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GLOBAL SOLUTIONS WITH INFINITE ENERGY FOR THE ONE-DIMENSIONAL ZAKHAROV SYSTEM

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Abstract. The one-dimensional Zakharov system is shown to have a unique global solution for data without finite energy. The proof uses the "I-method" introduced by Colliander, Keel, Staffilani, Takaoka, and Tao in connection with a refined bilinear Strichartz estimate.

1. Introduction

Consider the (1+1)-dimensional Cauchy problem for the Zakharov system

$$iu_t + u_{xx} = nu \tag{1.1}$$

$$n_{tt} - n_{xx} = (|u|^2)_{xx} \tag{1.2}$$

$$u(0) = u_0, \quad n(0) = n_0, \quad n_t(0) = n_1 \tag{1.3}$$

where u is a complex-valued and n a real-valued function defined for $(x,t) \in \mathbb{R}\times\mathbb{R}^+$. This Zakharov system was introduced in [19] to describe Langmuir turbulence in a plasma.

Our main result is the existence of a unique global solution for data without finite energy, more precisely we assume $u_0 \in H^s(\mathbb{R})$, $n_0 \in H^{s-1}(\mathbb{R})$, $A^{-1/2}n_1 \in H^{s-1}(\mathbb{R})$, where $1 > s > 5/6$, $A := -\frac{d^2}{dx^2}$.

This result can be proven by using the conservation laws, namely conservation of $\|u(t)\|$ and

$$E(u,n) := \|u_x(t)\|^2 + \frac{1}{2}(\|n(t)\|^2 + \|A^{-1/2}n_t(t)\|^2) + \int_{-\infty}^{\infty} n(t)|u(t)|^2\,dx$$

although under our assumptions these quantities are not finite, in general.

Results of this type were given in various situations in the previous years in the framework of the Fourier restriction norm method in most of the applications. One approach is to use Bourgain's trick to split the data into high and low frequency parts. He used it to prove global well-posedness for the (2+1)-and (3+1)-dimensional Schrödinger equations with rough data without finite energy [1, 2] and for the wave equation [3]. Later it was also used for other model equations [10, 11, 14, 15, 18]. Concerning the problem at hand the author had been able to show global well-posedness for data $(u_0, n_0, n_1) \in H^s \times L^2 \times \dot{H}^{-1}$ for $1 > s > 9/10$

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[17]. Remark here that no data $n_0 \notin L^2$ were admissible because in such a case the nonlinear part of $n(t)$ could not be shown to belong to L^2 which is necessary for this method. In contrast, the approach here allows data $n_0 \notin L^2$ and also less regular data u_0.

Another approach was initiated by Colliander, Keel, Staffilani, Takaoka and Tao in [6], called the I-method. The main idea is to use a modified energy functional which is also defined for less regular functions and not strictly conserved. When one is able to control its growth in time explicitly this allows to iterate a modified local existence theorem to continue the solution to any time T and moreover to estimate its growth in time. This method was successfully applied by these authors to several equations which have a scaling invariance with sometimes even optimal global well-posedness results. It was used in [6] to improve Bourgain's global well-posedness results [1, 2] for the (2+1)- and (3+1)-dimensional Schrödinger equation with a further improvement in [9]. Later it was applied to the (1+1)-dimensional derivative Schrödinger equation [4] with an (almost) optimal result in [5] and to the KdV and modified KdV equation with also optimal results in some cases [7, 8].

Although in our situation such a scaling argument does not work we are able to suitably modify the method to prove the above mentioned global existence result for the Zakharov system.

The paper is organized as follows. We transform the system in the usual way into a first order system. Then we apply the multiplier I_N for given $s < 1$ and $N >> 1$ to it, where $\widehat{I_N f}(\xi) := m_N(\xi)\widehat{f}(\xi)$. Here $m_N(\xi)$ is a smooth, radially symmetric, and nonincreasing function of $|\xi|$, defined by $m_N(\xi) = 1$ for $|\xi| \leq N$ and $m_N(\xi) = (\frac{N}{|\xi|})^{1-s}$ for $|\xi| \geq 2N$. We drop N from the notation for short and remark that $I : H^s \to H^1$ is a smoothing operator in the following sense:

$$\|u\|_{X_\varphi^{m,b}} \leq c\|Iu\|_{X_\varphi^{m+1-s,b}} \leq cN^{1-s}\|u\|_{X_\varphi^{m,b}}$$

Here we used the $X_\varphi^{m,b}$-spaces which are defined as follows: For an equation of the form $iu_t - \varphi(-i\partial_x)u = 0$, where φ is a measurable function, let $X_\varphi^{m,b}$ be the completion of $\mathcal{S}(\mathbb{R} \times \mathbb{R})$ with respect to

$$\begin{aligned} \|f\|_{X_\varphi^{m,b}} &:= \|\langle\xi\rangle^m\langle\tau\rangle^b\mathcal{F}(e^{it\varphi(-i\partial_x)}f(x,t))\|_{L^2_{\xi\tau}} \\ &= \|\langle\xi\rangle^m\langle\tau+\varphi(\xi)\rangle^b\widehat{f}(\xi,\tau)\|_{L^2_{\xi\tau}} \end{aligned}$$

For $\varphi(\xi) = \pm|\xi|$ we use the notation $X_\pm^{m,b}$ and for $\varphi(\xi) = |\xi|^2$ simply $X^{m,b}$. For a given time interval I we define $\|f\|_{X^{m,b}(I)} = \inf_{\tilde{f}_{|I}=f}\|\tilde{f}\|_{X^{m,b}}$ and similarly $\|f\|_{X_\pm^{m,b}(I)}$.

For the modified (by I multiplied) Zakharov system we then prove a local existence theorem by using the precise estimates given by [12] for the standard Zakharov system in connection with an interpolation type lemma in [8]. Our aim is to extract a factor T^δ with maximal δ from the nonlinear estimates in order to give an optimal lower bound for the local existence time T in terms of the norms of the data. Because the difference of the differentiability classes of the data is maximal (=1), one is forced also to use here the auxiliary spaces Y_φ^m (cf. [12]), defined by

$$\begin{aligned} \|f\|_{Y_\varphi^m} &:= \|\langle\xi\rangle^m\langle\tau\rangle^{-1}\mathcal{F}(e^{-it\varphi(-i\partial_x)}f(x,t))\|_{L^2_\xi L^1_\tau} \\ &= \|\langle\xi\rangle^m\langle\tau+\varphi(\xi)\rangle^{-1}\widehat{f}(\xi,\tau)\|_{L^2_\xi L^1_\tau} \end{aligned}$$

As is typical for the I-method one then has to consider in detail the modified energy functional $E(Iu, In)$ and to control its growth in time in dependence of the time interval and the parameter N (cf. the definition of I above). The increment of the energy has to be small for small time intervals and large N. Because the modified energy functional is somehow close to the original one here some sort of cancellation helps. An important tool is also a refined Strichartz estimate for the product of a wave and a Schrödinger part along the lines of Bourgain's improvements for the simpler pure Schrödinger case (cf. Lemma 3.2). This estimate for the modified energy functional can also control the growth of the corresponding norms of the solution of the problem during its time evolution. One iterates the local existence theorem with time steps of equal length in order to reach any given fixed time T. To achieve this one has to make the process uniform which can be done if s is close enough to 1 (namely $s > 5/6$).

We collect some elementary facts about the spaces $X_\varphi^{m,b}$ and Y_φ^m. The following interpolation property is well-known:
$X_\varphi^{(1-\Theta)m_0+\Theta m_1,(1-\Theta)b_0+\Theta b_1} = (X_\varphi^{m_0,b_0}, X_\varphi^{m_1,b_1})_{[\Theta]}$ for $\Theta \in [0,1]$.

