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DOMAIN GEOMETRY AND THE POHOZAEV IDENTITY

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Abstract. In this paper, we investigate the boundary between existence and nonexistence for positive solutions of Dirichlet problem $\Delta u + f(u) = 0$, where f has supercritical growth. Pohozaev showed that for convex or polar domains, no positive solutions may be found. Ding and others showed that for domains with non-trivial topology, there are examples of existence of positive solutions. The goal of this paper is to illuminate the transition from non-existence to existence of solutions for the nonlinear eigenvalue problem as the domain moves from simple (convex) to complex (non-trivial topology).

To this end, we present the construction of several domains in R^3 which are not starlike (polar) but still admit a Pohozaev nonexistence argument for a general class of nonlinearities. One such domain is a long thin tubular domain which is curved and twisted in space. It presents complicated geometry, but simple topology. The construction (and the lemmas leading to it) are new and combined with established theorems narrow the gap between non-existence and existence strengthening the notion that trivial domain topology is the ingredient for non-existence.

1. Introduction and background

A fundamental question in differential equations is whether or not a solution to the differential equation can be found. In the subject of nonlinear elliptic equations, Pohozaev provided a very useful tool in addressing this question. The Pohozaev variational identity has been very successful answering questions of solvability with respect to the nonlinearity and the domain. The authors have recently focused on the relation between domain geometry and problem solvability.

In this paper, we continue the thread of investigation by presenting some of the relations between the geometry of the domain and solvability of nonlinear eigenvalue problems. The essence of these problems may be captured into the following problem. Let Ω be an open bounded set in R^N, $N \geq 3$, with a smooth boundary $\partial\Omega$ (which means C^2 here). We seek $u : \Omega \to \mathbb{R}$ a positive solution to

$$\begin{gathered} \Delta u + f(u) = 0, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.1}$$

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where f has critical or supercritical growth, meaning, $f(u) \geq k\, u^{(N+2)/(N-2)}$ for some positive constant k. We ask the question "for a prescribed domain Ω and a nonlinearity f, can we find a positive solution u?" We restrict our focus to positive solutions and dimension $N \geq 3$; the latter ensuring the previous growth rate is defined. The case of $N = 2$ is well studied and existence of solutions has been demonstrated for general domains [2, 3, 4, 6, 7].

Pohozaev proved that there is no solution for polar or starlike domains [10]. A starlike domain is one that there is at least one point in the domain for which you can see the entire boundary (see Figure 1). On the other hand, Bahri and Coron, Ding [5, 1], have shown that a solution exists when $f(u) = u^{(N+2)/(N-2)}$ and the domain has nontrivial topology. Figure 1 gives an example of a domain with nontrivial topology. Between these two theorems is a vast complicated landscape of dimension, topology and growth rates.

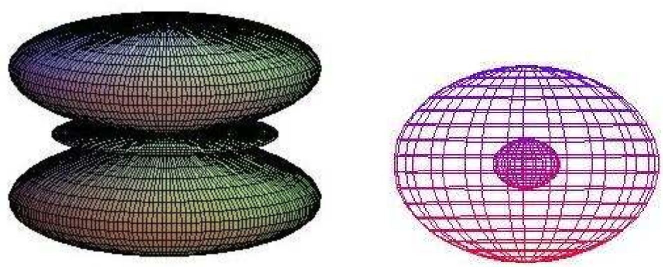

Figure 1. Starlike domain and a simple domain with nontrivial topology

The goal of this paper is to narrow the gap between Pohozaev's nonexistence result and Ding's existence result. It appears that the dominant factor is domain topology, not domain geometry. No proof of this assertion is offered here, only mounting evidence of examples for which domains with negative boundary curvature still present a nonexistence result. A rather interesting example is found in certain tubular domains constructed in Section 4. Our main result to this end is the construction of the required elements for a Pohozaev non-existence proof for curved tubular domains. In this direction, we prove three properties (Lemma's 4, 5, and 6) about the kernel of Pohozaev's variational identity. The lemmas and the resulting examples provides a base for building example domains for further exploration of the solvability question. We feel that the lemmas and examples are our main contribution in this paper; that they provide sufficient empirical evidence that the existence or non-existence of solutions depends on the domain being topologically trivial. For domain construction, dimension $N = 3$ is the difficult case and our examples will address this case. Before moving into the construction process, some background is useful.

For Pohozaev's nonexistence result to work, one needs to construct a vector field, $h : \Omega \to \mathbb{R}^N$, which locally is close to a radial vector field. Higher dimensions offer plenty of room and result in admitting more domains. In three dimensions, we must balance the restrictions arising from domain curvature and the space requirements

