}
\end{aligned}$$

Using the estimate (3.10) in the above equation, we get

$$\|(v_n - v_{n-1}, w_n - w_{n-1})\|_{L^\infty(\Omega_{T_0,2T_0})} \le C_T.6n(1/2)^{(n-2)} \tag{3.32}$$

where $C_T = \max[\|(r_0,s_0)\|_{L^\infty}, \|(r_B,s_B)\|_{L^\infty[0,T]}]$. Estimate (3.32) shows that (r_n,s_n) is Cauchy sequence in $\Omega_{T_0,2T_0}$ in the uniform norm and hence converges to a continuous function (r,s). Repeating this for a finite number of time intervals we get (r_n,s_n) converge uniformly to a continuous function (r,s) in Ω_T. Now passing to the limit in (3.13) we get (r,s) satisfies the integral equation

$$\begin{aligned}
r(x,t) &= \int_0^\infty r_0(y)p_\epsilon(x,y,t)\,dy + \int_0^t r_B(s)q_\epsilon(x,t,s)\,ds \\
&\quad - \int_0^t \int_0^\infty p_\epsilon(x,y,t-s)\lambda_1(r,s)(y,s)\partial_y r(y,s)\,dy\,ds, \\
s(x,t) &= \int_0^\infty s_0(y)p_\epsilon(x,y,t)\,dy + \int_0^t s_B(s)q_\epsilon(x,t,s)\,ds \\
&\quad - \int_0^t \int_0^\infty p_\epsilon(x,y,t-s)\lambda_2(r,s)(y,s)\partial_y s(y,s)\,dy\,ds.
\end{aligned}$$

From this integral representation it follows that (r,s) is once continuously differentiable in t and twice continuously differentiable in x and solves the problem (3.3)–(3.4). Further the estimate (3.1) and (3.2) follows from (3.10) and (3.15). The proof of the theorem is complete. □

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