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PERIODICITY AND STABILITY IN NEUTRAL NONLINEAR DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DELAY

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Abstract. We study the existence and uniqueness of periodic solutions and the stability of the zero solution of the nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))).$$

In the process we use integrating factors and convert the given neutral differential equation into an equivalent integral equation. Then we construct appropriate mappings and employ Krasnoselskii's fixed point theorem to show the existence of a periodic solution of this neutral differential equation. We also use the contraction mapping principle to show the existence of a unique periodic solution and the asymptotic stability of the zero solution provided that $Q(0,0) = G(t,0,0) = 0$.

1. Introduction

Theory of functional differential equations with delay has undergone a rapid development in the previous fifty years. We refer the readers to [1, 2, 6, 7, 11, 13] and the references therein for a wealth of reference materials on the subject. More recently researchers have given special attentions to the study of equations in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, so-called neutral differential equations. In particular, qualitative analysis such as periodicity and stability of solutions of neutral differential equations has been studied extensively by many authors. We refer to [3, 8, 9, 10, 12, 14, 15, 16, 17, 18] for some recent work on the subject of periodicity and stability of neutral equations. This reference list is not complete by any means.

Neutral differential equations have many applications. For example, these equations arise in the study of two or more simple oscillatory systems with some interconnections between them [4, 19], and in modelling physical problems such as vibration of masses attached to an elastic bar [19].

In the current paper, we study the existence of periodic solutions of the nonlinear system of differential equations

$$\frac{d}{dt}x(t) = -a(t)x(t) + \frac{d}{dt}Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))) \tag{1.1}$$

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where $a(t)$ is a continuous real-valued function. The functions $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous in their respective arguments. In the analysis we use the idea of integrating factor and convert equation (1.1) into an integral equation. Then we employ Krasnoselskii's fixed point theorem and show the existence of a periodic solution of (1.1) in Theorem 2.5. We also obtain the existence of a unique periodic solution in Theorem 2.6 employing the contraction mapping principle as the basic mathematical tool.

While most of the existing research on the qualitative analysis of neutral differential equations deal with constant delay, equation (1.1) contains a non-constant function $g(t)$ as the delay term. Related to our present work are articles [14, 15], where in [15] the author has studied the periodic solutions of a scalar neutral differential equation, and in [14] the authors considered the discrete analogue of [15]. We remark that equation (1.1) has a couple of features that are distinct from the features of the equation in [15]. The neutral term $\frac{d}{dt}Q(t, x(t-g(t)))$ of (1.1) allows nonlinearity in the derivative term $x'(t-g(t))$ (see Example 2.8). On the other hand, the neutral term $x'(t-g(t))$ in [15] enters linearly. As a result of these differences, the mathematical analysis used in this research to construct the mappings to employ fixed point theorems are different than that of [15]. In addition to the study of periodicity, in this research we obtain sufficient conditions for the asymptotic stability of the zero solution.

2. Existence of Periodic Solutions

For $T > 0$ let P_T be the set of all continuous scalar functions $x(t)$, periodic in t of period T. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

Since we are searching for the existence of periodic solutions for system (1.1), it is natural to assume that

$$a(t+T) = a(t), \quad g(t+T) = g(t) \tag{2.1}$$

with $g(t)$ being scalar, continuous, and $g(t) \geq g^* > 0$. Also, we assume

$$\int_0^T a(s)ds > 0. \tag{2.2}$$

Functions $Q(t,x)$ and $G(t,x,y)$ are periodic in t of period T. They are also globally Lipschitz continuous in x and in x and y, respectively. That is

$$Q(t+T,x) = Q(t,x),\ G(t+T,x,y) = G(t,x,y), \tag{2.3}$$

and there are positive constants E_1, E_2, E_3 such that

$$|Q(t,x) - Q(t,y)| \leq E_1\|x-y\| \tag{2.4}$$

and

$$|G(t,x,y) - G(t,z,w))| \leq E_2\|x-z\| + E_3\|y-w\|. \tag{2.5}$$

The next lemma is crucial to our results.

