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DECAY AND SYMMETRY OF POSITIVE SOLUTIONS OF ELLIPTIC SYSTEMS IN UNBOUNDED CYLINDERS

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Abstract. In this paper, we study the asymptotic behavior of positive solutions and apply the "improved moving plane" method to prove the symmetry of positive solutions of semilinear elliptic systems in unbounded cylinders.

1. Introduction

In studying differential equations it is often of interest to know if the solutions have symmetry, or perhaps monotonicity, in some direction. Questions of this kind have been investigated by Gidas-Ni-Nirenberg [6, 7] by Nirenberg [1, 2]. These articles are basically working on semilinear elliptic equations with Dirichlet or Neumann boundary values.

In our previous paper [3], we established the existence of positive solutions for a class of semilinear elliptic systems on unbounded domains. In this paper, we study the asymptotic behavior of positive solutions and symmetry of positive solutions of the elliptic systems of the form

$$\begin{gathered}
-\Delta u+u=g(v), \quad u>0 \text{ in } \mathbf{A}, \\
-\Delta v+v=f(u), \quad v>0 \text{ in } \mathbf{A}, \\
u=0, \quad v=0 \quad \text{on } \partial\mathbf{A}, \\
\lim_{|t|\to\infty} u(x,t)=0, \quad \lim_{|t|\to\infty} v(x,t)=0 \quad \text{uniformly in } x\in\Omega,
\end{gathered} \tag{1.1}$$

where $N=m+n\geq 2$, $n\geq 1$, (x,t) is the generic point of $\mathbb{R}^N$ with $x\in\mathbb{R}^m$ and $t\in\mathbb{R}^n$, $\Omega\subset\mathbb{R}^m$ is a smooth bounded $C^{1,1}$ domain, and $\mathbf{A}=\Omega\times\mathbb{R}^n$ is an unbounded cylinder in $\mathbb{R}^N$. Our results form a further development of the work by Figueiredo and Yang [5].

The method of "moving plane" was used in [5], as originally introduced by Alexandroff (see Hopf [9, Chap. 7]), later used by Serrin [12], and extensively used in recent times, after the work of [6] by Gidas-Ni-Nirenberg. Generally speaking, in applying the "moving plane" device it is important to first obtain the asymptotic behavior of solutions near ∞ in order to get the device started near ∞. In this paper, we apply the "improved moving plane" method given Li [10] and by Li and Ni [11], who make no assumption on the asymptotic behavior of positive solutions

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to prove that $u(x,t)$, $v(x,t)$ are radially symmetric in x and axially symmetric in t. In section 2, we establish the asymptotic behavior of positive solutions of the systems of semilinear elliptic equations (1.1).

2. Asymptotic behavior

Let λ_1 be the first eigenvalue and ϕ_1 the corresponding first positive eigenfunction of the Dirichlet problem $-\Delta\phi_1 = \lambda_1\phi_1$ in $\mathbf{\Omega}$, $\phi_1 = 0$ on $\partial\mathbf{\Omega}$.

The basic assumptions on the functions f and g are as follows:

- (H1) $f, g \in C^1(\mathbb{R},\mathbb{R})$, with $f(t) = g(t) = 0$ for $t \le 0$, $f(t) > 0$ and $g(t) > 0$ for $t > 0$.
- (H2) $f(t) = O(t^p)$, $g(t) = O(t^q)$ as $t \to 0$ for some $1 < p, q < \frac{N+2}{N-2}$ ($1 < p, q < \infty$ if $N = 2$).
- (H3) $f'(t)$, $g'(t)$ are nondecreasing for $t \ge 0$.

Proposition 2.1. *Assume that f, g satisfy (H1)–(H3). Let $u(x,t)$, $v(x,t)$ be C^2 positive solutions of the elliptic systems* (1.1). *Then for each $\varepsilon > 0$, there exist constants C_ε, $\overline{C_\varepsilon} > 0$ such that for $(x,t) \in \overline{\mathbf{A}}$,*

$$\overline{C_\varepsilon}\phi_1(x)e^{-\sqrt{1+\lambda_1}\,|t|}|t|^{-\frac{n-1}{2}-\varepsilon} \le u(x,t), v(x,t) \le C_\varepsilon\phi_1(x)e^{-\sqrt{1+\lambda_1}\,|t|}|t|^{-\frac{n-1}{2}+\varepsilon}.$$

Proof. We divide the proof into the following steps:

(1) We claim that for any $0 < \delta < 1 + \lambda_1$, there exists $\alpha_1 > 0$ such that

$$u(x,t) + v(x,t) \le \alpha_1\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|t|}, \quad \text{for } (x,t) \in \overline{\mathbf{A}}.$$

Without loss of generality, we assume $\delta < 1$. Since $\lim_{|t|\to\infty} u(x,t) = 0$ and $\lim_{|t|\to\infty} v(x,t) = 0$ uniformly in $x \in \mathbf{\Omega}$, we may choose $R_0 > 0$ large enough such that

$$\frac{f(u(x,t))}{u(x,t)} \le \delta, \quad \frac{g(v(x,t))}{v(x,t)} \le \delta, \quad \text{for } (x,t) \in \overline{\mathbf{A}}, |t| \ge R_0. \tag{2.1}$$