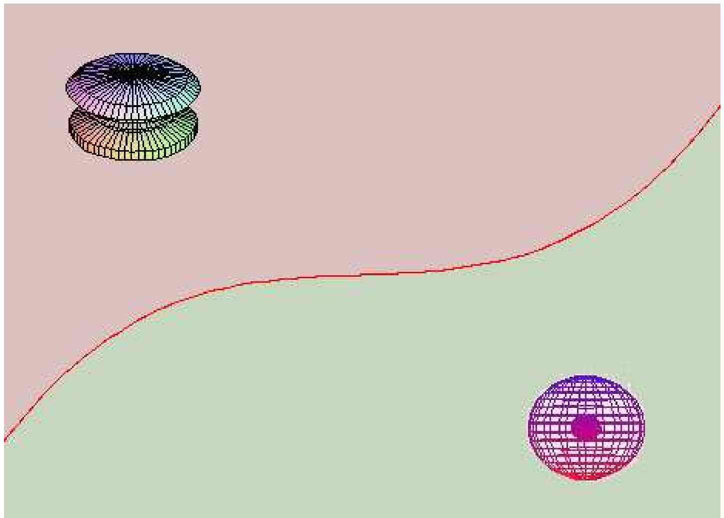

FIGURE 2. Question: What is the boundary between solution existence and non-existence?

of the radial nature of the vector field. The results that are available are constructed from bending or modifying the base vector field defined by $h(x) = [x_1, x_2, x_3] = x$.

As mentioned above, the Pohozaev Identity [10] is the principle tool used here to investigate the relation between domain geometry and solvability. Published in 1965, it has been a widely used variational identity in the study of divergence form elliptic equations. The original identity has been generalized extensively, and we will focus on the form published by Pucci and Serrin [11]. This paper examines elliptic problems in divergence form and provides a direct route to the Pohozaev Identity. Their result is reproduced below. The classical results of Pohozaev and Pucci-Serrin do not require that $N = 3$ and so are presented in the more general form. However, our constructions which follow do and so in Section 3 we restrict ourselves to $N = 3$.

Theorem 1.1 (Pucci-Serrin). *Let u be a C^2 solution to*

$$\begin{gathered} \operatorname{div}\{g(x, \nabla u)\} + f(x, u) = 0, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.2}$$

Further, let $h : \Omega \to \mathbb{R}^N$, where $h_i \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and $a \in \mathbb{R}$. Then u satisfies

$$\begin{aligned} &\int_\Omega \big\{ \operatorname{div}(h)[F(x, u) - G(x, \nabla u)] + h \cdot [F_x(x, u) - G_x(x, \nabla u)] \\ &- auf(x, u) + a\nabla u \cdot g(x, \nabla u) + \nabla u \cdot Dh\, g(x, \nabla u)\big\}\, dx \\ &= \int_{\partial\Omega} \big[\nabla u \cdot g(x, \nabla u) - G(x, \nabla u)\big](h \cdot \nu)\, dS \end{aligned} \tag{1.3}$$

where $g(x, s) = \partial G/\partial s$, $f(x, s) = \partial F/\partial s$.

In our case, application to Equation 1.1, Identity (1.3) becomes:

$$\begin{aligned} &\int_\Omega \{\operatorname{div}(h)F(u) - auf(u)\}\,dx \\ &= \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (h\cdot\nu)\,dS + \int_\Omega \big\{\big[\frac{1}{2}\operatorname{div}(h) - a\big]|\nabla u|^2 - \nabla u \cdot Dh\,\nabla u\big\}\,dx. \end{aligned} \tag{1.4}$$

For additional information on the development of variational identities leading to Pucci-Serrin's result and some applications see the references contained in [8, 9, 12].

2. Beyond convexity

To proceed with the analysis, we recall two geometric definitions. A domain is said to be convex if for any two arbitrary points in the domain, a line connecting the two points lies entirely in the domain. A domain is said to be starlike if there exists some point x_0 in the domain for which $(x - x_0)\cdot\nu > 0$, for all $x \in \partial\Omega$ and $\nu = \nu(x)$ is the boundary normal vector at the point x. Polar domains are often viewed as spheres or ellipsoids, but can be quite complicated and interesting in their own right (for example consider the spherical harmonic solutions to the Laplacian). Figure 3 presents an example of a geometrically complex polar domain.

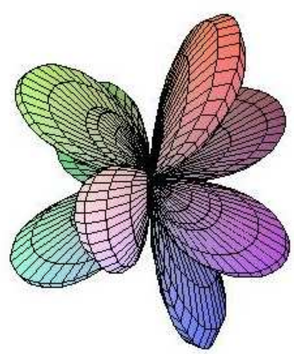

Figure 3. Non-convex starlike domain.

We can restate the starlike definition in the language of convex domains. A domain is said to be starlike if there exists a reference point in the domain such that for any point in the domain the line between the reference point and the arbitrary point lies in the domain.

For those not familiar with Pohozaev's result [10], it is presented here for the case where $f(u) = u^{(N+2)/(N-2)}$.

Theorem 2.1 (Pohozaev). *Assume that Ω is a smooth starlike domain, then there are no positive solutions to*

$$\begin{gathered} \Delta u + u^{(N+2)/(N-2)} = 0, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered}$$

The proof uses a form of the Pucci-Serrin identity and follows a proof by contradiction argument. We assume a positive solution u exists and select $f(u) = u^{(N+2)/(N-2)}$, $a = (N-2)/2$, $h(x) = x - x_0$ and plug into (1.4). We obtain

$$\frac{N}{\frac{N+2}{N-2}+1} - \frac{N-2}{2} > 0,$$

which leads to $0 > 0$, a contradiction.