Lemma 2.1. *Suppose* (2.1) *and* (2.3) *hold. If* $x(t) \in P_T$, *then* $x(t)$ *is a solution of equation* (1.1) *if and only if*

$$\begin{aligned} x(t) &= Q(t, x(t-g(t))) \\ &\quad + (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \int_{t-T}^{t} \Big[- a(u)Q(u, x(u-g(u))) \\ &\quad + G(u, x(u), x(u-g(u))) \Big] e^{-\int_u^t a(s)ds} du. \end{aligned} \tag{2.6}$$

Proof. Let $x(t) \in P_T$ be a solution of (1.1). First we write this equation as

$$\begin{aligned} \frac{d}{dt}\{x(t) - Q(t, x(t-g(t)))\} &= -a(t)\{x(t) - Q(t, x(t-g(t)))\} \\ &\quad - a(t)Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))). \end{aligned}$$

Multiply both sides of (1.1) with $e^{\int_0^t a(s)ds}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned} &\int_{t-T}^{t} \Big[\big(x(u) - Q(u, x(u-g(u)))\big) e^{\int_0^u a(s)ds} \Big]' du \\ &= \int_{t-T}^{t} \Big[- a(u)Q(u, x(u-g(u))) + G(u, x(u), x(u-g(u))) \Big] e^{\int_0^u a(s)ds} du. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &\Big(x(t) - Q(t, x(t-g(t)))\Big) e^{\int_0^t a(s)ds} \\ &- \Big(x(t-T) - Q(t-T, x(t-T-g(t-T)))\Big) e^{\int_0^{t-T} a(s)ds} \\ &= \int_{t-T}^{t} \Big[- a(u)Q(u, x(u-g(u))) + G(u, x(u), x(u-g(u))) \Big] e^{\int_0^u a(s)ds} du. \end{aligned}$$

Dividing both sides of the above equation by $\exp(\int_0^t a(s)ds)$ and the fact that $x(t) = x(t-T)$ and (2.1), we obtain

$$\begin{aligned} &x(t) - Q(t, x(t-g(t)) \\ &= (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \int_{t-T}^{t} \Big[- a(u)Q(u, x(u-g(u))) \\ &\quad + G(u, x(u), x(u-g(u))) \Big] e^{-\int_u^t a(s)ds} du. \end{aligned}$$

□

Define a mapping H by

$$\begin{aligned} (H\varphi)(t) &= Q(t, \varphi(t-g(t))) \\ &\quad + (1 - e^{-\int_{t-T}^{t} a(s)ds})^{-1} \int_{t-T}^{t} \Big[- a(u)Q(u, \varphi(u-g(u))) \\ &\quad + G(u, \varphi(u), \varphi(u-g(u))) \Big] e^{-\int_u^t a(s)ds} du. \end{aligned} \tag{2.7}$$

It is clear form (2.7) that $H : P_T \to P_T$ by the way it was constructed in Lemma 2.1.

Next we state Krasnoselskii's fixed point theorem which enables us to prove the existence of a periodic solution. For the proof of Krasnoselskii's fixed point theorem we refer the reader to [5].

Theorem 2.2 (Krasnoselskii). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that C and B map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) *$x, y \in \mathbb{M}$, implies $Cx + By \in \mathbb{M}$,*
(ii) *C is continuous and $C\mathbb{M}$ is contained in a compact set,*
(iii) *B is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = Cz + Bz$.