Let $(z_x, z_t) \in \partial\mathbf{A}$, and B be a small ball in $\mathbf{A}$ such that $(z_x, z_t) \in \partial B$. Since $\phi_1(x) > 0$ for $x \in \mathbf{\Omega}$, $\phi_1(z_x) = 0$, $u(x,t) > 0$, $v(x,t) > 0$ for $(x,t) \in B$, $u(z_x,z_t) = 0$, and $v(z_x,z_t) = 0$, by the Hopf boundary point lemma (see Gilbarg and Trudinger [8]), $\frac{\partial\phi_1}{\partial x}(z_x) < 0$, $\frac{\partial u}{\partial\nu}(z_x,z_t) < 0$, and $\frac{\partial v}{\partial\nu}(z_x,z_t) < 0$, where ν is the outward unit normal vector at (z_x,z_t). Thus

$$\lim_{(x,t)\to(z_x,z_t)} \frac{u(x,t)+v(x,t)}{\phi_1(x)} = \frac{\frac{\partial u}{\partial\nu}(z_x,z_t) + \frac{\partial v}{\partial\nu}(z_x,z_t)}{\frac{\partial\phi_1}{\partial x}(z_x)} > 0\,,$$

where $(x,t) \in \mathbf{A}$ and $(x,t) \to (z_x,z_t)$ normally. Note that $(u(x,t)+v(x,t))\phi_1^{-1}(x) > 0$ for $(x,t) \in \mathbf{A}$, thus

$$(u(x,t)+v(x,t))\phi_1^{-1}(x) > 0 \quad \text{for } (x,t) \in \overline{\mathbf{A}}.$$

Since $\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|t|}$, $u(x,t)$, and $v(x,t)$ are $C^1(\overline{\mathbf{A}})$, if we set

$$\alpha_1 = \sup_{(x,t)\in\overline{\mathbf{A}},\,|t|\le R_0} \left\{(u(x,t)+v(x,t))\phi_1^{-1}(x)e^{\sqrt{1+\lambda_1-\delta}\,|t|}\right\},$$

then $\alpha_1 > 0$ and

$$\alpha_1\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|t|} \ge u(x,t) + v(x,t), \quad \text{for } (x,t) \in \overline{\mathbf{A}}, |t| \le R_0. \tag{2.2}$$

Let $\Phi_1(x,t)=\alpha_1\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|t|}$ for $(x,t)\in\overline{\mathbf{A}}$. Then for $(x,t)\in\overline{\mathbf{A}}$, $|t|\geq R_0$, from (2.1), we have

$$\begin{aligned}
&\Delta(\Phi_1-u-v)(x,t)-(\Phi_1-u-v)(x,t)\\
&=[-\delta-\frac{\sqrt{1+\lambda_1-\delta}(n-1)}{|t|}]\Phi_1(x,t)+g(v)+f(u)\\
&\leq-\delta\Phi_1+\delta(u+v)\\
&=-\delta(\Phi_1-u-v).
\end{aligned}$$

Hence $\Delta(\Phi_1-u-v)(x,t)-(1-\delta)(\Phi_1-u-v)(x,t)\leq 0$ for $(x,t)\in\overline{\mathbf{A}}$, $|t|\geq R_0$. Then by the strong maximum principle, we obtain

$$u(x,t)+v(x,t)\leq\Phi_1(x,t),\quad\text{for }(x,t)\in\overline{\mathbf{A}},\ |t|\geq R_0. \tag{2.3}$$

Hence from (2.2) and (2.3), we get the claim.

(2) We claim that for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$u(x,t)+v(x,t)\leq C_\varepsilon\phi_1(x)e^{-\sqrt{1+\lambda_1}\,|t|}|t|^{-\frac{n-1}{2}+\varepsilon},\quad\text{for}(x,t)\in\overline{\mathbf{A}}.$$

Without loss of generality, we assume that $0<\varepsilon<\frac{n-1}{2}$. Now, given $\varepsilon>0$ and fixed $\delta>0$ as in part (1). By part (1), there exists $\alpha_1>0$ such that

$$u(x,t)+v(x,t)\leq\alpha_1\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|t|},\quad\text{for }(x,t)\in\overline{\mathbf{A}}.$$

From (2.1),

$$f(u)+g(v)\leq\delta\alpha_1\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|t|},\quad\text{for }(x,t)\in\overline{\mathbf{A}},\ |t|\geq R_0. \tag{2.4}$$

Let $m_\varepsilon=\frac{n-1}{2}-\varepsilon$ and

$$h(t)=-2\varepsilon\sqrt{1+\lambda_1}|t|^{-m_\varepsilon-1}+m_\varepsilon(m_\varepsilon-n+2)|t|^{-m_\varepsilon-2}.$$

