Electronic Journal of Differential Equations, Vol. 2005(2005), No. 144, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

NONLINEAR BOUNDARY CONDITIONS FOR ELLIPTIC EQUATIONS

PABLO AMSTER, MARÍA CRISTINA MARIANI, OSVALDO MÉNDEZ

Abstract. This work is devoted to the study of the elliptic equation $\Delta u = f(x, u)$ in a bounded domain $\Omega \subset \mathbb{R}^n$ with a nonlinear boundary condition. We obtain various existence results applying coincidence degree theory and the method of upper and lower solutions.

1. Introduction

In this paper we study the problem

$$\Delta u = f(x, u) \quad \text{in } \Omega \tag{1.1}$$

subject to the nonlinear boundary condition

$$\frac{\partial u}{\partial \nu} = g(x, u) \quad \text{on } \partial\Omega, \tag{1.2}$$

where $f, g : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions and $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$ domain. There is a vast body of literature concerning nonlinear problems with nonlinear boundary conditions, see e.g. [9] for a survey.

In section 2 we obtain solutions of (1.1)–(1.2) by the method of upper and lower solutions. The proof relies in the associated maximum principle and the unique solvability of the linear Robin problem

$$\begin{gathered} \Delta u - \lambda u = \varphi \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \mu u = \xi \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

for $\lambda, \mu > 0$ (see [8] and its references).

The method of super and subsolutions has been extensively used in nonlinear analysis, both for ODE's and PDE's problems. In particular, for elliptic problems with nonlinear boundary conditions, this method has been applied to obtain more general existence results for example in [5], [12]-[13]. However, the method presented here allows to relax the assumptions: firstly, less regularity is required for the domain Ω; secondly, only continuity is assumed for the nonlinearities f and g. More precisely, we prove Theorem 1.1 below.

2000 *Mathematics Subject Classification.* 35J25, 35J60.
Key words and phrases. Nonlinear elliptic problems; nonlinear boundary conditions; topological methods.

Submitted November 15, 2004. Published December 8, 2005.

Theorem 1.1. *Assume there exist $\alpha \le \beta$ such that*

$$\Delta\alpha \ge f(x,\alpha) \quad \text{in } \Omega, \quad \frac{\partial\alpha}{\partial\nu} \le g(x,\alpha) \quad \text{on } \partial\Omega$$

and

$$\Delta\beta \le f(x,\beta) \quad \text{in } \Omega, \quad \frac{\partial\beta}{\partial\nu} \ge g(x,\beta) \quad \text{on } \partial\Omega.$$

Then (1.1)–(1.2) *admits at least one solution u, with $\alpha \le u \le \beta$.*

In section 3 we study problem (1.1)–(1.2) for bounded f and g applying coincidence degree theory. We obtain an existence result under Landesman-Lazer type conditions ([6], [7]. For further applications to problems of resonant type see [11]). More precisely, we have

Theorem 1.2. *Assume that f and g are bounded, and define*

$$\limsup_{t\to\pm\infty} f(x,t) := f_s^{\pm}(x), \quad \liminf_{t\to\pm\infty} f(x,t) := f_i^{\pm}(x)$$
$$\limsup_{t\to\pm\infty} g(x,t) := g_s^{\pm}(x), \quad \liminf_{t\to\pm\infty} g(x,t) := g_i^{\pm}(x).$$

Then (1.1)–(1.2) *admits at least one solution, provided that one of the following assumptions holds:*

$$\int_{\partial\Omega} g_i^+ > \int_\Omega f_s^+ \quad \text{and} \quad \int_{\partial\Omega} g_s^- < \int_\Omega f_i^- \tag{1.4}$$

$$\int_{\partial\Omega} g_i^- > \int_\Omega f_s^- \quad \text{and} \quad \int_{\partial\Omega} g_s^+ < \int_\Omega f_i^+. \tag{1.5}$$

2. Upper and lower solutions

In this section we present a proof of Theorem 1.1. First we recall the following classical result.

Lemma 2.1. *Let $\lambda, \mu > 0$ and $\varphi \in C(\overline{\Omega})$, $\xi \in C(\partial\Omega)$. Then the Robin problem (1.3) admits a unique solution u. Furthermore, the operator $T : C(\overline{\Omega}) \times C(\partial\Omega) \to C(\overline{\Omega})$ given by $T(\varphi,\xi) = u$ is compact.*

Remark 2.2. If $\lambda = 0$ it is possible to extend Lemma 2.1 (except the last statement) to Lipschitz domains, in particular domains with corners or edges, considering φ in a suitable Sobolev space and ξ in the corresponding trace space. In this case it is also possible to replace μ with a function $g \in L^{n-1}(\Omega)$, see [8].