Many results have followed which generalize the nonlinearity used in the Pohozaev results. An observation about the vector function h points the way to generalizing the domain. One notes that the vector field need not be $x - x_0$ but just to exhibit the essential properties of a starlike field in a polar domain. This leads to the definition of h-starlike domains [9].

Definition 2.2. The domain Ω is said to be *h-starlike* if there exists a function $h : \Omega \to \mathbb{R}^N$, $h_i \in C^1(\overline{\Omega})$, and a positive number c with

$$\begin{gathered} \operatorname{div}(h)|y|^2 - 2y \cdot Dhy \ge c|y|^2, \quad x \in \overline{\Omega}, \quad y \in \mathbb{R}^N, \\ h \cdot \nu \ge 0, \quad x \in \partial\Omega, \end{gathered} \tag{2.1}$$

where Dh is the derivative map of h and ν is the outward unit normal to $\partial\Omega$.

For computational reasons, we will find it convenient to reformulate condition (2.1).

Definition 2.3. For a vector field h the Pohozaev trace is

$$P(h) = \operatorname{Trace}(Dh) - 2|\lambda_1|$$

where λ_1 is the largest eigenvalue (in magnitude) of the symmetrized Dh, namely $(Dh + Dh^T)/2$.

If $\inf_{x\in\Omega} P(h) \ge c > 0$, then the condition on h in the first inequality in (2.1) is satisfied since

$$\operatorname{div}(h)|y|^2 - 2y \cdot Dhy \ge P(h)|y|^2, \quad x \in \overline{\Omega}.$$

The next theorem can be established by letting $a = c/2$ in (1.4) and deleting the boundary integral—details can be found in [8].

Theorem 2.4 (Pohozaev)**.** *Assume that there exists an h-starlike function for the domain Ω with $\inf_{x\in\Omega} P(h) \ge c > 0$ and suppose that f satisfies*

$$\operatorname{div} h(x)F(t) - \frac{c}{2}tf(t) \le 0 \tag{2.2}$$

for $t \ge 0$ and $x \in \Omega$. Then there are no positive solutions to (1.1).

If $f(t) = t^p$ then $F(t) = t^{p+1}/(p+1)$ and (2.2) is satisfied if $b/(p+1) - c/2 \le 0$ or $p \ge 2b/c - 1$ where $b = \sup_{x\in\Omega} \operatorname{div} h(x)$. In particular, we have that if Ω is h-starlike for a vector field h, then (2.3) below has no positive solutions for $p \ge 2b/\inf_{x\in\Omega} P(h) - 1$.

$$\begin{gathered} \Delta u + u^p = 0, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{2.3}$$

Definition 2.5. For a given h-starlike domain Ω, let the set Q be the collection of vector fields for which (2.1) holds. We define the Pohozaev critical exponent as

$$P_c = \min_Q \frac{2\sup_{x\in\Omega} \operatorname{div} h(x)}{\inf_{x\in\Omega} P(h)} - 1 \tag{2.4}$$

Remark 2.6. In the subsequent sections we construct example domains and show that there exists at least one vector field on the domain for which $\inf_{x\in\Omega} P(h) > 0$. In light of the preceding discussion, it is automatic that there are no positive solutions for (2.3) on that domain when

$$p \geq \frac{2\sup_{x\in\Omega} \operatorname{div} h(x)}{\inf_{x\in\Omega} P(h)} - 1.$$

We make no attempt to compute the smallest such p; i.e., P_c.

Remark 2.7. It is clear that starlike domains are h-starlike. Consider a starlike domain in $\mathbb{R}^N$ with $h(x) = x$. Then

$$P_c \leq \frac{2N}{N-2} - 1 = \frac{N+2}{N-2}. \tag{2.5}$$

A result reminiscent of Theorem 2.1. Indeed, it is not difficult to show that we actually have equality in (2.5). One only needs to consider the possible eigenvalues of the symmetrized Dh.

3. Sectionally starlike domains

Several h-starlike domains are given in [9]. One example of h-starlike vectors fields which generate h-starlike domains is provided by the re-scaled radial field

$$h(x_1, x_2, \cdots, x_N) = (\epsilon x_1, x_2, x_3, \cdots, x_N).$$

Figure 4 gives two such sample domains for $N = 3$. For the remainder of the paper, we restrict ourselves to three dimensional domains: $N = 3$.

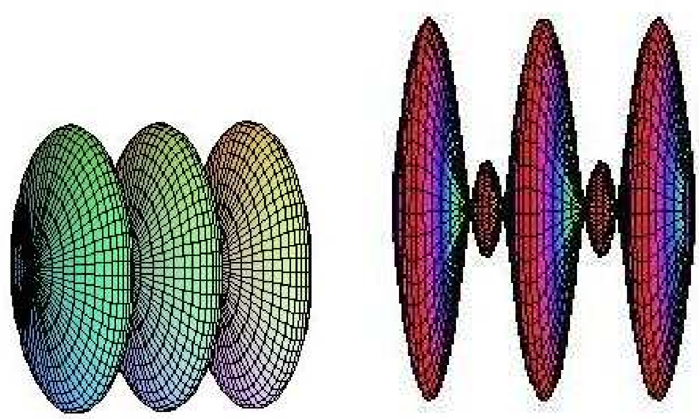

Figure 4. Non-starlike h-starlike domains (three and five disks).