We can choose $R_1>R_0$ such that for $|t|\geq R_1$,

$$e^{-\sqrt{1+\lambda_1}\,|t|}h(t)+\delta\alpha_1e^{-\sqrt{1+\lambda_1-\delta}\,|t|}\leq 0. \tag{2.5}$$

As in part (1), if we set

$$\alpha_2=\sup_{(x,t)\in\overline{\mathbf{A}},\,|t|\leq R_1}\left\{(u(x,t)+v(x,t))\phi_1^{-1}(x)e^{\sqrt{1+\lambda_1}\,|t|}|t|^{m_\varepsilon}\right\}+1,$$

then $\alpha_2>1$. Let $\Phi_2(x,t)=\alpha_2\phi_1(x)e^{-\sqrt{1+\lambda_1}\,|t|}|t|^{-m_\varepsilon}$ for $(x,t)\in\overline{\mathbf{A}}$, then

$$u(x,t)+v(x,t)\leq\Phi_2(x,t),\quad\text{for }(x,t)\in\overline{\mathbf{A}},\ |t|\leq R_1. \tag{2.6}$$

For $(x,t)\in\overline{\mathbf{A}}$, $|t|\geq R_1$, from (2.4) and (2.5), we have

$$\begin{aligned}
&\Delta(\Phi_2-u-v)(x,t)-(\Phi_2-u-v)(x,t)\\
&=h(t)|t|^{m_\varepsilon}\Phi_2(x,t)+g(v)+f(u)\\
&\leq\alpha_2\phi_1(x)e^{-\sqrt{1+\lambda_1}\,|t|}h(t)+\delta\alpha_1\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,|t|}\\
&\leq\phi_1(x)(e^{-\sqrt{1+\lambda_1}\,|t|}h(t)+\delta\alpha_1e^{-\sqrt{1+\lambda_1-\delta}\,|t|})\leq 0\,.
\end{aligned}$$

Then by the strong maximum principle,

$$u(x,t)+v(x,t)\leq\Phi_2(x,t),\quad\text{for }(x,t)\in\overline{\mathbf{A}},\ |t|\geq R_1. \tag{2.7}$$

Hence from (2.6) and (2.7), and since $u(x,t)$, $v(x,t)$ are positive solutions, it is straightforward that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$u(x,t), v(x,t) \le C_\varepsilon \phi_1(x) e^{-\sqrt{1+\lambda_1}\,|t|} |t|^{-\frac{n-1}{2}+\varepsilon}, \quad \text{for } (x,t) \in \overline{\mathbf{A}}.$$

(3) For a given $\varepsilon > 0$, let $\overline{m_\varepsilon} = \frac{n-1}{2} + \varepsilon$ and

$$k(t) = 2\varepsilon\sqrt{1+\lambda_1}|t|^{-1} + \overline{m_\varepsilon}(\overline{m_\varepsilon} - n + 2)|t|^{-2}.$$

We can choose $R_2 > 0$ such that $k(t) \ge 0$ for $|t| \ge R_2$. As in part (1), if we set

$$\beta = \inf_{(x,t)\in\overline{\mathbf{A}},\, |t|\le R_2} \big\{u(x,t)\phi_1^{-1}(x) e^{\sqrt{1+\lambda_1}\,|t|} |t|^{\overline{m_\varepsilon}}\big\},$$

then $\beta > 0$ and

$$\beta\phi_1(x) e^{-\sqrt{1+\lambda_1}\,|t|} |t|^{-\overline{m_\varepsilon}} \le u(x,t), \quad \text{for } (x,t) \in \overline{\mathbf{A}},\ |t| \le R_2. \tag{2.8}$$

Let $\Psi(x,t) = \beta\phi_1(x) e^{-\sqrt{1+\lambda_1}\,|t|} |t|^{-\overline{m_\varepsilon}}$ for $(x,t) \in \overline{\mathbf{A}}$. Then for $(x,t) \in \overline{\mathbf{A}}$, $|t| \ge R_2$, we have

$$\Delta(\Psi - u)(x,t) - (\Psi - u)(x,t) = k(t)\Psi(x,t) + g(v) \ge 0.$$

Then by the strong maximum principle,

$$u(x,t) \ge \Psi(x,t), \quad \text{for } (x,t) \in \overline{\mathbf{A}},\ |t| \ge R_2. \tag{2.9}$$

Hence from (2.8) and (2.9), for each $\varepsilon > 0$, there exists $\overline{C_\varepsilon} > 0$ such that

$$u(x,t) \ge \overline{C_\varepsilon}\phi_1(x) e^{-\sqrt{1+\lambda_1}\,|t|} |t|^{-\frac{n-1}{2}-\varepsilon}, \quad \text{for } (x,t) \in \overline{\mathbf{A}}.$$

For the positive solution $v(x,t)$ in the elliptic systems (1.1), we can get the same results as for $u(x,t)$. $\square$