Moreover, we shall use a maximum principle associated for the problem

Lemma 2.3. *Let $\lambda > 0$, $\mu \ge 0$, and assume that w satisfies*

$$\begin{gathered} \Delta w - \lambda w \ge 0 \quad \text{in } U, \\ \frac{\partial w}{\partial\nu} + \mu w \le 0, \quad \text{on } \Gamma_1, \\ w \le 0 \quad \text{on } \Gamma_2. \end{gathered}$$

where $U \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial U = \Gamma_1 \cup \Gamma_2$. Then $w \le 0$ in U.

Proof. Let $w^+ = \max\{w, 0\}$, and $U^+ = \{x \in U : w(x) > 0\}$. From our assumptions we have

$$0 \le \int_U (\Delta w - \lambda w) w^+ = -\int_{U^+} |\nabla w|^2 - \lambda \int_{U^+} w^2 + \int_{\partial U} w^+ \frac{\partial w}{\partial \nu}$$

Moreover, since $\frac{\partial w}{\partial \nu} \le -\mu w$ on Γ_1, we deduce that

$$\int_{\partial U} w^+ \frac{\partial w}{\partial \nu} = \int_{\Gamma_1} w^+ \frac{\partial w}{\partial \nu} \le -\mu \int_{\Gamma_1} w^+ w \le 0.$$

Hence $|U^+| = 0$, and the proof is complete. □

Proof of Theorem 1.1. Set two positive constants λ, μ, and define the function $P : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$P(x,u) = \begin{cases} \alpha(x) & \text{if } u < \alpha(x) \\ u & \text{if } \alpha(x) \le u \le \beta(x) \\ \beta(x) & \text{if } u > \beta(x). \end{cases}$$

Next, consider the compact fixed point operator $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ given by $Tv = u$, where u is the unique solution of the Robin problem

$$\Delta u - \lambda u = f(x, P(x,v)) - \lambda P(x,v) \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial \nu} + \mu u = g(x, P(x,v)) + \mu P(x,v) \quad \text{on } \partial\Omega.$$

Using Schauder Theorem, it is straightforward to prove that T has a fixed point u. We claim that $\alpha \le u \le \beta$, and hence u is a solution the problem. Indeed, let $U = \{x \in \Omega : u(x) > \beta(x)\}$. For $x \in U$, we have

$$\Delta u(x) - \lambda u(x) = f(x, \beta(x)) - \lambda \beta(x) \ge \Delta \beta(x) - \lambda \beta(x).$$

Moreover, if $x \in \partial U \cap \partial \Omega := \Gamma_1$, then

$$\frac{\partial u}{\partial \nu}(x) + \mu u(x) = g(x, \beta(x)) + \mu \beta(x) \le \frac{\partial \beta}{\partial \nu}(x) + \mu \beta(x).$$

Thus, if $w = u - \beta$ we deduce from Lemma 2.3 that $w \le 0$ in U, and hence U is empty. In the same way we show that $u \ge \alpha$, and the proof is complete. □

Example 2.4. In particular, when $\alpha \equiv R^-$ and $\beta \equiv R^+$ for some constants $R^- < R^+$, Theorem 1.1 guarantees the existence of solutions when f and g satisfy

$$f(x, R^+) \ge 0 \ge f(x, R^-) \quad \forall x \in \Omega$$
$$g(x, R^+) \le 0 \le g(x, R^-) \quad \forall x \in \partial\Omega.$$

For example, for $r > 0$ the problem

$$\Delta u = f(x,u) \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial \nu} + \varphi(x)|u|^r = p(x) \quad \text{on } \partial\Omega$$

admits at least one solution $R^- \le u \le R^+$, provided that

$$f(x, R^+) \ge 0 \ge f(x, R^-)$$

and $\varphi(x)|R^-|^r \le p(x) \le \varphi(x)|R^+|^r$.

3. Landesman-Lazer type conditions

For the sake of completeness, we summarize in this section the main aspects of coincidence degree theory. This technique has been applied to many problems, see e.g. [1] and [4]. For further details see [2, 10].