We note that to apply the above theorem we need to construct two mappings; one is contraction and the other is compact. Therefore, we express equation (2.7) as

$$(H\varphi)(t) = (B\varphi)(t) + (C\varphi)(t),$$

where $C, B : P_T \to P_T$ are given by

$$(B\varphi)(t) = Q(t, \varphi(t - g(t))) \tag{2.8}$$

and

$$\begin{aligned}(C\varphi)(t) = (1 - e^{-\int_{t-T}^t a(s)ds})^{-1} \int_{t-T}^t \Big[&- a(u)Q(u, \varphi(u - g(u))) \\ &+ G(u, \varphi(u), \varphi(u - g(u)))\Big] e^{-\int_u^t a(s)ds} du.\end{aligned} \tag{2.9}$$

To simplify notations, we introduce the following constants.

$$\eta = \max_{t\in[0,T]} |(1 - e^{-\int_{t-T}^t a(s)ds})^{-1}|, \quad \rho = \max_{t\in[0,T]} |a(t)|, \quad \gamma = \max_{u\in[t-T,t]} e^{-\int_u^t a(s)ds}. \tag{2.10}$$

Lemma 2.3. *If C is defined by* (2.9)*, then C is continuous and the image of C is contained in a compact set.*

Proof. Let C be defined by (2.9) and let $\varphi, \psi \in P_T$. Given $\epsilon > 0$, take $\delta = \epsilon/N$ with $N = \eta\gamma T(\rho E_1 + E_2 + E_3)$, where E_1, E_2 and E_3 are given by (2.4) and (2.5). Now, for $\|\varphi - \psi\| < \delta$. we obtain

$$\big\|C\varphi - C\psi\big\| \le \eta\gamma \int_0^T \big[\rho E_1\|\varphi - \psi\| + (E_2 + E_3)\|\varphi - \psi\|\big] du \le N\|\varphi - \psi\| < \epsilon.$$

This proves that C is continuous. To show that the image of C is contained in a compact set, we consider $D = \{\varphi \in P_T : \|\varphi\| \le R\}$, where R is a fixed positive constant. Let $\varphi_n \in D$ where n is a positive integer. Observe that in view of (2.4) and (2.5) we have

$$\begin{aligned}|Q(t, x)| &= |Q(t, x) - Q(t, 0) + Q(t, 0)| \\ &\le |Q(t, x) - Q(t, 0)| + |Q(t, 0)| \\ &\le E_1\|x\| + \alpha.\end{aligned}$$

Similarly,

$$\begin{aligned}|G(t, x, y)| &= |G(t, x, y) - G(t, 0, 0) + G(t, 0, 0)| \\ &\le |G(t, x, y) - G(t, 0, 0)| + |G(t, 0, 0)| \\ &\le E_2\|x\| + E_3\|y\| + \beta\end{aligned}$$

where $\alpha = \sup_{t\in[0,T]} |Q(t,0)|$ and $\beta = \sup_{t\in[0,T]} |G(t,0,0)|$. Hence, if C is given by (2.9) we obtain that

$$\|C\varphi_n\| \le L$$

for some positive constant L. Next we calculate $(C\varphi_n)'(t)$ and show that it is uniformly bounded. By making use of (2.2) and (2.3) we obtain by taking the derivative in (2.9) that

$$(C\varphi_n)'(t) = -a(t)C(\varphi_n(t)) - a(t)Q(t,\varphi_n(t-g(t))) + G(t,\varphi_n(t),\varphi_n(t-g(t))).$$

Thus, the above expression yields $\|(C\varphi_n)'\| \le F$, for some positive constant F. Thus the sequence $C\varphi_n$ is uniformly bounded and equi-continuous. Hence by Ascoli-Arzela's theorem $C(D)$ is compact. $\square$

Lemma 2.4. *If B is given by* (2.8) *with $E_1 < 1$, then B is a contraction.*

Proof. Let B be defined by (2.8). Then for $\varphi,\psi \in P_T$ we have

$$\begin{aligned}\|B(\varphi) - B(\psi)\| &= \sup_{t\in[0,T]} |B(\varphi) - B(\psi)| \\ &\le E_1 \sup_{t\in[0,T]} |\varphi(t-g(t)) - \psi(t-g(t)| \\ &\le E_1\|\varphi - \psi\|.\end{aligned}$$