We will refer to a domain as sectionally starlike (convex) if there exists a curve contained in the domain such that the domain intersected with the hyperplane normal to the tangent to the curve is starlike (convex). The previous domain is an example of a sectionally convex domain. Two questions arise here. Does Pohozaev's result extend directly to sectionally starlike domains? Is there some relation between h-starlike and sectionally starlike? The answer to the former question is no. A counter-example is a torus in 3D; problem (1.1) is solvable on a torus [8] for the critical exponent problem in R^3: $f(u) = u^{(N+2)/(N-2)} = u^5$. The latter question is addressed below.

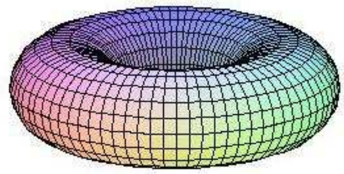

FIGURE 5. The torus

The solid torus may be generated by moving the center of a disk along a circle where the disk is orthogonal to the circular path. The curve that generates the torus, a circle, is a closed curve. The torus has nontrivial topology (there exists closed curves contained in the torus which cannot be shrunk to point and remain in the torus). Ding, Bahri and Coron, Lin and others have presented domains with nontrivial topology which demonstrate solvability of the critical exponent problem. The natural place to investigate is sectionally starlike domains with trivial topology.

The example domains shown in Figure 4 are volumes of revolution. Which means that there is a line segment which may act as the central curve for the sectional starlike definition. It is not surprising that a starlike region crossed with an interval produces an h-starlike domain. The more interesting question is whether the central axis can be curved. For this, we may give an affirmative answer.

We begin with the vector field defined by $h(x) = [\epsilon x, y, z]$. Note that this will shrink the x component and allow us to "bend" the domain. For clarity in this example, we select $\epsilon = 1/5$, but noting that this is only for illustration. Next, we may compute the Jacobian of h,

$$Dh = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues are $\{1/5, 1, 1\}$ and thus by Definition 2.3 the Pohozaev trace of h is

$$P(h) = \mathrm{Trace}(Dh) - 2|\lambda_1| = 1/5.$$

The vector field has positive Pohozaev trace. To simplify the construction, we look at the x-y cross-section. By following the flow lines, we may trace out curves for which the vector field is tangential. This satisfies the boundary condition for the h-starlike definition in the cross-section. We may construct the flow lines by solving the differential equations

$$\frac{dx}{dt} = \frac{1}{5}x, \quad \frac{dy}{dt} = y.$$

The orbits $y = kx^5$ are given in Figure 6. Figure 7 shows the boundary in the x-y plane and a sample domain constructed from this boundary. The z aspect of the domain turns out not to be very restricted (it is required to have a starlike cross-section). The essential requirement is to maintain the non-negative dot product

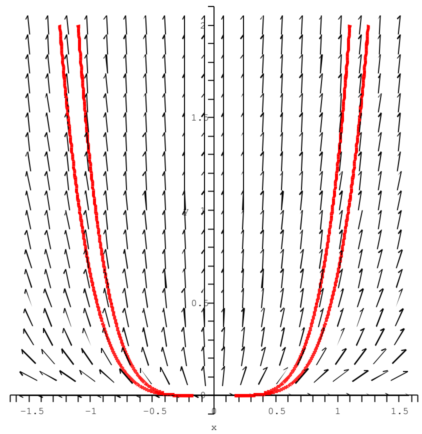

FIGURE 6. Vector field h with sample orbits.

between the boundary normal and the vector field. A surface with z values sufficiently large will produce the required dot product. Example 4.3 provides details on how to do this construction.

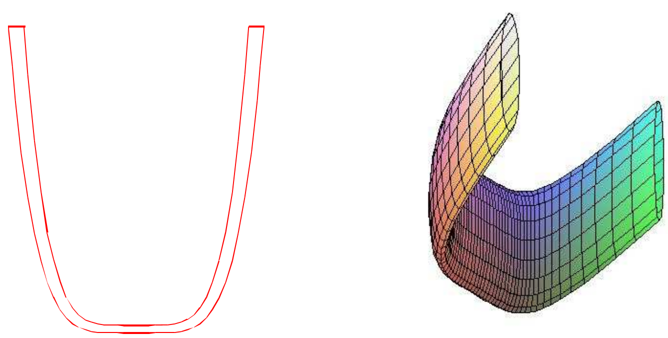

FIGURE 7. Boundary: (a) XY section, (b) 3D

Following a similar approach we may create other domains. Using *sin* and *cos* we can create vector fields which do not generate constant eigenvalues (over the domain). Starting with $h = [x, y(y - f(x)), z]$, $f(x) = \sin(x)$, we have

$$Dh = \begin{bmatrix} 1 & 0 & 0 \\ -y\cos(x) & 2y - \sin(x) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which yields the eigenvalues $\{1, 1, 2\,y - \sin(x)\}$ and the Pohozaev trace

$$P(h) = 2 + 2\,y - \sin(x) - 2\,\max\{1, 2\,y - \sin(x)\}.$$

Following the vector field, orbits can be traced out by integration. Flow lines (in yellow) and vector fields (in red) are given in Figure 8. The bounding curve (in blue) gives the region for which $p > .05$.