3. Symmetry of Positive Solutions

Let

$$\mathbf{S} = \{(x,t) \in B^{N-1}(0;R) \times \mathbb{R} : x = (x_1, \dots, x_{N-1}) \in B^{N-1}(0;R), t \in \mathbb{R}\},$$

where $B^{N-1}(0;R)$ is a ball with center at the origin and of radius R in $\mathbb{R}^{N-1}$. Now we consider the systems of semilinear elliptic equations

$$\begin{gathered} -\Delta u + u = g(v), \quad u > 0 \text{ in } \mathbf{S}, \\ -\Delta v + v = f(u), \quad v > 0 \text{ in } \mathbf{S}, \\ u = 0, \quad v = 0 \quad \text{on } \partial\mathbf{S}, \\ \lim_{|t|\to\infty} u(x,t) = 0, \quad \lim_{|t|\to\infty} v(x,t) = 0 \quad \text{uniformly in } x \in B^{N-1}(0;R). \end{gathered} \tag{3.1}$$

The purpose of this section is to apply the "improved moving plane" method to prove the symmetry of positive solutions of the elliptic systems (3.1) which makes no assumption on the asymptotic behavior of positive solution.

Theorem 3.1. *Assume that f, g satisfy (H1)–(H3). Let $u(x,t)$, $v(x,t)$ be C^2 positive solutions of the elliptic systems* (3.1). *Then $u(x,t)$, $v(x,t)$ are radially symmetric in x and axially symmetric in t; that is to say, $u(x,t-\sigma) = u(|x|, |t-\sigma|)$, $v(x,t-\sigma) = v(|x|, |t-\sigma|)$ for some σ.*

Part I. $u(x,t)$, $v(x,t)$ are radially symmetric in $x \in B^{N-1}(0;R)$.
Notation:

- $T_\lambda = \{(x,t) = (x_1, x_2, \dots, x_{N-1}, t) \in \mathbf{S} : x_1 = \lambda\}$
- $\Sigma_\lambda = \{(x,t) \in \mathbf{S} : x_1 < \lambda\}$
- For $(x,t) = (x_1, x_2, \dots, x_{N-1}, t) \in \mathbf{S}$, set $(x^\lambda, t) = (2\lambda - x_1, x_2, \dots, x_{N-1}, t)$; that is to say, (x^λ, t) is the reflection of (x,t) with respect to T_λ
- Let Λ be the collection of all $\lambda \in (-R, 0)$ such that

$$u(x,t) < u(x^\lambda, t), \quad v(x,t) < v(x^\lambda, t) \quad \text{for all } (x,t) \in \Sigma_\lambda,$$
$$u_{x_1}(x,t) > 0, \quad v_{x_1}(x,t) > 0 \quad \text{on } \mathbf{S} \cap T_\lambda.$$

In the sequel, we take $\lambda \le 0$. On the set $\overline{\Sigma_\lambda}$ we define the functions

$$U^\lambda(x,t) = u(x,t) - u(x^\lambda, t) \quad \text{and} \quad V^\lambda(x,t) = v(x,t) - v(x^\lambda, t).$$

Then

$$\begin{aligned} -\Delta U^\lambda(x,t) + U^\lambda(x,t) &= g(v(x,t)) - g(v(x^\lambda, t)), \\ -\Delta V^\lambda(x,t) + V^\lambda(x,t) &= f(u(x,t)) - f(u(x^\lambda, t)). \end{aligned}$$

By the mean value theorem,

$$\begin{aligned} g(v(x,t)) - g(v(x^\lambda, t)) &= g'(\psi_\lambda(x,t)) V^\lambda(x,t), \\ f(u(x,t)) - f(u(x^\lambda, t)) &= f'(\varphi_\lambda(x,t)) U^\lambda(x,t), \end{aligned}$$

where $\psi_\lambda(x,t)$ is a real number between $v(x,t)$ and $v(x^\lambda,t)$, $\varphi_\lambda(x,t)$ is some value between $u(x,t)$ and $u(x^\lambda,t)$, respectively. Let us denote $g'(\psi_\lambda(x,t)) = c_\lambda(x,t)$ and $f'(\varphi_\lambda(x,t)) = d_\lambda(x,t)$. So

$$\begin{aligned} -\Delta U^\lambda(x,t) + U^\lambda(x,t) &= c_\lambda(x,t) V^\lambda(x,t) \quad \text{in } \Sigma_\lambda, \\ -\Delta V^\lambda(x,t) + V^\lambda(x,t) &= d_\lambda(x,t) U^\lambda(x,t) \quad \text{in } \Sigma_\lambda. \end{aligned} \tag{3.2}$$

To prove Part I, we need the following lemmas.

Lemma 3.2. *For some* $0 < \delta < R$, $(-R, -R+\delta) \subset \Lambda$.

Proof. Note that by (H2), $\lim_{t\to 0^+} g'(t) = 0$ and $\lim_{t\to 0^+} f'(t) = 0$. Take $t_0 > 0$ such that if $0 < t \le t_0$, then $g'(t) < 1$ and $f'(t) < 1$. Since $\lim_{|x|\to R} u(x,t) = 0$ and $\lim_{|x|\to R} v(x,t) = 0$ uniformly in t, we can choose δ, $R > \delta > 0$ such that if $R - \delta < |x| < R$, $u(x,t) \le t_0$ and $v(x,t) \le t_0$ uniformly in t.