Hence B defines a contraction. $\square$

Theorem 2.5. *Suppose the hypothesis of Lemma 2.4. Let $\alpha = \sup_{t\in[0,T]} |Q(t,0)|$ and $\beta = \sup_{t\in[0,T]} |G(t,0,0)|$. Suppose* (2.1)-(2.5) *hold. Let J be a positive constant satisfying the inequality*

$$\alpha + E_1 J + \eta T\gamma\big[\rho(E_1 J + \alpha) + (E_2 + E_3)J + \beta\big] \le J. \tag{2.11}$$

Let $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \le J\}$. Then equation (1.1) *has a solution in M.*

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \le J\}$. By Lemma 2.3, C is continuous and $C\mathbb{M}$ is contained in a compact set. Also, from Lemma 2.4, the mapping B is a contraction and it is clear that $B : P_T \to P_T$. Next, we show that if $\varphi,\psi \in \mathbb{M}$, we have $\|C\phi + B\psi\| \le J$. Let $\varphi,\psi \in \mathbb{M}$ with $\|\varphi\|, \|\psi\| \le J$. Then

$$\begin{aligned}\big\|C\varphi + B\psi\big\| &\le E_1\|\psi\| + \alpha + \eta\gamma\int_0^T [|a(u)|(\alpha + E_1\|\varphi\|) + E_2\|\varphi\| + E_3\|\varphi\| + \beta]du \\ &\le \alpha + E_1 J + \eta T\gamma\Big[\rho(E_1 J + \alpha) + (E_2 + E_3)J + \beta\Big] \le J.\end{aligned}$$

We now see that all the conditions of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point z in $\mathbb{M}$ such that $z = Az + Bz$. By Lemma 2.1, this fixed point is a solution of (1.1). Hence (1.1) has a T-periodic solution. $\square$

Theorem 2.6. *Suppose* (2.1)-(2.5) *hold. If*

$$E_1 + \eta\gamma T(\rho E_1 + E_2 + E_3) < 1,$$

then equation (1.1) *has a unique T-periodic solution.*

Proof. Let the mapping H be given by (2.7). For $\varphi,\psi \in P_T$, in view of (2.7), we have

$$\big\|H\varphi - H\psi\big\| \le \big(E_1 + \eta\gamma T(\rho E_1 + E_2 + E_3)\big)\|\varphi - \psi\|.$$

This completes the proof by invoking the contraction mapping principle. $\square$

It is worth noting that Theorems 2.5 and 2.6 are not applicable to functions such as

$$G(t,\varphi(t),\varphi(t-g(t)))=f_1(t)\varphi^2(t)+f_2(t)\varphi^2(t-g(t))),$$

where $f_1(t), f_2(t)$ and $g(t)>0$ are continuous and periodic. To accommodate such functions, we state the following corollary, in which the functions G and Q are required to satisfy local Lipschitz conditions.

Corollary 2.7. *Suppose* (2.1)-(2.3) *hold and let α and β be the constants defined in Theorem 2.5. Let J be a positive constant and define $\mathbb{M}=\{\varphi\in P_T:\|\varphi\|\le J\}$. Suppose there are positive constants E_1^*, E_2^* and E_3^* so that for x,y,z and $w\in\mathbb{M}$ we have*

$$|Q(t,x)-Q(t,y)|\le E_1^*\|x-y\|,$$
$$|G(t,x,y)-G(t,z,w)|\le E_2^*\|x-z\|+E_3^*\|y-w\|.$$

If $E_1^*<1$ and $\|H\varphi\|\le J$, for $\varphi\in\mathbb{M}$, then (1.1) *has a T-periodic solution in $\mathbb{M}$. Moreover, if*

$$E_1^*+\eta\gamma T(\rho E_1^*+E_2^*+E_3^*)<1, \tag{2.12}$$

then (1.1) *has a unique solution in $\mathbb{M}$.*

Proof. Let $\mathbb{M}=\{\varphi\in P_T:\|\varphi\|\le J\}$. Let the mapping H be given by (2.7). Then the results follow immediately from Theorem 2.5 and Theorem 2.6. □

We remark that the constants E_j^*, $j=1,2,3$ may depend on J. Now we display an example.