If u is a solution of $iu_t + \varphi(-i\partial_x)u = 0$ with $u(0) = f$ and ψ is a cutoff function in $C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\psi \subset (-2,2)$, $\psi \equiv 1$ on $[-1,1]$, $\psi(t) = \psi(-t)$, $\psi(t) \geq 0$, $\psi_\delta(t) := \psi(\frac{t}{\delta})$, $0 < \delta \leq 1$, we have for $b > 0$:

$$\|\psi_1 u\|_{X_\varphi^{m,b}} \leq c\|f\|_{H^m}$$

If v is a solution of the problem $iv_t + \varphi(-i\partial_x)v = F$, $v(0) = 0$, we have for $b'+1 \geq b \geq 0 \geq b' > -1/2$

$$\|\psi_\delta v\|_{X_\varphi^{m,b}} \leq c\delta^{1+b'-b}\|F\|_{X_\varphi^{m,b'}}$$

and, if $b'+1 \geq b \geq 0 \geq b'$, we have

$$\|\psi_\delta v\|_{X_\varphi^{m,b}} \leq c(\delta^{1+b'-b}\|F\|_{X_\varphi^{m,b'}} + \delta^{\frac{1}{2}-b}\|F\|_{Y_\varphi^m})$$

(for a proof cf. [12], Lemma 2.1). Moreover, if $w(t) = \int_0^t e^{i(t-s)\varphi(-i\partial_x)}F(s)\,ds$ we have by [12, Lemma 2.2], especially (2.35), for $\delta \leq 1$

$$\|w\|_{C^0([0,\delta],H_x^1)} \leq c\|F\|_{Y_\varphi^1[0,\delta]} \tag{1.4}$$

Finally, if $1/2 > b > b' \geq 0$, $m \in \mathbb{R}$, we have the embedding

$$\|f\|_{X_\varphi^{m,b'}[0,\delta]} \leq c\delta^{b-b'}\|f\|_{X_\varphi^{m,b}[0,\delta]} \tag{1.5}$$

For the convenience of the reader we repeat the proof of [13], Lemma 1.10. The claimed estimate is an immediate consequence of the following

Lemma 1.1. *For $1/2 > b > b' \geq 0$, $0 < \delta \leq 1$, $m \in \mathbb{R}$ the following estimate holds:*

$$\|\psi_\delta f\|_{X_\varphi^{m,b'}} \leq c\delta^{b-b'}\|f\|_{X_\varphi^{m,b}}$$

Proof. The following Sobolev multiplication rule holds:

$$\|fg\|_{H_t^{b'}} \leq c\|f\|_{H_t^{\frac{1}{2}-(b-b')}}\|g\|_{H_t^b}$$

This rule follows easily by the Leibniz rule for fractional derivatives, using $J^s := \mathcal{F}^{-1}\langle\tau\rangle^s\mathcal{F}$:

$$\begin{aligned}\|fg\|_{H_t^{b'}} &\le c(\|(J^{b'}f)g\|_{L_t^2} + \|f(J^{b'}g)\|_{L_t^2}) \\ &\le c(\|J^{b'}f\|_{L_t^p}\|g\|_{L_t^{p'}} + \|f\|_{L_t^{q'}}\|J^{b'}g\|_{L_t^q})\end{aligned}$$

with $\frac{1}{p} = b$, $\frac{1}{p'} = \frac{1}{2} - b$, $\frac{1}{q'} = b - b'$, $\frac{1}{q} = \frac{1}{2} - (b - b')$. Sobolev's embedding theorem gives the claimed result. Consequently we get

$$\|\psi_\delta g\|_{H_t^{b'}} \le c\|\psi_\delta\|_{H_t^{\frac{1}{2}-(b-b')}}\|g\|_{H_t^b} \le c\delta^{b-b'}\|g\|_{H_t^b}$$

and thus

$$\begin{aligned}\|\psi_\delta f\|_{X_\varphi^{m,b'}} &= \|e^{it\varphi(-i\partial_x)}\psi_\delta f\|_{H_x^m\otimes H_t^{b'}} \\ &\le c\delta^{b-b'}\|e^{it\varphi(-i\partial_x)}f\|_{H_x^m\otimes H_t^b} \\ &= c\delta^{b-b'}\|f\|_{X_\varphi^{m,b}}\end{aligned}$$

Fundamental are the following linear Strichartz type estimates for the Schrödinger equation (cf. e.g. [12, Lemma 2.4]):

$$\|e^{it\partial_x^2}\psi\|_{L_t^q(I,L_x^r(\mathbb{R}))} \le c\|\psi\|_{L_x^2(\mathbb{R})}$$

and

$$\|u\|_{L_t^q(I,L_x^r(\mathbb{R}))} \le c\|u\|_{X^{0,\frac{1}{2}+}(I)}$$

if $0 \le \frac{2}{q} = \frac{1}{2} - \frac{1}{r}$, especially

$$\|u\|_{L_{xt}^6} \le c\|u\|_{X^{0,\frac{1}{2}+}}$$

which by interpolation with the trivial case $\|u\|_{L_{xt}^2} = \|u\|_{X^{0,0}}$ gives:

$$\|u\|_{L_{xt}^p} \le c\|u\|_{X^{0,\frac{3}{2}(\frac{1}{2}-\frac{1}{p})+}}$$

if $2 < p \le 6$. For the wave equation we only use $\|n_\pm\|_{L_t^\infty L_x^2} \le c\|n_\pm\|_{X_\pm^{0,\frac{1}{2}+}}$. $\square$

We use the notation $\langle\lambda\rangle := (1+\lambda^2)^{1/2}$. Let $a\pm$ denote a number slightly larger (resp., smaller) than a.

2. Local existence

The system (1.1)–(1.3) has the following two quantities conserved: $M := \|u(t)\|$ and

$$E(u,n) := \|A^{1/2}u(t)\|^2 + 1/2(\|n(t)\|^2 + \|V(t)\|^2) + \int_{-\infty}^{+\infty} n(t)|u(t)|^2\,dx$$

where $V_x := -n_t$ and $A := -\frac{d^2}{dx^2}$.
The system (1.1)–(1.3) is now transformed into a first order system in t as follows: with $n_\pm := n \pm iA^{-1/2}n_t$, i.e. $n = \frac{1}{2}(n_+ + n_-)$, $2iA^{-1/2}n_t = n_+ - n_-$, and $\overline{n_+} = n_-$ this gives

$$iu_t + u_{xx} = \frac{1}{2}(n_+ + n_-)u \tag{2.1}$$

$$in_{\pm t} \mp A^{1/2}n_\pm = \pm A^{1/2}(|u|^2) \tag{2.2}$$

$$u(0) = u_0\,, \quad n_\pm(0) = n_{\pm 0} := n_0 \pm iA^{-1/2}n_1\,. \tag{2.3}$$

The energy is given by

$$E(u,n_+) = \|A^{1/2}u\|^2 + \frac{1}{2}\|n_+\|^2 + \frac{1}{2}\int (n_+ + \overline{n_+})|u|^2\,dx$$

By Gagliardo-Nirenberg,

$$\begin{aligned}\int n|u|^2\,dx &\le \frac{1}{4}\int n^2\,dx + c\int |u|^4\,dx\\ &\le \frac{1}{4}\|n\|^2 + c\|u_x\|\|u\|^3\\ &\le \frac{1}{4}(\|n\|^2 + \|u_x\|^2) + c_0\|u\|^6\end{aligned}$$

This implies

$$\|A^{1/2}u\|^2 + \|n\|^2 + \|V\|^2 \le c(E + \|u\|^6) = c_0(E + M^6) \tag{2.4}$$

and

$$E \le c_0(\|A^{1/2}u\|^2 + \|n\|^2 + \|V\|^2 + M^6) \tag{2.5}$$

We want to apply the I-method (for the definition of I see the introduction). A crucial role is played by the modified energy $E(Iu, In_+)$ for the system

$$iIu_t + Iu_{xx} = \frac{1}{2}I[(n_+ + n_-)u] \tag{2.6}$$

$$iIn_{\pm t} \mp A^{1/2}In_\pm = \pm IA^{1/2}(|u|^2) \tag{2.7}$$

$$Iu(0) = Iu_0, \quad In_\pm(0) = In_{\pm 0} = I(n_0 \pm iA^{-1/2}n_1), \tag{2.8}$$

namely

$$E(Iu, In_+) := \|Iu_x\|^2 + \frac{1}{2}\|In_+\|^2 + \frac{1}{2}\int I(n_+ + \overline{n_+})|Iu|^2\,dx$$

which is not conserved but its growth is controllable. An elementary but lengthy calculation shows

$$\begin{aligned}&\frac{d}{dt}E(Iu, In_+)\\ &= \mathrm{Re}\langle I(n_+ + \overline{n_+})Iu - I((n_+ + \overline{n_+})u), Iu_t\rangle + \mathrm{Re}\langle In_+, iA^{1/2}(|Iu|^2 - I(|u|^2))\rangle\end{aligned} \tag{2.9}$$

If $I = id$ this again shows the conservation of $E(u,n_+)$.

Before considering this modified energy in detail we give a local existence result for the system (2.6)–(2.8), which essentially uses the bilinear estimates given by [12] for their local existence result of the Zakharov system.