We claim that if $-R < \lambda < -R + \delta$, then $U^\lambda(x,t) \le 0$ and $V^\lambda(x,t) \le 0$ in Σ_λ. On the contrary, suppose there exists λ such that $-R < \lambda < -R + \delta$, $U^\lambda(x,t) > 0$ for some $(x,t) \in \Sigma_\lambda$, or $V^\lambda(x,t) > 0$ for some $(x,t) \in \Sigma_\lambda$. Since $\lim_{|t|\to\infty} U^\lambda(x,t) = 0$ and $\lim_{|t|\to\infty} V^\lambda(x,t) = 0$ uniformly in $x \in B^{N-1}(0;R)$, $U^\lambda(x,t)$ achieves its maximum at $(x_0,t_0) \in \Sigma_\lambda$ and $V^\lambda(x,t)$ achieves its maximum at $(x_1,t_1) \in \Sigma_\lambda$. Then

$$\begin{aligned} \nabla U^\lambda(x_0,t_0) &= 0, \quad \{U^\lambda_{ij}(x_0,t_0)\} \le 0, \\ \nabla V^\lambda(x_1,t_1) &= 0, \quad \{V^\lambda_{ij}(x_1,t_1)\} \le 0. \end{aligned}$$

Since $\Delta U^\lambda(x_0,t_0) \le 0$ and $\Delta V^\lambda(x_1,t_1) \le 0$, from the elliptic systems (3.2) it follows that

$$\begin{aligned} c_\lambda(x_0,t_0) V^\lambda(x_0,t_0) &\ge U^\lambda(x_0,t_0) > 0, \\ d_\lambda(x_1,t_1) U^\lambda(x_1,t_1) &\ge V^\lambda(x_1,t_1) > 0. \end{aligned}$$

From (H1)–(H3) it follows that $c_\lambda(x_0,t_0) \ge 0$ and $d_\lambda(x_1,t_1) \ge 0$. Then we obtain that $V^\lambda(x_0,t_0) > 0$ and $U^\lambda(x_1,t_1) > 0$. Moreover, if $-R < \lambda < -R+\delta$, since $V^\lambda(x_0,t_0) > 0$, $U^\lambda(x_1,t_1) > 0$, then from (H3), $c_\lambda(x_0,t_0) < 1$, $d_\lambda(x_1,t_1) < 1$, and get

$$\begin{aligned} V^\lambda(x_0,t_0) &> c_\lambda(x_0,t_0)V^\lambda(x_0,t_0) \ge U^\lambda(x_0,t_0),\\ U^\lambda(x_1,t_1) &> d_\lambda(x_1,t_1)U^\lambda(x_1,t_1) \ge V^\lambda(x_1,t_1). \end{aligned} \tag{3.3}$$

Since $V^\lambda(x_1,t_1) \ge V^\lambda(x_0,t_0)$, from (3.3) it follows that $U^\lambda(x_1,t_1) > U^\lambda(x_0,t_0)$, we come to a contradiction. So for $-R < \lambda < -R+\delta$, $U^\lambda(x,t) \le 0$ and $V^\lambda(x,t) \le 0$ in Σ_λ. Applying the maximum principle and the Hopf boundary point lemma, for $-R < \lambda < -R+\delta$, we get $U^\lambda(x,t) < 0$ in Σ_λ, $U^\lambda_{x_1}(x,t) > 0$ for $(x,t) \in \mathbf{S} \cap T_\lambda$, $V^\lambda(x,t) < 0$ in Σ_λ, and $V^\lambda_{x_1}(x,t) > 0$ for $(x,t) \in \mathbf{S} \cap T_\lambda$. Hence $u_{x_1}(x,t) > 0$ and $v_{x_1}(x,t) > 0$ for $(x,t) \in \mathbf{S} \cap T_\lambda$. Then $(-R,-R+\delta) \subset \Lambda$. □

Lemma 3.3. *If $(-R,\lambda] \subset \Lambda$, then there exists $\tau > 0$ such that $[\lambda,\lambda+\tau) \subset \Lambda$.*

Proof. Suppose not. Then there exist a decreasing sequence $\lambda_k \to \lambda$ and a sequence $\{(x_{k0},t_{k0})\}$ of points in Σ_{λ_k} such that $U^{\lambda_k}(x_{k0},t_{k0}) = u(x_{k0},t_{k0}) - u(x^{\lambda_k}_{k0},t_{k0}) > 0$, or a sequence $\{(x_{k1},t_{k1})\}$ of points in Σ_{λ_k} such that $V^{\lambda_k}(x_{k1},t_{k1}) = v(x_{k1},t_{k1}) - v(x^{\lambda_k}_{k1},t_{k1}) > 0$. There exists a subsequence $\{(x_{k0},t_{k0})\}$ such that $x_{k0} \to \overline{x}_0 \in \overline{B^{N-1}(0;R)}$ or a subsequence $\{(x_{k1},t_{k1})\}$ such that $x_{k1} \to \overline{x}_1 \in \overline{B^{N-1}(0;R)}$. There may arise two possibilities: Cases 1 and 2 below.
Case 1. $|t_{k0}| \to \infty$. As shown in Lemma 3.2, we assume that

$$\begin{gathered} U^{\lambda_k}(x_{k0},t_{k0}) = \max_{(x,t)\in\overline{\Sigma}_{\lambda_k}} U^{\lambda_k}(x,t),\\ \nabla U^{\lambda_k}(x_{k0},t_{k0}) = 0, \quad \{U^{\lambda_k}_{ij}(x_{k0},t_{k0})\} \le 0. \end{gathered}$$