Example 2.8. *For small positive ε_1 and ε_2, we consider the perturbed Van Der Pol equation*

$$x'=-(2+\sin(\omega t))x(t)+\varepsilon_1\frac{d}{dt}\big(\sin(\omega t)x^2(t-g(t))\big)+\varepsilon_2\big(\cos(\omega t)+x^2(t)\big) \tag{2.13}$$

where $g(t)$ is nonnegative, continuous and $\frac{2\pi}{\omega}$-periodic for ω is positive. So we have

$$a(t)=2+\sin(\omega t),\quad Q(t,x(t-g(t)))=\varepsilon_1\sin(\omega t)x^2(t-g(t))$$

and

$$G(t,x(t),x(t-g(t)))=\varepsilon_2\big(\cos(\omega t)+x^2(t)\big).$$

Define $\mathbb{M}=\{\phi\in P_{\frac{2\pi}{\omega}}:\|\phi\|\le J\}$, where J is a positive constant. For $\varphi\in\mathbb{M}$, we have

$$\begin{aligned}\|H\varphi\|&=\|Q(t,\varphi(t-g(t)))\\&\quad+(1-e^{-\int_{t-T}^t a(s)ds})^{-1}\int_{t-T}^t\Big[-a(u)Q(u,\varphi(u-g(u)))\\&\quad+G(u,\varphi(u),\varphi(u-g(u)))\Big]e^{-\int_u^t a(s)ds}du\|\\&\le\varepsilon_1J^2+\frac{2\pi}{\omega}\Big(1-e^{-\frac{4\pi}{\omega}}\Big)^{-1}\Big[3\varepsilon_1J^2+\varepsilon_2J^2+\varepsilon_2\Big].\end{aligned}$$

Thus, the inequality

$$\varepsilon_1J^2+\frac{2\pi}{\omega}\big(1-e^{-\frac{4\pi}{\omega}}\big)^{-1}\big[3\varepsilon_1J^2+\varepsilon_2J^2+\varepsilon_2\big]\le J, \tag{2.14}$$

which is satisfied for small ω, ε_1 and ε_2, implies $\|H\varphi\|\le J$. Hence, (2.13) *has a $\frac{2\pi}{\omega}$-periodic solution, by Corollary 2.7.*

For the uniqueness of the solution we let $\varphi, \psi \in \mathbb{M}$. From (2.13) we see that

$$\eta = \left(1 - e^{-\frac{4\pi}{\omega}}\right)^{-1}, \quad \rho \le 3, \quad \gamma \le 1.$$

Also $\alpha = 0$, $\beta = \varepsilon_2$, $E_1^* = 2\varepsilon_1 J$, $E_3^* = 0$, $E_2^* = 2\varepsilon_2 J$, where J is given by (2.14). *If*

$$2\varepsilon_1 J + \frac{2\pi}{\omega}\left(1 - e^{-\frac{4\pi}{\omega}}\right)^{-1}\left[6\varepsilon_1 J + 2\varepsilon_2 J\right] < 1$$

is satisfied for small ε_1 and ε_2, then (2.13) *has a unique $\frac{2\pi}{\omega}$-periodic solution.*

3. Stability

Lyapunov functions and functionals have been successfully used to obtain boundedness, stability and the existence of periodic solutions of differential equations, differential equations with functional delays and functional differential equations. In the study of differential equations with functional delays by using Lyapunov functionals, many difficulties arise if the delay is unbounded or if the differential equation in question has unbounded terms. In [17], the third author using fixed point theory, studied the asymptotic stability of the zero solution of the scalar neutral differential equation

$$x'(t) = -a(t)x(t) + c(t)x'(t - g(t)) + q\big(t, x(t), x(t - g(t))\big), \tag{3.1}$$

where $a(t), b(t), g(t)$ and q are continuous in their respective arguments. It is clear that (1.1) is more general than (3.1).

This section is mainly concerned with the asymptotic stability of the zero solution of (1.1). We assume that the functions Q and G are continuous, as before. Also, we assume that $g(t)$ is continuous, $g(t) \ge g^* > 0$ and $Q(t, 0) = G(t, 0, 0) = 0$. The techniques used in this section are adapted from the paper of [2].