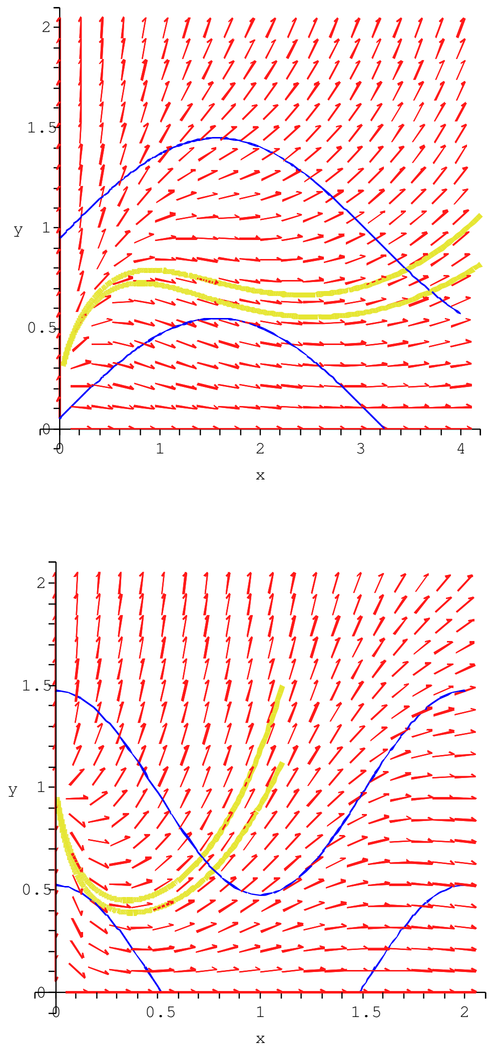

FIGURE 8. Flow lines for $h = [x, y(y - f(x)), z]$ (a) $f(x) = \sin(x)$, (b) $f(x) = \cos(x)$

To construct sample domains for Figure 8, we generate a tubular neighborhood of a curve bounded between the flow lines. This construction is done for both domains and presented in Figure 9. Endcaps for the domain are not generated but could be placed on in a smooth manner producing a C^2 boundary.

We have managed to construct three domains which are h-starlike but not starlike. Is it possible to construct domains like building with Lego's? In other words,

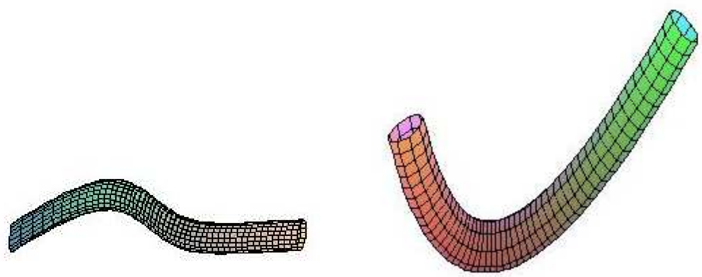

FIGURE 9. 3D domain for (a) $f(x) = \sin(x)$, (b) $f(x) = \cos(x)$

can we start connecting these domains to construct complicated twisted geometries? We address these questions in the next section.

4. Tubular neighborhoods

The sample domains constructed previously were tubular neighborhoods of given space curves. The radius of the neighborhood was selected to ensure that the Pohozaev trace was strictly positive on the closure of the domain. We now embark on connecting two tubular domains in R^3 to construct domains which curve in the plane and rise out of the plane of curvature (torsion).

We assume that the two domains can be connected in a smooth fashion. This means we can cut the end of the domains to be connected so that the end-section is a planar section, has the same cross-sectional shape, and is orthogonal to the generating space curve. To join two domains (see Figure 10), we need to ensure

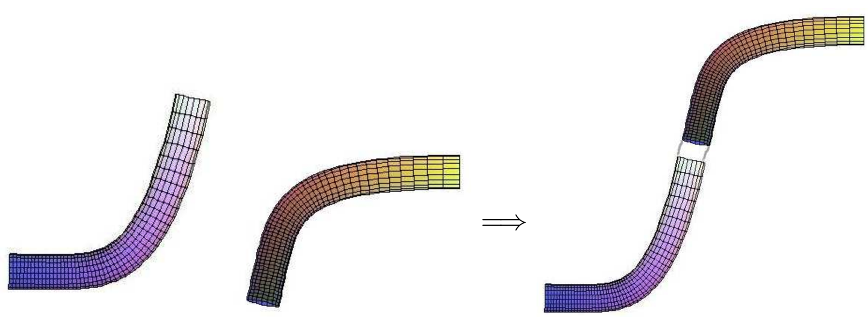

FIGURE 10. Join two domains

that the Pohozaev trace on the interface (or the transition region) is defined and positive. First, we need a result about the Pohozaev trace.