Proposition 2.1. *Assume $s \ge 1/2$. Let $(u_0, n_{+0}, n_{-0}) \in H^s \times H^{s-1} \times H^{s-1}$ be given. Then there exists a positive number $\delta \sim \frac{1}{(\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2} + \|In_{-0}\|_{L^2})^{4+}}$ such that the system (2.6)–(2.8) has a unique local solution in the time interval $[0,\delta]$ with the property (dropping from now on $[0,\delta]$ from the notation):*

$$\|Iu\|_{X^{1,\frac{1}{2}}} + \|In_+\|_{X_+^{0,\frac{1}{2}+}} + \|In_-\|_{X_-^{0,\frac{1}{2}+}} \le c(\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2} + \|In_{-0}\|_{L^2})$$

This solution also belongs to $C^0([0,\delta], H^1_x(\mathbb{R}))$ and

$$\|Iu\|_{C^0([0,\delta],H^1_x(\mathbb{R}))} \le c(\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2} + \|In_{-0}\|_{L^2})$$

Proof. We use the corresponding integral equations to define a mapping $S = (S_0, S_1)$ by

$$S_0(Iu(t)) = Ie^{it\partial_x^2}u_0 + \frac{1}{2}\int_0^t e^{i(t-s)\partial_x^2} I(u(s)((n_+(s)+n_-(s)))\,ds$$
$$S_1(In_\pm(t)) = Ie^{itA^{1/2}}n_{\pm 0} \pm i\int_0^t e^{\mp i(t-s)A^{1/2}}A^{1/2}I(|u(s)|^2)\,ds$$

We use [12, Lemma 4.3] to conclude for $s \geq 1/2$,

$$\|n_\pm u\|_{X^{s,-\frac{1}{2}}} \leq c\|n_\pm\|_{X_\pm^{s-1,\frac{3}{8}+}}\|u\|_{X^{s,\frac{3}{8}+}}\,.$$

Similarly [12, Lemma 4.5] shows

$$\|n_\pm u\|_{Y^s} \leq c\|n_\pm\|_{X_\pm^{s-1,\frac{3}{8}+}}\|u\|_{X^{s,\frac{3}{8}+}}\,.$$

Finally, [12, Lemma 4.4] shows that for $s \geq 0$,

$$\|A^{1/2}(|u|^2)\|_{X_\pm^{s-1,-\frac{1}{2}+}} \leq c\|u\|^2_{X^{s,\frac{1}{4}++}}$$

These estimates imply similar estimates including the I-operator by the interpolation lemma of [8], namely

$$\|I(n_\pm u)\|_{X^{1,-\frac{1}{2}}} + \|I(n_\pm u)\|_{Y^1} \leq c\|In_\pm\|_{X_\pm^{0,\frac{3}{8}+}}\|Iu\|_{X^{1,\frac{3}{8}+}}$$

and

$$\|IA^{1/2}(|u|^2)\|_{X_\pm^{0,-\frac{1}{2}+}} \leq c\|Iu\|^2_{X^{1,\frac{1}{4}++}}$$

where c is independent of N.

The same estimates also hold true for functions defined on $[0,\delta]$ only, and for such functions we can also use the embedding (5). This gives

$$\|I(n_\pm u)\|_{X^{1,-\frac{1}{2}}} + \|I(n_\pm u)\|_{Y^1} \leq c\|In_\pm\|_{X_\pm^{0,\frac{1}{2}+}}\|Iu\|_{X^{1,\frac{1}{2}}}\delta^{\frac{1}{4}-}$$

and

$$\|IA^{1/2}(|u|^2)\|_{X_\pm^{0,-\frac{1}{2}+}} \leq c\|Iu\|^2_{X^{1,\frac{1}{2}}}\delta^{\frac{1}{2}-}$$

Using these estimates the integral equations lead to (remark here that one needs the space Y^1, cf. [12, Lemma 2.1]):

$$\|S_0(Iu)\|_{X^{1,\frac{1}{2}}} \leq c\|Iu_0\|_{H^1} + c(\|In_+\|_{X_+^{0,\frac{1}{2}+}} + \|In_-\|_{X_-^{0,\frac{1}{2}+}})\|Iu\|_{X^{1,\frac{1}{2}}}\delta^{\frac{1}{4}-}\,,$$
$$\|S_1(In_\pm)\|_{X_\pm^{0,\frac{1}{2}+}} \leq c\|In_{\pm 0}\|_{L^2} + c\|Iu\|^2_{X^{1,\frac{1}{2}}}\delta^{\frac{1}{2}-}$$

The standard contraction argument gives the existence of a unique solution on $[0,\delta]$ with

$$\|Iu\|_{X^{1,\frac{1}{2}}} + \|In_+\|_{X_+^{0,\frac{1}{2}+}} + \|In_-\|_{X_-^{0,\frac{1}{2}+}} \leq 2c(\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2} + \|In_{-0}\|_{L^2})$$

provided

$$c\delta^{\frac{1}{4}-}(\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2} + \|In_{-0}\|_{L^2}) < 1$$

Concerning the property $Iu \in C^0([0,\delta],H^1_x)$ we refer to [12, Lemma 2.2], (use the first integral equation and $I(un_\pm) \in Y^1$). Moreover (1.4) gives

$$\begin{aligned}
\|Iu\|_{C^0([0,\delta],H^1)} &\le \|Iu_0\|_{H^1} + c(\|I(n_+u)\|_{Y^1} + \|I(n_-u)\|_{Y^1}) \\
&\le \|Iu_0\|_{H^1} + c\delta^{\frac14-}(\|In_+\|_{X_+^{0,\frac12+}} + \|In_-\|_{X_-^{0,\frac12+}})\|Iu\|_{X^{1,\frac12}} \\
&\le \|Iu_0\|_{H^1} + c\frac{(\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2} + \|In_{-0}\|_{L^2})^2}{\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2} + \|In_{-0}\|_{L^2}} \\
&\le c(\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2} + \|In_{-0}\|_{L^2})
\end{aligned}$$

□

3. A bilinear Strichartz estimate

Lemma 3.1.

$$\|(D_x^{1/2}u)n_\pm\|_{L^2_{xt}} \le c\|n_\pm\|_{X_\pm^{0,\frac12+}}\|u\|_{X^{0,\frac12+}}$$

Proof. We split the domain of integration into two parts (a) $\operatorname{supp}\widehat{u} \subset \{|\xi| \ge 2\}$ (b) $\operatorname{supp}\widehat{u} \subset \{|\xi| \le 2\}$

(a) We assume $\operatorname{supp}\widehat{m_\pm} \subset \{\xi \ge 0\}$ (the other part $\operatorname{supp}\widehat{m_\pm} \subset \{\xi \le 0\}$ can be treated similarly) and $\operatorname{supp}\widehat{v} \subset \{|\xi| \ge 2\}$. In this region we have

$$\begin{aligned}
&\|e^{it\partial_x^2}D_x^{1/2}ve^{\pm it|\partial_x|}m_\pm\|^2_{L^2_{xt}} \\
&= \int d\xi dt \Big|\int_{\xi=\xi_1+\xi_2,\xi_2\ge0,|\xi_1|\ge2} e^{-it\xi_1^2 \pm it|\xi_2|}\widehat{v}(\xi_1)\widehat{m_\pm}(\xi_2)|\xi_1|^{1/2}d\xi_1\Big|^2 \\
&= \int d\xi dt \int_{\xi=\xi_1+\xi_2=\eta_1+\eta_2,\eta_2,\xi_2\ge0,|\xi_1|,|\eta_1|\ge2} e^{-it(\xi_1^2\pm|\xi_2|-\eta_1^2\mp|\eta_2|)}\widehat{v}(\xi_1)\overline{\widehat{v}(\eta_1)} \qquad (3.1) \\
&\quad \times \widehat{m_\pm}(\xi_2)\overline{\widehat{m_\pm}(\eta_2)}|\xi_1|^{1/2}|\eta_1|^{1/2}d\xi_1 d\eta_1 \\
&= \int d\xi \int d\xi_1 d\eta_1 \delta(P(\eta_1))\widehat{v}(\xi_1)\overline{\widehat{v}(\eta_1)}\widehat{m_\pm}(\xi_2)\overline{\widehat{m_\pm}(\eta_2)}|\xi_1|^{1/2}|\eta_1|^{1/2}
\end{aligned}$$

with

$$\begin{aligned}
P(\eta_1) &:= \xi_1^2 \pm |\xi_2| - \eta_1^2 \mp |\xi - \eta_1| = \xi_1^2 \pm |\xi_2| - \eta_1^2 \mp |\eta_2| \\
&= \xi_1^2 \pm \xi_2 - \eta_1^2 \mp \eta_2 = \xi_1^2 \pm (\xi - \xi_1) - \eta_1^2 \mp (\xi - \eta_1) \\
&= \xi_1^2 - \eta_1^2 \mp (\xi_1 - \eta_1) = (\xi_1 - \eta_1)[(\xi_1 + \eta_1) \mp 1]
\end{aligned}$$

This function has the simple zeroes $\eta_1 = \xi_1$ and $\eta_1 = \pm1 - \xi_1$. Moreover $P'(\eta_1) = -2\eta_1 \pm 1$, thus $|P'(\eta_1)| \sim |\eta_1|$ in our region $|\eta_1| \ge 2$. Using the well-known identity

$$\int \delta(P(\eta_1))f(\eta_1)\,d\eta_1 = \sum \frac{f(x_k)}{|P'(x_k)|}$$

where x_k denotes the simple zeroes of P, we remark that in our case for the zeroes we have $|\eta_1| \sim |\xi_1|$, and therefore the factor $|\xi_1|^{1/2}|\eta_1|^{1/2}$ cancels with $|P'(x_k)|$.