To arrive at the correct mapping, we rewrite (1.1) as in the proof of Lemma 2.1, multiply both sides by $e^{\int_0^t a(s)ds}$ and then integrate from 0 to t to obtain

$$\begin{aligned} x(t) &= Q(t, x(t - g(t))) + \Big(x(0) - Q(0, x(-g(0)))\Big)e^{-\int_0^t a(s)ds} \\ &\quad + \int_0^t \Big[- a(u)Q(u, x(u - g(u))) \\ &\quad + G(u, x(u), x(u - g(u)))\Big] e^{-\int_u^t a(s)ds} du. \end{aligned} \tag{3.2}$$

Thus, we see that $x(t)$ is a solution of (1.1) if and only if it satisfies (3.2). Let $\psi(t) : (-\infty, 0] \to \mathbb{R}$ be a given continuous bounded initial function. We say $x(t) := x(t, 0, \psi)$ is a solution of (1.1) if $x(t) = \psi(t)$ for $t \le 0$ and satisfies (1.1) for $t \ge 0$.

We say the zero solution of (1.1) is stable at t_0 if for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\big[\psi : (-\infty, t_0] \to \mathbb{R}$ with $\|\psi\| < \delta$ on $(-\infty, t_0]$, implies $|x(t, t_0, \psi)| < \varepsilon$.

Without loss of generality, we will state and prove our results by starting at $t_0 = 0$. Let C be the space of all continuous functions from $\mathbb{R} \to \mathbb{R}$ and define the set S by

$$\begin{aligned} S = \Big\{&\varphi : \mathbb{R} \to \mathbb{R} : \varphi(t) = \psi(t) \text{ if } t \le 0, \varphi(t) \to 0, \text{ as } t \to \infty, \\ &\varphi \in C \text{ and } \varphi \text{ is bounded}\Big\}. \end{aligned}$$

Then, $(S, \|\cdot\|)$ is a complete metric space where $\|\cdot\|$ is the supremum norm. For the next theorem we impose the following conditions.

$$\int_0^t a(s)\,ds > 0 \quad \text{and} \quad e^{-\int_0^t a(s)ds} \to 0, \quad \text{as } t \to \infty, \tag{3.3}$$

there is an $\alpha > 0$ such that

$$E_1 + \int_0^t \Big[|a(u)|E_1 + E_2 + E_3\Big] e^{-\int_u^t a(s)ds} du \le \alpha < 1, \quad t \ge 0, \tag{3.4}$$

$$t - g(t) \to \infty, \quad \text{as } t \to \infty, \tag{3.5}$$

$$Q(t, 0) \to 0, \quad \text{as } t \to \infty. \tag{3.6}$$

Theorem 3.1. *If* (2.4),(2.5), (3.3) - (3.6) *hold, then every solution* $x(t, 0, \psi)$ *of* (1.1) *with small continuous initial function* $\psi(t)$*, is bounded and approaches zero as* $t \to \infty$*. Moreover, the zero solution is stable at* $t_0 = 0$*.*

Proof. Define the mapping $P : S \to S$ by

$$\big(P\varphi\big)(t) = \begin{cases} \psi(t) & \text{if } t \le 0, \\ Q(t, \varphi(t - g(t))) + \big[\psi(0) - Q(0, \psi(-g(0)))\big] e^{-\int_0^t a(s)ds} \\ + \int_0^t \Big[- a(u)Q(u, \varphi(u - g(u))) \\ +G(u, \varphi(u), \varphi(u - g(u)))\Big] e^{-\int_u^t a(s)ds} du, & \text{if } t \ge 0\,. \end{cases}$$