Lemma 4.1. *The Pohozaev trace is superadditive,*

$$P(h_1 + h_2) \geq P(h_1) + P(h_2)$$

Proof. For real symmetric (more generally normal) matrices A and B, the spectral radius $r(A)$ is the same as the spectral norm. Consequently, we have subadditivity $r(A+B) \le r(A) + r(B)$ and we have that $r(A) = |\lambda_1|$, where λ_1 is the largest eigenvalue (in magnitude).

$$\begin{aligned}
&\operatorname{div}(h_1 + h_2) - 2r([Dh_1/2 + Dh_1^t/2] + [Dh_2/2 + Dh_2^t/2]) \\
&\ge \operatorname{div}((h_1) + \operatorname{div}(h_2) - 2r(Dh_1/2 + Dh_1^t/2) - 2r(Dh_2/2 + Dh_2^t/2),
\end{aligned}$$

and thus $P(h_1 + h_2) \ge P(h_1) + P(h_2)$. □

Lemma 4.2. *The Pohozaev trace is invariant under rigid rotations of the vector field.*

Proof. Let h_r be a rotation of h, and set $h_r = M \circ h \circ M^{-1}$, where M is an orthogonal matrix. Next, compute

$$P(h_r) = \operatorname{Tr}(M \circ Dh \circ M^{-1}) - |\lambda_1(M \circ Dh \circ M^{-1})| = \operatorname{Tr}(Dh) - |\lambda_1(Dh)| = P(h),$$

where as before $\lambda_1(Dh)$ is the largest eigenvalue of Dh in magnitude. □

The following example shows how one can use a transition function to move from the zero field to $[x/5, y, z]$.

Example 4.3. Define

$$p(x) = \begin{cases} 0 & x \le 0 \\ -2x^3 + 3x^2 & 0 < x < 1 \\ 1 & x \ge 1 \end{cases},$$

and set $h := p(x)[x/5, y, z]$. Figure 11 shows the values of the Pohozaev trace of h.

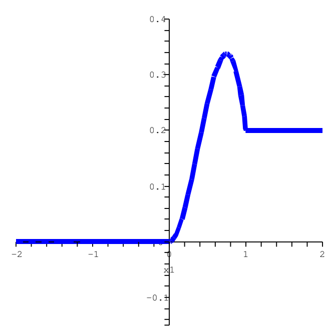

Figure 11. Pohozaev trace of $p(x)h(x, y, z)$; horizontal axis is x

We give a piecewise definition of a parameterization of a tube in $\mathbb{R}^3$. Let $t \in [0, 2\pi]$; For $0 \le s \le 1$ define $r(s,t) = [s, \cos(t), 1 + \sin(t)]$ and for $s \ge 1$ define $r(s,t) = [s, \cos(t), 1 + \sin(t) + a(s-1)^5]$ with $a > 0$ a parameter. For $s \ge 1$, we use the normal vector field

$$n := r_s \times r_t = [-5\sin(t)a(s-1)^4, \cos(t), \sin(t)].$$

With this normal,

$$h \cdot n = 1 + \sin(t) - a\sin(t)(s-1)^4. \tag{4.1}$$

Analyzing formula (4.1) we discover that the additional restriction $s \le 1 + (2/a)^{1/4}$ must be imposed in order to ensure that $h \cdot n \ge 0$. At $s = 1 + (2/a)^{1/4}$ the tube attains a height of $2 + 2^{5/4}/a^{1/4}$. Figure 12 illustrates the case where $a = 2/3^4$. Note: for $0 \le s \le 1$, the cylindrical portion of the tube, $h \cdot n = p(s)[1 + \sin(t)] \ge 0$.

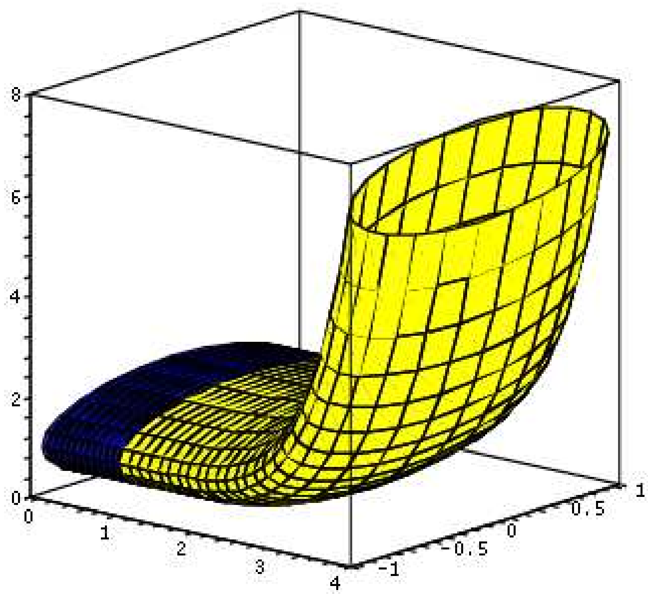

FIGURE 12. Parameterized Tube

Using the method of example 4.3, we can construct two tubes as illustrated in figure 13 and glue them together along their cylindrical portions ($0 \le s \le 1$ in example 4.3). It is of course permissible by Lemma 4.2 to first rotate either of the tubes about the axis of the cylindrical portion prior to gluing. In either case, the Pohozaev trace on the union of the tubes will be positive and bounded away from zero by Lemma 4.1.