From $\lim_{|t_{k0}|\to\infty} u(x_{k0},t_{k0}) = 0$, as in Lemma 3.2, we obtain a contradiction. The same argument applies to $V^{\lambda_k}(x_{k1},t_{k1})$.
Case 2. $t_{k0} \to \bar{t}_0$. We have $(x_{k0},t_{k0}) \to (\overline{x}_0,\bar{t}_0) \in \overline{\Sigma_\lambda}$. Thus $U^\lambda(\overline{x}_0,\bar{t}_0) \ge 0$. Clearly $(\overline{x}_0,\bar{t}_0) \notin \Sigma_\lambda$ since $U^\lambda(x,t) < 0$ in Σ_λ. If $(\overline{x}_0,\bar{t}_0) \in T_\lambda$, then $u_{x_1}(\overline{x}_0,\bar{t}_0) < 0$, which contradicts to $\lambda \in \Lambda$. Moreover, $(\bar{x}_0,\bar{t}_0) \notin \partial\mathbf{S} \cap \overline{\Sigma_\lambda}$ since if $(\bar{x}_0,\bar{t}_0) \in \partial\mathbf{S} \cap \overline{\Sigma_\lambda}$ then $0 = u(\bar{x}_0,\bar{t}_0) \ge u(\bar{x}^\lambda_0,\bar{t}_0) > 0$, a contraction. We conclude that Case 2 is impossible. The same argument applies to $V^{\lambda_k}(x_{k1},t_{k1})$. □

Proof of Part I. Let $\mu = \sup\{\lambda \in (-R,0) : (-R,\lambda) \subset \Lambda\}$. Then $\mu \notin \Lambda$. If not, by Lemma 3.3 we would have $[\mu,\mu+\tau) \subset \Lambda$, which contradicts to the definition of μ. We claim that $\mu = 0$. Suppose not, $\mu \in (-R,0)$. By continuity we have $u(x,t) \le u(x^\mu,t)$ and $v(x,t) \le v(x^\mu,t)$ for all $(x,t) \in \Sigma_\mu$, then by the maximum principle we have $u(x,t) \equiv u(x^\mu,t)$ and $v(x,t) \equiv v(x^\mu,t)$ for all $(x,t) \in \Sigma_\mu$, which is impossible. Thus $\mu = 0$. By reversing the x_1 axis, we conclude that $u(x,t)$ and $v(x,t)$ are symmetric with respect to the hyperplane T_0, $u_{x_1}(x,t) < 0$ and $v_{x_1}(x,t) < 0$ for $x_1 > 0$. Since the x_1 direction can be chosen arbitrarily, we conclude that $u(x,t)$ and $v(x,t)$ are radially symmetric in $x \in B^{N-1}(0;R)$. □

Part II. $u(x,t)$, $v(x,t)$ are axially symmetric with respect to some hyperplane $t = \sigma$.

Notation:

- $S_\theta = \{(x,t) \in \mathbf{S} : x \in B^{N-1}(0;R), t = \theta\}$
- $\Gamma_\theta = \{(x,t) \in \mathbf{S} : x \in B^{N-1}(0;R),\ t < \theta\}$

- For any $(x,t) \in \mathbf{S}$, set $(x,t^\theta) = (x, 2\theta - t)$ that is to say, (x,t^θ) is the reflection of (x,t) with respect to S_θ
- Let Θ be the collection of all $\theta \in \mathbb{R}$ such that

$$\begin{gathered} u(x,t) < u(x,t^\theta), \quad v(x,t) < v(x,t^\theta) \quad \text{for all } (x,t) \in \Gamma_\theta, \\ u_t(x,t) > 0, \quad v_t(x,t) > 0 \quad \text{on } \mathbf{S} \cap S_\theta. \end{gathered}$$

On $\overline{\Gamma_\theta}$, we define the functions

$$M^\theta(x,t) = u(x,t) - u(x,t^\theta) \quad \text{and} \quad N^\theta(x,t) = v(x,t) - v(x,t^\theta).$$

Then

$$\begin{aligned} -\Delta M^\theta(x,t) + M^\theta(x,t) &= g(v(x,t)) - g(v(x,t^\theta)), \\ -\Delta N^\theta(x,t) + N^\theta(x,t) &= f(u(x,t)) - f(u(x,t^\theta)). \end{aligned}$$