Let X and Y be real normed spaces, $L : D \subset X \to Y$ a linear Fredholm mapping of index 0, and $N : X \to Y$ continuous.

Next, set two continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Ran}(P) = \ker(L)$ and $\ker(Q) = \mathrm{Ran}(L)$ and an isomorphism $J : \mathrm{Ran}(Q) \to \ker(L)$. It is readily seen that

$$L_P := L|_{D\cap\ker(P)} : D \cap \ker(P) \to \mathrm{Ran}(L)$$

is one-to-one; denote its inverse by K_P. If $\mathcal{B}$ is a bounded open subset of X, N is called L-compact on $\mathcal{B}$ if $QN(\mathcal{B})$ is bounded and $K_P(I-Q)N : \mathcal{B} \to X$ is compact.

The following continuation theorem was proved in Mawhin [10].

Theorem 3.1. *Let L be a Fredholm mapping of index zero and N be L-compact on a bounded domain $\mathcal{B} \subset X$. Suppose:*

1. *$Lx \neq \lambda Nx$ for each $\lambda \in (0, 1]$ and each $x \in \partial\mathcal{B}$.*
2. *$QNx \neq 0$ for each $x \in \ker(L) \cap \partial\mathcal{B}$.*
3. *$d(JQN, \mathcal{B} \cap \ker(L), 0) \neq 0$, where d denotes the Brouwer degree.*

Then the equation $Lx = Nx$ has at least one solution in $D \cap \mathcal{B}$.

In this context, we may consider $X = H^1(\Omega)$, $D = H^2(\Omega)$, $Y = L^2(\Omega) \times H^{-1/2}(\partial\Omega)$ and L, N the operators given by

$$Lu = \Big(\Delta u, \frac{\partial u}{\partial\nu}|_{\partial\Omega}\Big),$$
$$Nu = \Big(f(\cdot, u), g(\cdot, u|_{\partial\Omega})\Big).$$

We recall that the operator N is defined in $X = H^1$. Hence, the trace of a function u is well defined, and we can see that $g(\cdot, u|_{\partial\Omega}) \in H^{-1/2}(\partial\Omega)$.

Since f and g are bounded, it is immediate to prove that N is well defined and continuous. Moreover,

$$\ker(L) = \mathbb{R},$$
$$\mathrm{Ran}(L) = \{(\varphi, \xi) \in Y : \int_\Omega \varphi = \int_{\partial\Omega} \xi\}.$$

Thus L is a Fredholm mapping of index zero, and we may consider the projectors $P : X \to X$ and $Q : Y \to Y$ given by

$$Pu = \overline{u} := \frac{1}{|\Omega|}\int_\Omega u,$$
$$Q(\varphi, \xi) = \frac{1}{|\Omega| + |\partial\Omega|}\Big(\int_\Omega \varphi - \int_{\partial\Omega} \xi, \int_{\partial\Omega} \xi - \int_\Omega \varphi\Big).$$

Also define $J : \mathrm{Ran}(Q) \to \ker(L)$ as

$$J(c, -c) = c.$$

Hence, for $(\varphi, \xi) \in \operatorname{Ran}(L)$ it follows that $K_P(\varphi, \xi)$ is the unique solution $u \in H^2(\Omega)$ of the problem

$$\begin{gathered}\Delta u = \varphi \\ \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \xi \\ \overline{u} = 0.\end{gathered}$$

We shall use the following estimate.

Lemma 3.2. *There exists a constant c such that*

$$\|u\|_{H^2} \le c\|\Delta u\|_{L^2} \tag{3.1}$$

for every $u \in H^2(\Omega)$ such that

$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad \overline{u} = 0.$$

Proof. ¿From the Neumann condition for u we have that $\int_{\partial\Omega} u \frac{\partial u}{\partial \nu} = 0$, and using Green's identity, $-\int_\Omega u\Delta u = \int_\Omega |\nabla u|^2$. It follows that

$$\|\nabla u\|_{L^2}^2 \le \|u\|_{L^2}\|\Delta u\|_{L^2}.$$

On the other hand, as $\lambda_0 = 0$ is the first eigenvalue of the problem

$$-\Delta u = \lambda u, \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0,$$

with eigenfunction $u \equiv 1$, it follows for the second eigenvalue λ_1 that

$$\lambda_1 = \inf_{\overline{u}=0, u\neq 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}.$$

Hence $\|u\|_{L^2} \le \frac{1}{\sqrt{\lambda_1}}\|\nabla u\|_{L^2}$, and it follows that

$$\|\nabla u\|_{L^2} \le \frac{1}{\sqrt{\lambda_1}}\|\Delta u\|_{L^2}, \quad \|u\|_{L^2} \le \frac{1}{\lambda_1}\|\Delta u\|_{L^2}.$$

Let us recall the following result (see e.g. [3]): There exists a constant c such that if u is a weak solution of the problem $\Delta u - u = f$, with homogeneous Neumann condition, then $\|u\|_{H^2} \le c\|f\|_{L^2}$.