Thus we can estimate (3.1) using Schwarz' inequality by

$$
\begin{aligned}
& c\int d\xi \int d\xi_1 |\widehat{v}(\xi_1)\overline{\widehat{v}(\xi_1)}\widehat{m_\pm}(\xi-\xi_1)\overline{\widehat{m_\pm}(\xi-\xi_1)}| \\
& + c\int d\xi \int d\xi_1 |\widehat{v}(\xi_1)\overline{\widehat{v}(\pm 1-\xi_1)}\widehat{m_\pm}(\xi-\xi_1)\overline{\widehat{m_\pm}(\xi-(\pm 1-\xi_1))}| \\
& \le c\|\widehat{v}\|_{L^2}^2 \|\widehat{m_\pm}\|_{L^2}^2 = c\|v\|_{L^2}^2\|m_\pm\|_{L^2}^2
\end{aligned}
$$

and the claimed estimate follows directly in the region $\operatorname{supp}\widehat{u} \subset \{|\xi| \ge 2\}$ (cf. e.g. [18, Lemma 1.4], [13, Lemma 2.1], [16, Section 3]).
(b) In the region $\operatorname{supp}\widehat{u} \subset \{|\xi| \le 2\}$ we have

$$
\begin{aligned}
\|(D_x^{1/2}u)n_\pm\|_{L^2_{xt}} &\le \|D_x^{1/2}u\|_{L^2_tL^\infty_x}\|n_\pm\|_{L^\infty_tL^2_x} \le c\|D_x^{1/2}u\|_{L^2_tH_x^{\frac12+}}\|n_\pm\|_{X_\pm^{0,\frac12+}} \\
&\le c\|u\|_{L^2_tL^2_x}\|n_\pm\|_{X_\pm^{0,\frac12+}} \le c\|u\|_{X^{0,\frac12+}}\|n_\pm\|_{X_\pm^{0,\frac12+}}
\end{aligned}
$$

□

Lemma 3.2.

$$
\|(D_x^{1/2}u)n_\pm\|_{L^2_{xt}} \le c\|n_\pm\|_{X_\pm^{0,\frac12+}}\|u\|_{X^{0+,\frac12}}\,.
$$

For the proof of the above lemma, we interpolate the estimate of the previous lemma with

$$
\begin{aligned}
\|(D_x^{1/2}u)n_\pm\|_{L^2_{xt}} &\le \|n_\pm\|_{L^\infty_tL^2_x}\|D_x^{1/2}u\|_{L^2_tL^\infty_x} \le c\|n_\pm\|_{X_\pm^{0,\frac12+}}\|u\|_{L^2_tH^{1+}_x} \\
&\le c\|n_\pm\|_{X_\pm^{0,\frac12+}}\|u\|_{X^{1+,0}}\,.
\end{aligned}
$$

Lemma 3.3.

$$
\|(D_x^{1/2}u)n_\pm\|_{L^{2+}_tL^2_x} \le c\|n_\pm\|_{X_\pm^{0,\frac12}}\|u\|_{X^{0+,\frac12}}\,.
$$

Proof. By Sobolev's embedding theorem and Strichartz' estimate we have

$$
\begin{aligned}
\|(D_x^{1/2}u)n_\pm\|_{L^{4-}_tL^2_x} &\le c\|n_\pm\|_{L^{\infty-}_tL^2_x}\|D_x^{1/2}u\|_{L^4_tL^\infty_x} \\
&\le c\|n_\pm\|_{X_\pm^{0,\frac12-}}\|D_x^{1/2}u\|_{L^4_tH_x^{\frac14+,4}} \\
&\le c\|n_\pm\|_{X_\pm^{0,\frac12-}}\|u\|_{X^{\frac34+,\frac38+}}
\end{aligned}
$$

Interpolation with Lemma 3.1 gives the claimed result. □

A variant of this lemma is given next.

Lemma 3.4.

$$
\|\widehat{(D_x^{1/2}u)} * \widehat{n_\pm}\|_{L^2_\xi L^{2-}_\tau} \le c\|n_\pm\|_{X_\pm^{0,\frac12+}}\|u\|_{X^{0+,\frac12}}
$$

Proof. On the one hand, Lemma 3.1 gives

$$
\|\widehat{(D_x^{1/2}u)} * \widehat{n_\pm}\|_{L^2_{\xi\tau}} \le c\|n_\pm\|_{X_\pm^{0,\frac12+}}\|u\|_{X^{0,\frac12+}}\,. \tag{3.2}
$$

On the other hand Young's inequality shows

$$
\|\widehat{(D_x^{1/2}u)} * \widehat{n_\pm}\|_{L^2_\xi L^{\frac43}_\tau} \le c\|\widehat{n_\pm}\|_{L^2_\xi L^1_\tau}\|\widehat{D_x^{1/2}u}\|_{L^1_\xi L^{\frac43}_\tau} \tag{3.3}
$$

Now by Schwarz' inequality

$$\begin{aligned}\|\widehat{n_\pm}\|_{L^2_\xi L^1_\tau} &= \|\int |\widehat{n_\pm}(\xi,\tau)|\langle\tau\pm|\xi|\rangle^{\frac12+}\langle\tau\pm|\xi|\rangle^{-\frac12-}d\tau\|_{L^2_\xi}\\ &\le c\|\widehat{n_\pm}(\xi,\tau)\langle\tau\pm|\xi|\rangle^{\frac12+}\|_{L^2_{\xi\tau}} = c\|n_\pm\|_{X^{0,\frac12+}}\end{aligned}$$

and by Hölder's inequality in τ and Schwarz' inequality in ξ:

$$\begin{aligned}\|\widehat{D_x^{1/2}u}\|_{L^1_\xi L^{\frac43}_\tau} &= \|\widehat{D_x^{1/2}u}(\xi,\tau)\langle\tau+\xi^2\rangle^{\frac14+}\langle\tau+\xi^2\rangle^{-\frac14-}\|_{L^1_\xi L^{\frac43}_\tau}\\ &\le c\|\widehat{D_x^{1/2}u}(\xi,\tau)\langle\tau+\xi^2\rangle^{\frac14+}\|_{L^1_\xi L^2_\tau}\\ &= c\|\widehat{D_x^{1/2}u}(\xi,\tau)\langle\tau+\xi^2\rangle^{\frac14+}\langle\xi\rangle^{\frac12+}\langle\xi\rangle^{-\frac12-}\|_{L^1_\xi L^2_\tau}\\ &\le c\|\widehat{D_x^{1/2}u}(\xi,\tau)\langle\tau+\xi^2\rangle^{\frac14+}\langle\xi\rangle^{\frac12+}\|_{L^2_{\xi\tau}}\\ &\le c\|u\|_{X^{1+,\frac14+}}\end{aligned}$$

Interpolating (3.3) and (3.2) we get the result. □

We also need a bilinear Strichartz' refinement for the pure Schrödinger problem. We have the well-known

Lemma 3.5. *If u_1,u_2 fulfill $|\xi_1| \gg |\xi_2| \ge 1$ for $\xi_i \in \operatorname{supp}\widehat{u_i}$ $(i=1,2)$, the following estimate holds:*

$$\|(D_x^{1/2}u_1)u_2\|_{L^2_{xt}} \le c\|u_1\|_{X^{0,\frac12+}}\|u_2\|_{X^{0,\frac12+}}\,.$$

The proof the lemma above can be found in [4, Lemma 7.1].

We also have the following variant.