The parametrization given in example 4.3 has the advantage that the computations for $h \cdot n$ are straight-forward. However, it has the disadvantage that the cross-sections of the tube normal to the generating curve are not circular on both ends. As a consequence, it is difficult to attach other tubes to the far end ($s = 1 + (2/a)^{1/4}$).

Example 4.4. For $0 \le s \le 1$ define

$$r(s,t) := [s, R\sin(t), R + R\cos(t)]$$

and for $1 \le s \le s_0$ define

$$r(s,t) := [s, 0, R + a(s-1)^5] + R\sin(t)[0,1,0] + \frac{R\cos(t)}{\sqrt{1 + 25a^2(s-1)^8}}[5a(s-1)^4, 0, -1].$$

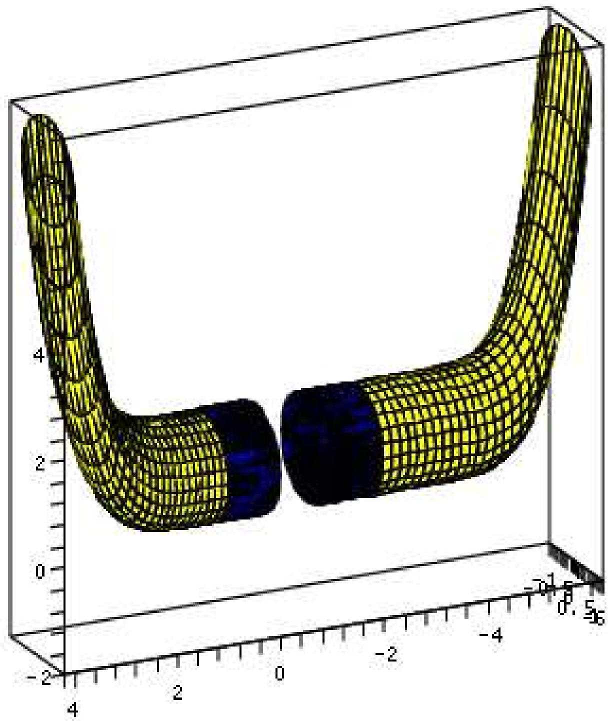

Figure 13. Gluing tubes

The largest s such that $h \cdot n \geq 0$ for all t is

$$s_{\max} := 1 + \frac{1}{45}\sqrt[4]{91125\,\frac{\sqrt[3]{5300+81\sqrt{2451}}}{a} + \frac{20867625}{a\sqrt[3]{5300+81\sqrt{2451}}} + \frac{1822500}{a}}.$$

Let $v(t) := r(s_0, t)$, we attach a short right-cylindrical tube to the far end of the tube by the parametrization

$$r(s,t) := v(t) + [s, 0, 5a(s_0-1)^4 s]\,, \quad s_0 < s \leq s_1. \tag{4.2}$$

If $s_0 < s_{\max}$ and if s_1 is close enough to s_0, then $h \cdot n \geq 0$ on the entire tube. Figure 14 illustrates the case where $a = 1/500$, $R = 1/2$, $s_0 = 5.5$ and $s_1 = 5.7$: it can be checked that s_{max} is larger than 5.9 and that $h \cdot n \geq 0$ on this tube, i.e. for $0 \leq s \leq 5.7$ and all t.

This provides the building blocks for curved and twisted tubular domains with a sample given in Figure 15 (b). At some point in the process of joining tubes it will be necessary to kill off a vector field h in the direction of growth. This will result in a negative Pohozaev trace. Fortunately, the Pohozaev trace has the scale linearity property, $P(ch) = cP(h)$. Thus we can remedy this problem by making the (positive) Pohozaev trace associated with each subsequent tube large enough.

It is tempting to think that one can create an h-starlike vector field on a torus. It is not possible using this construction approach to join the free ends of the tube. Taking a planar section which is orthogonal to the center line, we see that the