Since we already know that $\|u\|_{H^1} \le c\|\Delta u\|_{L^2}$, we may define $f = \Delta u - u$ and hence

$$\|u\|_{H^2} \le c\|f\|_{L^2} \le c(\|\Delta u\|_{L^2} + \|u\|_{L^2}) \le C\|\Delta u\|_{L^2}\,.$$

Thus, the proof is complete. □

Lemma 3.3. *Let L and N be as before and assume that f and g are bounded. Then N is L-compact on $\mathcal{B}$ for any bounded domain $\mathcal{B} \subset H^1(\Omega)$.*

Proof. If $\|w\|_{H^1} \le R$, and $(\varphi, \xi) = (I-Q)N(w)$, it follows that $\|\varphi\|_{L^2} + \|\xi\|_{L^2(\partial\Omega)} \le C$ for some constant C depending only on R. Let $u = K_P(\varphi, \xi)$, and define

$$c_\xi = \frac{1}{|\Omega|}\int_{\partial\Omega} \xi,$$

and v_ξ the unique solution of the problem

$$\begin{gathered}\Delta v_\xi = c_\xi\\ \frac{\partial v_\xi}{\partial \nu}\Big|_{\partial\Omega} = \xi\\ \overline{v}_\xi = 0.\end{gathered}$$

Then

$$\|u - v_\xi\|_{H^2} \le c\|\varphi - c_\xi\|_{L^2} \le C$$

for some constant C depending only on R. Moreover, $\|v_\xi\|_{L^2} \le c\|\nabla v_\xi\|_{L^2}$, and by Cauchy-Schwarz inequality and the continuity of the trace function $Tr : H^1(\Omega) \to L^2(\partial\Omega)$ we have

$$\|\Delta v_\xi\|_{L^2}.\|v_\xi\|_{L^2} \ge \|\nabla v_\xi\|_{L^2}^2 - C\|v_\xi\|_{H^1}$$

for some constant C. Since $\|\Delta v_\xi\|_{L^2} \le C$ for some constant C depending only on R, it follows that $\|v_\xi\|_{H^2} \le C$ for some constant C depending only on R. Hence, the norm of u is bounded by a constant depending only on R, and the result follows from the compactness of the imbedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$. □

Proof of Theorem 1.2. We shall prove that if R is large enough, then the assumptions of Theorem 3.1 are fulfilled for $\mathcal{B} = B_R(0) \subset H^1(\Omega)$.

Step 1: There exists a constant R such that if $Lu = \lambda Nu$ for some $\lambda \in (0, 1]$, then $\|u\|_{H^1} < R$. Indeed this is so, otherwise there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that

$$Lu_n = \lambda_n Nu_n, \quad \|u_n\|_{H^1} \to +\infty$$

with $\lambda_n \in (0, 1]$. Let $v_n = u_n - \overline{u_n}$. Then v_n satisfies

$$\begin{gathered}\Delta v_n = \lambda_n f(x, u_n)\\ \frac{\partial v_n}{\partial \nu}\Big|_{\partial\Omega} = \lambda_n g(x, u_n|_{\partial\Omega})\\ \overline{v_n} = 0.\end{gathered} \tag{3.2}$$

As in the proof of the previous lemma, it follows that $\|v_n\|_{H^2} \le C$ for some constant C, and hence $|\overline{u_n}| \to +\infty$. Taking a subsequence, assume for example that $\overline{u_n} \to +\infty$. Integrating (3.2), as $\lambda_n \ne 0$ we obtain:

$$\int_\Omega f(x, u_n) = \int_{\partial\Omega} g(x, u_n).$$

If (1.4) holds, taking a subsequence if necessary we get, by Fatou's Lemma,

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega f(x, u_n) &\le \int_\Omega \limsup_{n\to\infty} f(x, u_n)\\ &= \int_\Omega f_s^+\\ &< \int_{\partial\Omega} g_i^+\\ &\le \liminf_{n\to\infty} \int_{\partial\Omega} g(x, u_n),\end{aligned}$$

a contradiction. The proof for $\overline{u_n} \to -\infty$ is analogous. In the same way, we obtain a contradiction if (1.5) holds.