Lemma 3.6. *Under the assumptions of Lemma 3.5 we have:*

$$\|(D_x^{1/2}u_1)u_2\|_{L^{2+}_t L^2_x} \le c\|u_1\|_{X^{0+,\frac12}}\|u_2\|_{X^{0,\frac12}}$$

The proof is similar to that of Lemma 3.3.

Remark: All the estimates in this section remain true, if any of the functions on the left-hand sides of the estimates are replaced by their complex conjugates.

4. Estimates for the modified energy

The main step towards global existence is an exact control of the increment of the modified energy.

Proposition 4.1. *Let $(u,n_\pm)$ be a solution of (2.1)–(2.3) on $[0,\delta]$ in the sense of Proposition 2.1. Then the following estimate holds (for $N\ge 1$, $s>3/4$):*

$$\begin{aligned}&|E(Iu(\delta),In_+(\delta)) - E(Iu(0),In_+(0))|\\ &\le c((N^{-\frac12+}\delta^{\frac12-} + N^{-\frac32+}\delta^{0+})\|In_+\|_{X^{0,\frac12+}_+}\|Iu\|^2_{X^{1,\frac12}}\\ &+ (N^{-3+} + N^{-1+}\delta^{\frac12-})\|In_+\|^2_{X^{0,\frac12+}_+}\|Iu\|^2_{X^{1,\frac12}})\,.\end{aligned}$$

Proof. Using (2.9) and replacing Iu_t by (2.6), we have to show

$$\Big|\int_0^\delta\int In_+A^{1/2}(|Iu|^2 - I(|u|^2))dxdt\Big| \le \frac{c}{N^{1-}}\delta^{\frac12-}\|In_+\|_{X^{0,\frac12+}_+}\|Iu\|^2_{X^{1,\frac12}} \tag{4.1}$$

and

$$\begin{aligned} &\Big|\int_0^\delta \int (Iu)_{xx}(I(n_+u) - In_+Iu)dxdt\Big| \\ &\le c(N^{-\frac{1}{2}+}\delta^{\frac{1}{2}-} + N^{-\frac{3}{2}+}\delta^{0+})\|In_+\|_{X_+^{0,\frac{1}{2}+}}\|Iu\|^2_{X^{1,\frac{1}{2}}} \end{aligned} \tag{4.2}$$

as well as

$$\begin{aligned} &\Big|\int_0^\delta \int I(n_+u)(I(n_+u) - In_+Iu)dxdt\Big| \\ &\le c(\frac{1}{N^{3-}} + N^{-1+}\delta^{\frac{1}{2}-})\|In_+\|^2_{X_+^{0,\frac{1}{2}+}}\|Iu\|^2_{X^{1,\frac{1}{2}}} \end{aligned} \tag{4.3}$$

Here and in the sequel we assume, without loss of generality, that the Fourier transforms of all these functions to be nonnegative, ignore the appearance of complex conjugates, use dyadic decompositions with respect to the frequencies $|\xi_j| \sim N_j = 2^k$ $(k = 0, 1, 2, \dots)$. To sum over the dyadic pieces at the end, we need to have extra factors N_j^{0-} everywhere.

We start with (4.1) which follows from

$$\begin{aligned} &\int_0^\delta \int_* \widehat{n_+}(\xi_1,t)|\xi_2+\xi_3|^{1/2}\Big|\frac{m(\xi_2+\xi_3)-m(\xi_2)m(\xi_3)}{m(\xi_2)m(\xi_3)}\Big|\widehat{u_2}(\xi_2,t)\widehat{u_3}(\xi_3,t)d\xi dt \\ &\le \frac{c}{N^{1-}}\delta^{\frac{1}{2}-}\|n_+\|_{X_+^{0,\frac{1}{2}+}}\|u_2\|_{X^{1,\frac{1}{2}}}\|u_3\|_{X^{1,\frac{1}{2}}} . \end{aligned} \tag{4.4}$$

Here and in the sequel, * denotes integration over the set $\sum_{i=1}^3 \xi_i = 0$ (or $\sum_{i=1}^4 \xi_i = 0$). The symmetry in ξ_2, ξ_3 allows to assume $N_2 \ge N_3$, and moreover we can assume $N_2 \ge N$, because otherwise the symbol is $\equiv 0$. The condition $\sum_{i=1}^3 \xi_i = 0$ implies $N_1 \le cN_2$. Thus $N_2 \sim N_{max}$, where $N_{max} := \max(N_1, N_2, N_3)$.

We have $|\frac{m(\xi_2+\xi_3)-m(\xi_2)m(\xi_3)}{m(\xi_2)m(\xi_3)}| \le \frac{c}{|m(\xi_2)||m(\xi_3)|} \le c\left\langle (\frac{N_2}{N})^{1/2}\right\rangle$ and thus the bound

$$\begin{aligned} &c\int_0^\delta \int_* \widehat{n_+}(\xi_1,t)|\xi_2|^{1/2}\widehat{u_2}(\xi_2,t)\widehat{u_3}(\xi_3,t)d\xi dt\langle(\frac{N_2}{N})^{1/2}\rangle \\ &\le c\|n_+D_x^{1/2}u_2\|_{L^2_{xt}}\|u_3\|_{L^2_{xt}}\langle(\frac{N_2}{N})^{\frac{1}{2}}\rangle \\ &\le c\|n_+\|_{X_+^{0,\frac{1}{2}+}}\|u_2\|_{X^{0+,\frac{1}{2}}}\|u_3\|_{X^{0,0}}\langle(\frac{N_2}{N})^{1/2}\rangle \\ &\le c\|n_+\|_{X_+^{0,\frac{1}{2}+}}\frac{1}{N_2^{1-}}\|u_2\|_{X^{1,\frac{1}{2}}}\frac{1}{N_3}\delta^{\frac{1}{2}-}\|u_3\|_{X^{1,\frac{1}{2}}}\langle(\frac{N_2}{N})^{1/2}\rangle \\ &\le \frac{c}{N^{1-}}N_{max}^{0-}\delta^{\frac{1}{2}-}\|n_+\|_{X_+^{0,\frac{1}{2}+}}\|u_2\|_{X^{1,\frac{1}{2}}}\|u_3\|_{X^{1,\frac{1}{2}}} \end{aligned} \tag{4.5}$$

by the bilinear Strichartz estimate. This implies (4.4). Next we prove (4.2) which is implied by

$$\begin{aligned} &\int_0^\delta \int_* \Big|\frac{m(\xi_2+\xi_3)-m(\xi_2)m(\xi_3)}{m(\xi_2)m(\xi_3)}\Big|\widehat{u_1}(\xi_1,t)\widehat{u_2}(\xi_2,t)\widehat{n_+}(\xi_3,t)\,d\xi dt \\ &\le c(N^{-\frac{1}{2}+}\delta^{\frac{1}{2}-} + N^{-\frac{3}{2}+}\delta^{0+})\|u_1\|_{X^{-1,\frac{1}{2}}}\|u_2\|_{X^{1,\frac{1}{2}}}\|n_+\|_{X_+^{0,\frac{1}{2}+}} \end{aligned} \tag{4.6}$$

Case 1: $N_2 \sim N_3 \geq cN$. Then $N_1 \leq cN_2$ as above. The multiplier is estimated by $\frac{c}{m(\xi_2)^2} \leq c(\frac{N_2}{N})^{\frac{1}{2}-\epsilon}$, so that we get the bound

$$\begin{aligned}
&c\|n_+ D_x^{1/2} u_2\|_{L^2_{xt}} \frac{1}{N_2^{1/2}} \|u_1\|_{L^2_{xt}} (\frac{N_2}{N})^{\frac{1}{2}-\epsilon} \\
&\leq c\|n_+\|_{X_+^{0,\frac{1}{2}+}} \|u_2\|_{X^{0+,\frac{1}{2}}} \frac{N_1}{N_2^{1/2}} \delta^{\frac{1}{2}-} \|u_1\|_{X^{-1,\frac{1}{2}}} (\frac{N_2}{N})^{\frac{1}{2}-\epsilon} \\
&\leq c\|n_+\|_{X_+^{0,\frac{1}{2}+}} \|u_2\|_{X^{1,\frac{1}{2}}} \frac{N_1}{N_2^{\frac{3}{2}-}} \delta^{\frac{1}{2}-} \|u_1\|_{X^{-1,\frac{1}{2}}} (\frac{N_2}{N})^{\frac{1}{2}-\epsilon}
\end{aligned}$$

which implies (4.6).
Case 2: $N_1 \sim N_2 \geq cN$, thus $N_3 \leq cN_1$. The symbol is majorized by $\frac{c}{m(\xi_2)m(\xi_3)} \leq c\langle(\frac{N_2}{N})^{\frac{1}{2}-\epsilon}\rangle$, which can be handled as in Case 1.
Case 3: $N_1 \sim N_3 \geq cN$, $N_2 << N_1 \sim N_3$.
Subcase a: $N_2 \leq N$. By the mean value theorem we have