By the mean value theorem,

$$\begin{aligned} g(v(x,t)) - g(v(x,t^\theta)) &= g'(\xi_\theta(x,t))N^\theta(x,t), \\ f(u(x,t)) - f(u(x,t^\theta)) &= f'(\zeta_\theta(x,t))M^\theta(x,t), \end{aligned}$$

where $\xi_\theta(x,t)$ is a real number between $v(x,t)$ and $v(x,t^\theta)$, $\zeta_\theta(x,t)$ is some value between $u(x,t)$ and $u(x,t^\theta)$, respectively. Let us denote $g'(\xi_\theta(x,t)) = e_\theta(x,t)$ and $f'(\zeta_\theta(x,t)) = f_\theta(x,t)$. So

$$\begin{aligned} -\Delta M^\theta(x,t) + M^\theta(x,t) &= e_\theta(x,t)N^\theta(x,t) \quad \text{in } \Gamma_\theta, \\ -\Delta N^\theta(x,t) + N^\theta(x,t) &= f_\theta(x,t)M^\theta(x,t) \quad \text{in } \Gamma_\theta. \end{aligned} \tag{3.4}$$

To prove Part II, we need the following lemmas.

Lemma 3.4. *There exists $\theta_0 > 0$, such that either $(-\infty, -\theta_0] \subset \Theta$ or $u(x,t) \equiv u(x,t^{-\theta_0})$ and $v(x,t) \equiv v(x,t^{-\theta_0})$ in $\Gamma_{-\theta_0}$.*

Proof. Note that by (H2), $\lim_{t\to 0^+} g'(t) = 0$ and $\lim_{t\to 0^+} f'(t) = 0$. Take $t_0 > 0$ such that if $0 < t \le t_0$, then $g'(t) < 1$ and $f'(t) < 1$. Since $\lim_{|t|\to\infty} u(x,t) = 0$ and $\lim_{|t|\to\infty} v(x,t) = 0$ uniformly in $x \in B^{N-1}(0;R)$, we can choose $\theta_0 > 0$ such that if $t \le -\theta_0$, $u(x,t) \le t_0$ and $v(x,t) \le t_0$ uniformly in $x \in B^{N-1}(0;R)$.

We claim that if $\theta \le -\theta_0$, then $M^\theta(x,t) \le 0$ and $N^\theta(x,t) \le 0$ in Γ_θ.

On the contrary, suppose that there exists θ such that $\theta \le -\theta_0$, $M^\theta(x,t) > 0$ for some $(x,t) \in \Gamma_\theta$, or $N^\theta(x,t) > 0$ for some $(x,t) \in \Gamma_\theta$. Since $\lim_{t\to-\infty} M^\theta(x,t) = 0$ and $\lim_{t\to-\infty} N^\theta(x,t) = 0$ uniformly in $x \in B^{N-1}(0;R)$, $M^\theta(x,t)$ achieves its maximum at $(x_2,t_2) \in \Gamma_\theta$ and $N^\theta(x,t)$ achieves its maximum at $(x_3,t_3) \in \Gamma_\theta$. Then

$$\begin{aligned} \nabla M^\theta(x_2,t_2) = 0, &\quad \{M^\theta_{ij}(x_2,t_2)\} \le 0, \\ \nabla N^\theta(x_3,t_3) = 0, &\quad \{N^\theta_{ij}(x_3,t_3)\} \le 0. \end{aligned}$$

Since $\Delta M^\theta(x_2,t_2) \le 0$ and $\Delta N^\theta(x_3,t_3) \le 0$, from elliptic systems (3.4) it follows that

$$\begin{aligned} e_\theta(x_2,t_2)N^\theta(x_2,t_2) &\ge M^\theta(x_2,t_2) > 0, \\ f_\theta(x_3,t_3)M^\theta(x_3,t_3) &\ge N^\theta(x_3,t_3) > 0. \end{aligned}$$

From (H1)–(H3) it follows that $e_\theta(x_2,t_2) \geq 0$ and $f_\theta(x_3,t_3) \geq 0$. Then we obtain that $N^\theta(x_2,t_2) > 0$ and $M^\theta(x_3,t_3) > 0$. Moreover, if $\theta \leq -\theta_0$, since $N^\theta(x_2,t_2) > 0$, $M^\theta(x_3,t_3) > 0$, then from (H3), $e_\theta(x_2,t_2) < 1$, $f_\theta(x_3,t_3) < 1$, and get

$$\begin{aligned} N^\theta(x_2,t_2) &> e_\theta(x_2,t_2)N^\theta(x_2,t_2) \geq M^\theta(x_2,t_2),\\ M^\theta(x_3,t_3) &> f_\theta(x_3,t_3)M^\theta(x_3,t_3) \geq N^\theta(x_3,t_3). \end{aligned} \tag{3.5}$$