Step 2: For R is large enough, if $u \in \partial\mathcal{B}_R \cap \ker(L)$ then $QN(u) \ne 0$ and

$$d(JQN, \mathcal{B} \cap \ker(L), 0) \ne 0.$$

Indeed, for $u \in \mathcal{B}_R \cap \ker(L) = [-R, R]$, by definition we have that

$$JQN(u) = \frac{1}{|\Omega| + |\partial\Omega|}\Big(\int_\Omega f(x,u) - \int_{\partial\Omega} g(x,u)\Big).$$

In the same way as before, if (1.4) holds, we obtain

$$\limsup_{n\to\infty} \int_\Omega f(x,u_n) - \liminf_{n\to\infty} \int_{\partial\Omega} g(x,u_n) < 0$$

for $u_n \to +\infty$ and

$$\liminf_{n\to\infty} \int_\Omega f(x,u_n) - \limsup_{n\to\infty} \int_{\partial\Omega} g(x,u_n) > 0$$

for $u_n \to -\infty$. Thus, for R large enough

$$\int_\Omega f(x,R) - \int_{\partial\Omega} g(x,R) < 0 < \int_\Omega f(x,-R) - \int_{\partial\Omega} g(x,-R),$$

and the result holds. The proof is analogous if (1.5) holds. □

Acknowledgments. The authors want to thank Professor Alfonso Castro and the anonymous referee for their careful reading of the manuscript and their fruitful remarks.

References

[1] H. Fang, Z. C. Wang, *Periodic Boundary Value Problems of Impulsive Differential Equations,* Applied Mathematics E-Notes, 1 (2001), 77-85.
[2] R. Gaines, J. Mawhin, *Lecture Notes in Mathematics,* Vol 586, Springer-Verlag, Berlin, 1997.
[3] P. Grisvard, *Elliptic problems in nonsmooth domains,* Pitman (1985).
[4] S. Iyase, *Non-resonant oscillations for some fourth-order differential equations with delay,* Math. Proc. of the Royal Irish Academy, 99A (1) 1999, 113-121.
[5] F. Inkmann, *Existence and multiplicity theorems for semilinear elliptic equations with nonlinear boundary conditions*, Indiana Univ. Math J. 31 (1982), 213-221.
[6] E. Landesman and A. Lazer: Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1970), 609-623.
[7] A. Lazer, *A Second look at the first result of Landesman-Lazer Type,* Electron. J. Diff. Eqns., Conf. 05 (2000), 113–119
[8] L. Lanzani and O. Méndez, *The Poisson's problem for the Laplacian with Robin boundary condition in non-smooth domains,* to appear in Revista Matemática Iberoamericana
[9] J. Lions and E. Magenes, *Non-Homogeneous boundary value problems and applications, I, II.,* Springer Verlag, 1972.
[10] J. Mawhin, *Topological degree methods in nonlinear boundary value problems,* NSF-CBMS Regional Conference in Mathematics no. 40, American Mathematical Society, Providence, RI, (1979).
[11] J. Mawhin, *Landesman-Lazer conditions for boundary value problems: A nonlinear version of resonance,* Bol. de la Sociedad Española de Mat.Aplicada 16 (2000), 45-65.
[12] J. Mawhin, K. Schmitt, *Upper and lower solutions and semilinear second order elliptic equations with non-linear boundary conditions*, Proc. Royal Soc. Edinburgh 97A (1984), 199-207.
[13] J. Mawhin, K. Schmitt, *Corrigendum: Upper and lower solutions and semilinear second order elliptic equations with non-linear boundary conditions*, Proc. Royal Soc. Edinburgh 100A (1985), 361.

Pablo Amster
Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina, Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET)
E-mail address: pamster@dm.uba.ar

María Cristina Mariani
Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003-8001, USA
E-mail address: mmariani@nmsu.edu

Osvaldo Méndez
Department of Mathematics, 500W University Ave., 124 Bell Hall, The University of Texas at El Paso, El Paso, TX 79968, USA
E-mail address: mendez@math.utep.edu