$$\Big|\frac{m(\xi_2+\xi_3) - m(\xi_2)m(\xi_3)}{m(\xi_2)m(\xi_3)}\Big| = \Big|\frac{m(\xi_2+\xi_3) - m(\xi_3)}{m(\xi_3)}\Big| \leq c\Big|\frac{(\nabla m)(\xi_3)}{m(\xi_3)}\xi_2\Big| \leq c\frac{N_2}{N_3}$$

and we get the bound

$$\begin{aligned}
&c\|n_+ D_x^{1/2} u_1\|_{L^2_{xt}} N_1^{-\frac{1}{2}} \|u_2\|_{L^2_{xt}} \frac{N_2}{N_3} \\
&\leq c\|n_+\|_{X_+^{0,\frac{1}{2}+}} \|u_1\|_{X^{0+,\frac{1}{2}}} N_1^{-\frac{1}{2}} \|u_2\|_{X^{0,0}} \frac{N_2}{N_3} \\
&\leq c\|n_+\|_{X_+^{0,\frac{1}{2}+}} \|u_1\|_{X^{-1,\frac{1}{2}}} N_1^{1+} N_1^{-\frac{1}{2}} N_2^{-1} \delta^{\frac{1}{2}-} \|u_2\|_{X^{1,\frac{1}{2}}} \frac{N_2}{N_3} \\
&\leq cN_1^{-\frac{1}{2}+} \delta^{\frac{1}{2}-} \|n_+\|_{X_+^{0,\frac{1}{2}+}} \|u_1\|_{X^{-1,\frac{1}{2}}} \|u_2\|_{X^{1,\frac{1}{2}}}
\end{aligned}$$

which implies (4.6).
Subcase b: $|\xi_1| \sim |\xi_3| >> |\xi_2| \geq N$. This is the technically most complicated region where we want to use algebraic manipulations on the Fourier side with respect to τ and ξ and have also to take into account the characteristic function $\psi(t)$ of the time interval $[0,\delta]$. The problem is that $\widehat{\psi}(\tau) = \frac{1}{\sqrt{2\pi}} \frac{e^{i\tau\delta}-1}{i\tau} \notin L^1_\tau$, but fortunately $\in L^{1+}_\tau$. We perform no dyadic decompositions at all here.

We estimate the multiplier by $\frac{c}{|m(\xi_2)|} \leq c|\xi_2|^{1/2} N^{-\frac{1}{2}}$. Thus our aim is to give the following bound

$$\begin{aligned}
&\int_0^\delta \int_* \widehat{u_1}(\xi_1,t)|\xi_2|^{1/2}\widehat{u_2}(\xi_2,t)\widehat{n_+}(\xi_3,t)d\xi dt \\
&\leq cN^{-1+}\delta^{0+}\|u_1\|_{X^{-1,\frac{1}{2}}}\|u_2\|_{X^{1,\frac{1}{2}}}\|n_+\|_{X_+^{\frac{1}{2}+}}
\end{aligned} \tag{4.7}$$

which would imply (4.6). Abusing notation we denote the Fourier transform with respect to x and t also by $\widehat{\ }$. The left-hand side is bounded by

$$\int_{**} \widehat{u_1}(\xi_1,\tau_1)|\widehat{\psi}(\tau_0)||\xi_2|^{1/2}\widehat{u_2}(\xi_2,\tau_2)\widehat{n_+}(\xi_3,\tau_3)\, d\xi d\tau \tag{4.8}$$

Here ** denotes integration over $\sum_{i=1}^3 \xi_i = \sum_{i=0}^3 \tau_i = 0$. Remark again that without loss of generality $\widehat{u_1}, \widehat{u_2}, \widehat{n_+} \geq 0$. The crucial algebraic inequality in our region is the following:

$$|\xi_1| \le c\Big(\langle \tau_1 + |\xi_1|^2\rangle^{1/2} + \langle \tau_2 + |\xi_2|^2\rangle^{1/2} + \langle \tau_3 + |\xi_3|\rangle^{1/2} + |\tau_0|^{1/2}\Big)$$

We consider 4 cases according to which of the terms on the right-hand side is dominant.

Region 1: $\langle \tau_1 + |\xi_1|^2\rangle^{1/2}$ dominant. We get the following bound for (4.8):

$$\begin{aligned}
&c\int_{**} \langle \tau_1 + |\xi_1|^2\rangle^{1/2} |\xi_1|^{-1}\widehat{u_1}(\xi_1,\tau_1)|\widehat{\psi}(\tau_0)||\xi_2|^{1/2}\widehat{u_2}(\xi_2,\tau_2)\widehat{n_+}(\xi_3,\tau_3)\,d\xi d\tau\\
&\le c\|u_1\|_{X^{-1,\frac12}}\|\mathcal{F}^{-1}(|\widehat{\psi}|)(D_x^{1/2}u_2)n_+\|_{L^2_{xt}}\\
&\le c\|u_1\|_{X^{-1,\frac12}}\|\mathcal{F}^{-1}(|\widehat{\psi}|)\|_{L_t^{\infty-}}\|(D_x^{1/2}u_2)n_+\|_{L_t^{2+}L_x^2}\\
&\le c\delta^{0+}\|u_1\|_{X^{-1,\frac12}}\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{X^{0+,\frac12}}\\
&\le c\delta^{0+}N^{-1+}\|u_1\|_{X^{-1,\frac12}}\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}
\end{aligned}$$

by Lemma 3.3 and by Hausdorff-Young, which gives

$$\|\mathcal{F}^{-1}(|\widehat{\psi}|)\|_{L_t^{\infty-}} \le c\|\widehat{\psi}\|_{L_\tau^{1+}} \le c\delta^{0+}$$

as one easily calculates.

Region 2: $\langle \tau_2 + |\xi_2|^2\rangle^{1/2}$ dominant. As before, we estimate (4.8) by

$$\begin{aligned}
&c\int_{**} |\xi_1|^{-1}\widehat{u_1}(\xi_1,\tau_1)|\widehat{\psi}(\tau_0)|\langle \tau_2 + |\xi_2|^2\rangle^{1/2}|\xi_2|^{1/2}\widehat{u_2}(\xi_2,\tau_2)\widehat{n_+}(\xi_3,\tau_3)\,d\xi d\tau\\
&\le c\|u_2\|_{X^{\frac12,\frac12}}\|\mathcal{F}^{-1}(|\widehat{\psi}|)(D_x^{-1}u_1)n_+\|_{L^2_{xt}}\\
&\le c\|u_2\|_{X^{\frac12,\frac12}}\|\mathcal{F}^{-1}(|\widehat{\psi}|)\|_{L_t^{\infty-}}\|(D_x^{-1}u_1)n_+\|_{L_t^{2+}L_x^2}\\
&\le c\delta^{0+}\|u_2\|_{X^{\frac12,\frac12}}\|n_+\|_{X_+^{0,\frac12+}}\|u_1\|_{X^{-\frac32+,\frac12}}\\
&\le c\delta^{0+}N^{-\frac12}\|u_2\|_{X^{1,\frac12}}\|n_+\|_{X_+^{0,\frac12+}}\|u_1\|_{X^{-1,\frac12}}N^{-\frac12+}\,,
\end{aligned}$$

where we have used Lemma 3.3 again.