It is clear that for $\varphi \in S$, $P\varphi$ is continuous. Let $\varphi \in S$ with $\|\varphi\| \le K$, for some positive constant K. Let $\psi(t)$ be a small given continuous initial function with $\|\psi\| < \delta, \delta > 0$. Then using (3.4) in the definition of $(P\varphi)(t)$, we have

$$\begin{aligned} \|(P\varphi)(t)\| &\le E_1 K + \big|\big(\psi(0) - Q(0, \psi(-g(0)))\big)\big| e^{-\int_0^t a(s)ds} \\ &\quad + \int_0^t \big[|a(u)|E_1 + E_2 + E_3\big] e^{-\int_u^t a(s)ds} du K \\ &\le \big(1 + E_1\big)\delta + E_1 K + \int_0^t \big[|a(u)|E_1 + E_2 + E_3\big] e^{-\int_u^t a(s)ds} du K \\ &\le \big(1 + E_1\big)\delta + \alpha K, \end{aligned} \tag{3.7}$$

which implies $\|\big(P\varphi\big)(t)\| \le K$, for the right δ. Thus, (3.7) implies that $(P\varphi)(t)$ is bounded. Next we show that $\big(P\varphi\big)(t) \to 0$ as $t \to \infty$. The second term on the right side of $\big(P\varphi\big)(t)$ tends to zero, by condition (3.3). Also, the first term on the right side tends to zero, because of (3.5), (3.6) and the fact that $\varphi \in S$. Left to show that the integral term goes to zero as $t \to \infty$.

Let $\epsilon > 0$ be given and $\varphi \in S$ with $\|\varphi\| \le K$, $K > 0$. Then, there exists a $t_1 > 0$ so that for $t > t_1$, $|\varphi(t - g(t))| < \epsilon$. Due to condition (3.3), there exists a $t_2 > t_1$ such that for $t > t_2$ implies that $\exp(-\int_{t_1}^t a(s)ds) < \frac{\epsilon}{\alpha K}$. Thus for $t > t_2$, we have

$$
\begin{aligned}
&\Big|\int_0^t \Big[-a(u)Q(u,\varphi(u-g(u))) + G(u,\varphi(u),\varphi(u-g(u)))\Big] e^{-\int_u^t a(s)ds} du\Big| \\
&\le K\int_0^{t_1} \big[|a(u)|E_1 + E_2 + E_3\big] e^{-\int_u^t a(s)ds} du \\
&\quad + \epsilon \int_{t_1}^t \big[|a(u)|E_1 + E_2 + E_3\big] e^{-\int_u^t a(s)ds} du \\
&\le K e^{-\int_{t_1}^t a(s)ds} \int_0^{t_1} \big[|a(u)|E_1 + E_2 + E_3\big] e^{-\int_u^{t_1} a(s)ds} du + \alpha\epsilon \\
&\le \alpha K e^{-\int_{t_1}^t a(s)ds} + \alpha\epsilon \le \epsilon + \alpha\epsilon.
\end{aligned}
$$

Hence, $\big(P\varphi\big)(t) \to 0$ as $t \to \infty$. It remains to show that $\big(P\varphi\big)(t)$ is a contraction under the supremum norm. Let $\zeta, \eta \in S$. Then

$$
\begin{aligned}
\big|(P\zeta)(t) - (P\eta)(t)\big| &\le \Big\{E_1 + \int_0^t \big[|a(u)|E_1 + E_2 + E_3\big] e^{-\int_u^t a(s)ds} du\Big\} \|\zeta - \eta\| \\
&\le \alpha\|\zeta - \eta\|.
\end{aligned}
$$

Thus, by the contraction mapping principle, P has a unique fixed point in S which solves (1.1), bounded and tends to zero as t tends to infinity. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing K by ϵ. This completes the proof. $\square$

Example 3.2. Let ψ be a small continuous initial function, $\psi : (-\infty, 0] \to \mathbb{R}$ with $\|\psi\| \le \delta$, for positive δ. For small ε_1 and ε_2, we consider the nonlinear neutral differential equation,

$$
x'(t) = -2x(t) + \varepsilon_1 \frac{d}{dt} x^2(t - (\frac{t}{2})) + \varepsilon_2\big(x^2(t) + x^2(t - (\frac{t}{2}))\big). \tag{3.8}
$$