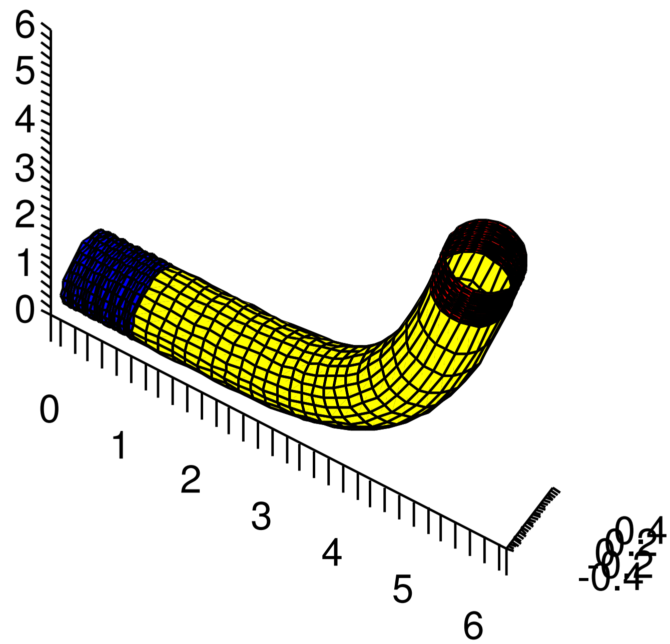

FIGURE 14. Parameterized Tube

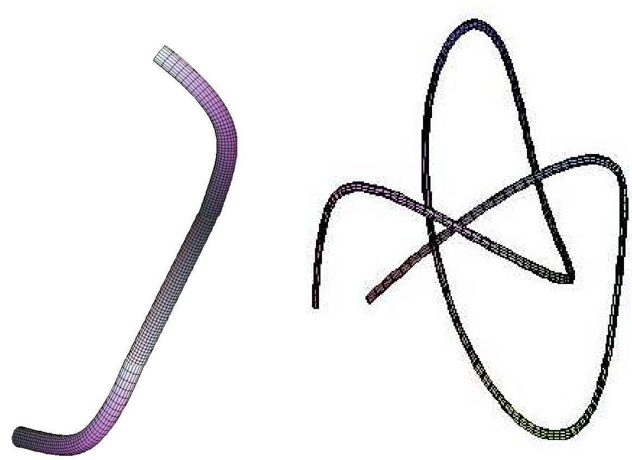

FIGURE 15. (a) Join two domains (b) Pseudo-knotted domain

vector field would have to be reflected across the section as it moves through the transition region. In other words, the flow induced by the vector field points inward at each end of the tube. At the ends the vector field must be tangential to the end section which has zero Pohozaev trace. Thus the resulting field is not h-starlike. Our transition region only has monodirectional flow. This is more than a deficit in the construction; recall that the circular torus cannot have a h-starlike field by a previous result [8]. However, we don't expect that topologically nontrivial domains will admit h-starlike vector fields.

Conclusion. In this paper, we have examined a prototypical nonlinear elliptic problem. The problem has a rich history and results abound. In our case, we follow a line of inquiry based on an integral identity attributed to Pohozaev. The Pohozaev integral identity is used to demonstrate that there cannot be a positive solution to the supercritical growth problem on starlike domains [10]. One of the two main assumptions was that the domain needed to be starlike.

Previous work by the authors has extended the domain to something known as h-starlike. Once we can show a domain is h-starlike, the nonexistence proof is automatic. A number of complicated domains have been presented and we have constructed h-starlike vector fields on them. It is immediate that there is no solution for the elliptic problem on the given domain when the growth constant is larger than the Pohozaev critical exponent. The plethora of trivial topology domains presented here all indicate our proposition: existence vs non-existence boils down to simple vs not-simple topology.

The many papers based on Pohozaev's original result have addressed generalizing the nonlinearity, the operator and the domains. In addition, a priori estimates are possible with the same integral identities and can be used for existence and uniqueness theorems. We have found these integral identities to be very useful in the study of elliptic partial differential equations.

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5. Appendix: Graphics and numerics notes

All of the graphics, the analytical and numerical computations for this paper were completed on Maple 9. For example, the computations used to generate the second image in Figure 8 are listed below.

```
with(linalg):with(plots):with(DEtools):
f:=(y-cos(Pi*x)):
h:=vector([x,y*f,z]):
Dh:=jacobian(h,[x,y,z]):
ev:=eigenvals(Dh):
p:=trace(Dh)-2*max(ev):
h1:=vector([h[1],h[2]]):
n:=normalize(h1):
A:=implicitplot([p=0.05],x=0..2,y=0..2,numpoints=1000,color=blue):
A1:=implicitplot([p=0.2],x=0..2,y=0..2,numpoints=1000,color=blue):
A2:=implicitplot([p=0.4],x=0..2,y=0..2,numpoints=1000,color=blue):
A3:=implicitplot([p=0.6],x=0..2,y=0..2,numpoints=1000,color=blue):
V:=[x(t),y(t)]:
XY:=x=0..2,y=0..2:
F:=[diff(x(t),t)=x(t),diff(y(t),t)=y(t)*(y(t)-cos(Pi*x(t)))]:
L:=[[x(0)=.15,y(0)=.5],[x(0)=.15,y(0)=.55]]:
B:=DEplot(F,V,t=0..2,L,XY,stepsize=.01):
BB:=DEplot(F,V,t=0..-3,L,XY,stepsize=.01):
```

```
display({A,A1,A2,A3,B,BB,pp});
```

Using the `tubeplot` command we can gain the 3D domain image (once we have a curve which runs down the center of the domain. A secondary but handy feature is to have Maple generate LaTeX for the output of the commands. The package used is `codegen` and it is called by entering `with(codegen)`. After this, for example, entering `latex(Dh)` will produce the latex command for typesetting the Maple computed Jacobian matrix.

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