Region 3: $\langle \tau_3 + |\xi_3|^2\rangle^{1/2}$ dominant. Using Lemma 3.6, we control (4.8) by:

$$\begin{aligned}
&c\int_{**} |\xi_1|^{-1}\widehat{u_1}(\xi_1,\tau_1)|\widehat{\psi}(\tau_0)||\xi_2|^{1/2}\widehat{u_2}(\xi_2,\tau_2)\langle \tau_3 + |\xi_3|^2\rangle^{\frac12+}\widehat{n_+}(\xi_3,\tau_3)\,d\xi d\tau\\
&\le c\|n_+\|_{X_+^{0,\frac12+}}\|\mathcal{F}^{-1}(|\widehat{\psi}|)(D_x^{-1}u_1)(D_x^{1/2}u_2)\|_{L^2_{xt}}\\
&\le c\|n_+\|_{X_+^{0,\frac12+}}\|\mathcal{F}^{-1}(|\widehat{\psi}|)\|_{L_t^{\infty-}}\|(D_x^{-1}u_1)(D_x^{1/2}u_2)\|_{L_t^{2+}L_x^2}\\
&\le c\delta^{0+}\|n_+\|_{X_+^{0,\frac12+}}\|u_1\|_{X^{-\frac32+,\frac12}}\|u_2\|_{X^{\frac12,\frac12}}\\
&\le c\delta^{0+}\|n_+\|_{X_+^{0,\frac12+}}N^{-\frac12+}\|u_1\|_{X^{-1,\frac12}}\|u_2\|_{X^{1,\frac12}}N^{-\frac12}\,.
\end{aligned}$$

Region 4: $|\tau_0|^{1/2}$ dominant. The upper bound for (4.8) here is

$$\begin{aligned}
&c\int_{**} |\xi_1|^{-1}\widehat{u_1}(\xi_1,\tau_1)|\tau_0|^{1/2}|\widehat{\psi}(\tau_0)||\xi_2|^{1/2}\widehat{u_2}(\xi_2,\tau_2)\widehat{n_+}(\xi_3,\tau_3)\,d\xi d\tau\\
&\le c\|\widehat{D_x^{-1}u_1}\|_{L^2_{\xi_1}L^{1+}_{\tau_1}}\||\tau|^{1/2}|\widehat{\psi}| * \widehat{D_x^{1/2}u_2} * \widehat{n_+}\|_{L^2_\xi L^{\infty-}_\tau}
\end{aligned}$$

by Hölder. The first factor is estimated as follows by Hölder with respect to τ_1:

$$\begin{aligned}\|\widehat{D_x^{-1}u_1}\|_{L^2_{\xi_1}L^{1+}_{\tau_1}} &= \|\widehat{D_x^{-1}u_1}\,\langle\tau_1+\xi_1^2\rangle^{1/2}\langle\tau_1+\xi_1^2\rangle^{-\frac12}\|_{L^2_{\xi_1}L^{1+}_{\tau_1}}\\ &\le \|\widehat{D_x^{-1}u_1}\,\langle\tau_1+\xi_1^2\rangle^{1/2}\|_{L^2_{\xi_1\tau_1}} \le c\|u_1\|_{X^{-1,\frac12}}\,.\end{aligned}$$

The second factor is bounded by Young's inequality by

$$\begin{aligned}c\||\tau|^{1/2}|\widehat{\psi}|\|_{L^{2+}_\tau}\|\widehat{D_x^{1/2}u_2}*\widehat{n_+}\|_{L^2_\xi L^{2--}_\tau} &\le \delta^{0+}\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{X^{0+,\frac12}}\\ &\le c\delta^{0+}N^{-1+}\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}\,.\end{aligned}$$

Here we used Lemma 3.4 and the bound $\||\tau|^{1/2}|\widehat{\psi}|\|_{L^{2+}_\tau}\le c\delta^{0+}$, which is easily checked. Thus we get (4.7) in all regions.

Finally we prove (4.3). It is implied by

$$\begin{aligned}&\int_0^\delta\int_*\Big|\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)}\cdot\frac{m(\xi_3+\xi_4)-m(\xi_3)m(\xi_4)}{m(\xi_3)m(\xi_4)}\Big|\\ &\times\widehat{n_+}(\xi_1,t)\widehat{u_2}(\xi_2,t)\widehat{n_+}(\xi_3,t)\widehat{u_4}(\xi_4,t)d\xi dt\\ &\le c\Big(\frac{1}{N^{3-}}+N^{-1+}\delta^{\frac12-}\Big)\|n_+\|^2_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}\|u_4\|_{X^{1,\frac12}}\,.\end{aligned}\tag{4.9}$$

Case 1: $N_1\sim N_3\ge cN, N_1\sim N_3 >> N_2,N_4$
Subcase a: $N_4\le N$

$$\Big|\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)}\Big|\le\frac{c}{m(\xi_2)}\le c\langle(\frac{N_2}{N})^{\frac12-\epsilon}\rangle$$

$$\Big|\frac{m(\xi_3+\xi_4)-m(\xi_3)m(\xi_4)}{m(\xi_3)m(\xi_4)}\Big|=\Big|\frac{m(\xi_3+\xi_4)-m(\xi_3)}{m(\xi_3)}\Big|\le\Big|\frac{(\nabla m)(\xi_3)}{m(\xi_3)}\xi_4\Big|\le\frac{cN_4}{N_3}$$

by the mean value theorem. Thus we get the bound

$$\begin{aligned}&c\frac{N_4}{N_3}\|n_+\|_{L^\infty_tL^2_x}\|u_2\|_{L^2_tL^\infty_x}\|n_+D_x^{1/2}u_4\|_{L^2_{xt}}\frac{1}{N_4^{1/2}}\langle(\frac{N_2}{N})^{\frac12-\epsilon}\rangle\\ &\le c\frac{N_4}{N_3}\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{L^2_tH_x^{\frac12+}}\|n_+\|_{X_+^{0,\frac12+}}\|u_4\|_{X^{0+,\frac12}}\frac{1}{N_4^{1/2}}\langle(\frac{N_2}{N})^{\frac12-\epsilon}\rangle\\ &\le c\frac{N_4}{N_3}\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}\frac{1}{N_2^{\frac12-}}\delta^{\frac12-}\|n_+\|_{X_+^{0,\frac12+}}\|u_4\|_{X^{1,\frac12}}\frac{1}{N_4^{\frac32-}}\langle(\frac{N_2}{N})^{\frac12-\epsilon}\rangle\end{aligned}$$

which implies (4.9).
Subcase b: $|\xi_1|\sim|\xi_3|\ge cN, |\xi_1|\sim|\xi_3|>>|\xi_2|,|\xi_4|\,,\ |\xi_2|,|\xi_4|\ge N$. In this case we avoid any dyadic decomposition and estimate as follows:

$$\Big|\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)}\Big|\le\frac{c}{|m(\xi_2)|}\le c(\frac{|\xi_2|}{N})^{\frac12-},$$

$$\Big|\frac{m(\xi_3+\xi_4)-m(\xi_3)m(\xi_4)}{m(\xi_3)m(\xi_4)}\Big|\le\frac{c}{|m(\xi_4)|}\le c(\frac{|\xi_4|}{N})^{\frac12-}$$

Thus we get the bound

$$\begin{aligned}
&\int_0^\delta \int_* \frac{|\xi_2|^{\frac12-}}{N^{\frac12-}} \frac{|\xi_4|^{\frac12-}}{N^{\frac12-}} \widehat{n_+}(\xi_1,t)(|\xi_2|^{\frac12-}\widehat{u_2}(\xi_2,t)) \\
&\times \frac{1}{|\xi_2|^{\frac12-}} \widehat{n_+}(\xi_3,t)(|\xi_4|^{\frac12-}\widehat{u_4}(\xi_4,t)) \frac{1}{|\xi_4|^{\frac12-}}\, d\xi dt \\
&\le \frac{c}{N^{1-}} \|n_+\|_{L_t^\infty L_x^2} \|D_x^{\frac12-} u_2\|_{L_t^2 L_x^\infty} \|n_+\|_{L_t^\infty L_x^2} \|D_x^{\frac12-} u_4\|_{L_t^2 L_x^\infty} \\
&\le \frac{c}{N^{1-}} \|n_+\|_{X_+^{0,\frac12+}} \|u_2\|_{X^{1,\frac12}} \delta^{\frac12-} \|n_+\|_{X_+^{0,\frac12+}} \|u_4\|_{X^{1,\frac12}} \delta^{\frac12-}
\end{aligned}$$

which is sufficient, because no dyadic decomposition was performed.
Subcase c: $|\xi_1| \sim |\xi_3| \ge cN, |\xi_1| \sim |\xi_3| \gg |\xi_2|, |\xi_4|\,, |\xi_4| \ge N \ge |\xi_2|$. We again perform no dyadic decomposition and estimate as follows:

$$\begin{gathered}
\Big|\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)}\Big| \le c, \\
\Big|\frac{m(\xi_3+\xi_4)-m(\xi_3)m(\xi_4)}{m(\xi_3)m(\xi_4)}\Big| \le \frac{c}{|m(\xi_4)|} \le c\Big(\frac{|\xi_4|}{N}\Big)^{1/2}
\end{gathered}$$