Since $N^\theta(x_3,t_3) \geq N^\theta(x_2,t_2)$, from (3.5) it follows that $M^\theta(x_3,t_3) > M^\theta(x_2,t_2)$. We come to a contradiction. So for $\theta \leq -\theta_0$, $M^\theta(x,t) \leq 0$ and $N^\theta(x,t) \leq 0$ in Γ_θ. As a consequence of the maximum principle and the Hopf boundary point lemma, either $M^{-\theta_0}(x,t) \equiv 0$, $N^{-\theta_0}(x,t) \equiv 0$ in $\Gamma_{-\theta_0}$ or for $\theta \leq -\theta_0$, $M^\theta(x,t) < 0$ in Γ_θ, $M^\theta_t(x,t) > 0$ for $(x,t) \in \mathbf{S} \cap S_\theta$, $N^\theta(x,t) < 0$ in Γ_θ, and $N^\theta_t(x,t) > 0$ for $(x,t) \in \mathbf{S} \cap S_\theta$. Hence either $(-\infty,-\theta_0] \subset \Theta$ or $u(x,t) \equiv u(x,t^{-\theta_0})$ and $v(x,t) \equiv v(x,t^{-\theta_0})$ in $\Gamma_{-\theta_0}$. □

Lemma 3.5. *If $(-\infty,\theta] \subset \Theta$, then there exists $\varepsilon > 0$ such that $[\theta,\theta+\epsilon) \subset \Theta$.*

Proof. Suppose not. Then there exist a decreasing sequence $\theta_k \to \theta$ and a sequence $\{(x_{k2},t_{k2})\}$ of points in Γ_{θ_k} such that $M^{\theta_k}(x_{k2},t_{k2}) = u(x_{k2},t_{k2}) - u(x_{k2},t^{\theta_k}_{k2}) > 0$, or a sequence $\{(x_{k3},t_{k3})\}$ of points in Γ_{θ_k} such that $N^{\theta_k}(x_{k3},t_{k3}) = v(x_{k3},t_{k3}) - v(x_{k3},t^{\theta_k}_{k3}) > 0$. There exists a subsequence $\{(x_{k2},t_{k2})\}$ such that $x_{k2} \to \bar{x}_2 \in \overline{B^{N-1}(0;R)}$ or a subsequence $\{(x_{k3},t_{k3})\}$ such that $x_{k3} \to \overline{x}_3 \in \overline{B^{N-1}(0;R)}$. There may arise two possibilities: Cases 1 and 2 below.
Case 1. $t_{k2} \to -\infty$. As shown in Lemma 3.4, we assume

$$\begin{gathered} M^{\theta_k}(x_{k2},t_{k2}) = \max_{(x,t)\in\overline{\Gamma}_{\theta_k}} M^{\theta_k}(x,t),\\ \nabla M^{\theta_k}(x_{k2},t_{k2}) = 0, \quad \{M^{\theta_k}_{ij}(x_{k2},t_{k2})\} \leq 0. \end{gathered} \tag{3.6}$$

From $\lim_{t_{k2}\to-\infty} u(x_{k2},t_{k2}) = 0$, as in Lemma 3.4, we obtain a contradiction. The same argument applies to $N^{\theta_k}(x_{k3},t_{k3})$.
Case 2. $t_{k2} \to \bar{t}_2$. We have $(x_{k2},t_{k2}) \to (\bar{x}_2,\bar{t}_2) \in \overline{\Gamma_\theta}$. Thus $M^\theta(\bar{x}_2,\bar{t}_2) \geq 0$. Clearly $(\bar{x}_2,\bar{t}_2) \notin \Gamma_\theta$ since $M^\theta(x,t) < 0$ in Γ_θ. If $(\bar{x}_2,\bar{t}_2) \in S_\theta$, then $u_t(\bar{x}_2,\bar{t}_2) < 0$, which contradicts $\theta \in \Theta$. Moreover, $(\bar{x}_2,\bar{t}_2) \notin \partial\mathbf{S} \cap \overline{\Gamma_\theta}$. Note that $M^\theta(x,t)$ satisfies the elliptic systems (3.4), and by the Hopf boundary point lemma, we obtain $\frac{\partial}{\partial\nu}M^\theta(\bar{x}_2,\bar{t}_2) < 0$. On the other hand, taking the limit in (3.6), we obtain $\nabla M^\theta(\bar{x}_2,\bar{t}_2) = 0$, a contradiction. We conclude that Case 2 is impossible. The same argument applies to $N^{\theta_k}(x_{k3},t_{k3})$. □

Proof of Part II. Let $\sigma = \sup\{\theta \in \mathbb{R} : (-\infty,\theta) \subset \Theta\}$. Then $\sigma \notin \Theta$. If not, by Lemma 3.5 we would have $[\sigma,\sigma+\epsilon) \subset \Theta$, which contradicts to the definition of σ. By continuity we have $u(x,t) \leq u(x,t^\sigma)$ and $v(x,t) \leq v(x,t^\sigma)$ for all $(x,t) \in \Gamma_\sigma$, then by the maximum principle we have $u(x,t) \equiv u(x,t^\sigma)$ and $v(x,t) \equiv v(x,t^\sigma)$ for all $(x,t) \in \Gamma_\sigma$. This proves $u(x,t)$ and $v(x,t)$ are symmetric with respect to the hyperplane $t = \sigma$ for all $(x,t) \in \mathbf{S}$. □

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