Suppose

$$
0 < 4(2\varepsilon_1 + \varepsilon_2)\delta(1 + \varepsilon_1\, \delta) < 1, \tag{3.9}
$$

and define the set S by

$$
S = \big\{\varphi : \mathbb{R} \to \mathbb{R}|\ \varphi(t) = \psi(t) \text{ if } t \le 0, \varphi(t) \to 0, \text{as } t \to \infty, \varphi \in C \text{ and } \|\varphi\| \le K\big\},
$$

for positive constant K satisfying the inequality

$$
\frac{1 - \sqrt{1 - 4(2\varepsilon_1 + \varepsilon_2)\delta(1 + \varepsilon_1\, \delta)}}{2(2\varepsilon_1 + \varepsilon_2)} < K < \frac{1}{2(2\varepsilon_1 + \varepsilon_2)}. \tag{3.10}
$$

Then every solution $x(t, 0, \psi)$ of (3.8) is bounded and approaches 0 as $t \to \infty$.

Proof. It is clear from (3.9) that inequality (3.10) is well defined. Define

$$
(P\varphi)(t) = \begin{cases} \psi(t) & \text{if } t \le 0, \\ = \varepsilon_1\varphi^2(t/2) + \big(\psi(0) - \varepsilon_1\ \psi^2(0)\big)e^{-2t} + \int_0^t \big(-2\varepsilon_1\varphi^2(u/2) & \\ +\varepsilon_2(\varphi^2(u) + \varphi^2(u/2))\big)e^{-2(t-u)} du & \text{if } t \ge 0] \,. \end{cases}
$$

Then, for $\varphi \in S$ with $\|\varphi\| \le K$, we have

$$
\begin{aligned}
\|P\varphi)\| &\le \varepsilon_1\, K^2 + (1 + \varepsilon_1\, \delta)\delta + 2(\varepsilon_1 + \varepsilon_2)K^2 \int_0^t e^{-2(t-u)}\, du \\
&\le (2\varepsilon_1 + \varepsilon_2)K^2 + \delta(1 + \varepsilon_1\, \delta).
\end{aligned}
$$

In order for P to map S into itself, we need to ask that, using (3.11),

$$(2\varepsilon_1+\varepsilon_2)K^2+\delta(1+\varepsilon_1\ \delta)\le K. \tag{3.11}$$

But inequality (3.11) is satisfied for K satisfying (3.10), by noting that

$$\frac{1}{2(2\varepsilon_1+\varepsilon_2)}<\frac{1+\sqrt{1-4(2\varepsilon_1+\varepsilon_2)\delta(1+\varepsilon_1\ \delta)}}{2(2\varepsilon_1+\varepsilon_2)}.$$

Thus, we have shown that if $\varphi\in S$, then $\|\big(P\varphi\big)\|\le K$. It is obvious that conditions (3.3), (3.5) and (3.6) are satisfied. Left to show that that P defines a contraction mapping on the metric space S. Let $\zeta,\eta\in S$. Then

$$\begin{aligned}\big|(P\zeta)(t)-(P\eta)(t)\big| &\le 2\varepsilon_1\ K\ \|\zeta-\eta\|+\big(4\varepsilon_1K+4\varepsilon_2K\big)\int_0^t e^{-2(t-u)}\ du\ \|\zeta-\eta\|\\ &\le \big[2\varepsilon_1K+2K(\varepsilon_1+\varepsilon_2)\big]\|\zeta-\eta\|\\ &=2(2\varepsilon_1+\varepsilon_2)K\|\zeta-\eta\|.\end{aligned}$$

By condition (3.10), we have

$$\Big|(P\zeta)(t)-(P\eta)(t)\Big|\le\alpha\|\zeta-\eta\|,\quad \alpha\in(0,1).$$

Hence, by Theorem 3.1, every solution $x(t,0,\psi)$ of (3.8) with small continuous initial function $\psi(t):(-\infty,\ 0]\to\mathbb{R}$, is in S, bounded and approaches zero as $t\to\infty$. □

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