Thus we get the bound

$$\begin{aligned}
&\int_0^\delta \int_* \frac{|\xi_4|^{1/2}}{N^{1/2}} \widehat{n_+}(\xi_1,t)\widehat{u_2}(\xi_2,t))\widehat{n_+}(\xi_3,t)(|\xi_4|^{1/2}\widehat{u_4}(\xi_4,t))\frac{1}{|\xi_4|^{1/2}}\, d\xi dt \\
&\le \frac{c}{N^{1/2}} \|n_+\|_{L_t^\infty L_x^2} \|u_2\|_{L_t^2 L_x^\infty} \|n_+ D_x^{1/2} u_4\|_{L^2_{xt}} \\
&\le \frac{c}{N^{1/2}} \|n_+\|_{X_+^{0,\frac12+}} \|u_2\|_{L_t^2 H_x^{\frac12+}} \|n_+\|_{X_+^{0,\frac12+}} \|u_4\|_{X^{0+,\frac12}} \\
&\le \frac{c}{N^{1/2}} \|n_+\|_{X_+^{0,\frac12+}} \delta^{\frac12-} \|u_2\|_{X^{1,\frac12}} \|n_+\|_{X_+^{0,\frac12+}} \frac{1}{N^{1-}} \|u_4\|_{X^{1,\frac12}} \\
&\le \frac{c}{N^{\frac32-}} \delta^{\frac12-} \|n_+\|^2_{X_+^{0,\frac12+}} \|u_2\|_{X^{1,\frac12}} \|u_4\|_{X^{1,\frac12}}
\end{aligned}$$

Case 2: $N_2 \sim N_4 \ge cN, N_2 \sim N_4 \gg N_1, N_3$. We have

$$\begin{gathered}
\Big|\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)}\Big| \le \frac{c}{|m(\xi_1)|} \le c\langle\Big(\frac{N_1}{N}\Big)^{1/2}\rangle \le c\Big(\frac{N_4}{N}\Big)^{1/2}, \\
\Big|\frac{m(\xi_3+\xi_4)-m(\xi_3)m(\xi_4)}{m(\xi_3)m(\xi_4)}\Big| \le \frac{c}{|m(\xi_3)|} \le c\langle\Big(\frac{N_3}{N}\Big)^{1/2}\rangle \le c\Big(\frac{N_2}{N}\Big)^{1/2}.
\end{gathered}$$

This gives the bound

$$\begin{aligned}
&c\|n_+ D_x^{1/2} u_2\|_{L^2_{xt}} N_2^{-1/2} \|n_+ D_x^{1/2} u_4\|_{L^2_{xt}} N_4^{-1/2} \Big(\frac{N_4}{N}\Big)^{1/2} \Big(\frac{N_2}{N}\Big)^{1/2} \\
&\le \frac{c}{N} \|n_+\|_{X_+^{0,\frac12+}} \|u_2\|_{X^{0+,\frac12}} \|n_+\|_{X_+^{0,\frac12+}} \|u_4\|_{X^{0+,\frac12}} \\
&\le \frac{c}{N} N_2^{-1+} N_4^{-1+} \|n_+\|^2_{X_+^{0,\frac12+}} \|u_2\|_{X^{1,\frac12}} \|u_4\|_{X^{1,\frac12}}
\end{aligned}$$

which implies (4.9).

Case 3: $N_1 \sim N_2 \geq cN$, $N_1 \sim N_2 \gg N_3, N_4$. Using

$$|\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)}| \leq \frac{c}{|m(\xi_1)|^2} \leq c(\frac{N_1}{N})^{1/2},$$

$$|\frac{m(\xi_3+\xi_4)-m(\xi_3)m(\xi_4)}{m(\xi_3)m(\xi_4)}| \leq \frac{c}{|m(\xi_3)m(\xi_4)|} \leq c\langle(\frac{N_3}{N})^{1/2}\rangle\langle(\frac{N_4}{N})^{1/2}\rangle,$$

we get the bound

$$\begin{aligned}
&c\|n_+D_x^{1/2}u_2\|_{L^2_{xt}}N_2^{-1/2}\|n_+\|_{L^\infty_t L^2_x}\|u_4\|_{L^2_t L^\infty_x}\langle(\frac{N_3}{N})^{1/2}\rangle\langle(\frac{N_4}{N})^{1/2}\rangle(\frac{N_1}{N})^{1/2}\\
&\leq c\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{X^{0+,\frac12}}N_2^{-1/2}\|n_+\|_{X_+^{0,\frac12+}}\|u_4\|_{L^2_tH_x^{\frac12+}}\langle(\frac{N_3}{N})^{1/2}\rangle\langle(\frac{N_4}{N})^{1/2}\rangle(\frac{N_1}{N})^{1/2}\\
&\leq c\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}N_2^{-\frac32+}\|n_+\|_{X_+^{0,\frac12+}}\|u_4\|_{X^{1,\frac12}}\delta^{\frac12-}N_4^{-\frac12+}\\
&\quad\times\langle(\frac{N_2}{N})^{1/2}\rangle\langle(\frac{N_4}{N})^{1/2}\rangle(\frac{N_1}{N})^{1/2}\\
&\leq cN^{-\frac32+}(N_1N_2N_3N_4)^{0-}\delta^{\frac12-}\|n_+\|^2_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}\|u_4\|_{X^{1,\frac12}}
\end{aligned}$$

Case 4: $N_2 \sim N_3 \geq cN, N_2 \sim N_3 \gg N_1, N_4$. Using

$$|\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)}| \leq \frac{c}{|m(\xi_1)|} \leq c\langle(\frac{N_1}{N})^{1/2}\rangle,$$

$$|\frac{m(\xi_3+\xi_4)-m(\xi_3)m(\xi_4)}{m(\xi_3)m(\xi_4)}| \leq \frac{c}{|m(\xi_4)|} \leq c\langle(\frac{N_4}{N})^{1/2}\rangle$$

we get the bound

$$\begin{aligned}
&c\|n_+D_x^{1/2}u_2\|_{L^2_{xt}}N_2^{-1/2}\|n_+\|_{L^\infty_t L^2_x}\|u_4\|_{L^2_t L^\infty_x}\langle(\frac{N_1}{N})^{1/2}\rangle\langle(\frac{N_4}{N})^{1/2}\rangle\\
&\leq c\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}N_2^{-\frac32+}\|n_+\|_{X_+^{0,\frac12+}}\|u_4\|_{X^{1,\frac12}}\delta^{\frac12-}N_4^{-\frac12+}\\
&\quad\times\langle(\frac{N_2}{N})^{1/2}\rangle\langle(\frac{N_4}{N})^{1/2}\rangle\\
&\leq cN^{-\frac32+}(N_1N_2N_3N_4)^{0-}\delta^{\frac12-}\|n_+\|^2_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}\|u_4\|_{X^{1,\frac12}}
\end{aligned}$$

Case 5: $N_3 \sim N_4 \geq cN, N_3 \sim N_4 >> N_1, N_2$. Using

$$|\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)}| \leq \frac{c}{|m(\xi_1)m(\xi_2)|} \leq c\langle(\frac{N_1}{N})^{1/2}\rangle\langle(\frac{N_2}{N})^{1/2}\rangle,$$

$$|\frac{m(\xi_3+\xi_4)-m(\xi_3)m(\xi_4)}{m(\xi_3)m(\xi_4)}| \leq \frac{c}{|m(\xi_4)|^2} \leq c(\frac{N_4}{N})^{1/2},$$

we get the bound

$$\begin{aligned}
&c\|n_+\|_{L^\infty_t L^2_x}\|u_2\|_{L^2_t L^\infty_x}\|n_+D_x^{1/2}u_4\|_{L^2_{xt}}N_4^{-1/2}\langle(\frac{N_1}{N})^{1/2}\rangle\langle(\frac{N_2}{N})^{1/2}\rangle(\frac{N_4}{N})^{1/2}\\
&\leq c\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}N_2^{-\frac12+}\delta^{\frac12-}\|n_+\|_{X_+^{0,\frac12+}}\|u_4\|_{X^{1,\frac12}}N_4^{-\frac32+}\\
&\quad\times\langle(\frac{N_1}{N})^{1/2}\rangle\langle(\frac{N_2}{N})^{1/2}\rangle(\frac{N_4}{N})^{1/2}\\
&\leq cN^{-\frac32+}(N_1N_2N_3N_4)^{0-}\delta^{\frac12-}\|n_+\|^2_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}\|u_4\|_{X^{1,\frac12}}
\end{aligned}$$

Case 6: $N_1 \sim N_4 \geq cN, N_1 \sim N_4 >> N_2, N_3$.

$$|\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)}| \leq \frac{c}{|m(\xi_2)|} \leq c\langle(\frac{N_2}{N})^{1/2}\rangle\,,$$
$$|\frac{m(\xi_3+\xi_4)-m(\xi_3)m(\xi_4)}{m(\xi_3)m(\xi_4)}| \leq \frac{c}{|m(\xi_3)|} \leq c\langle(\frac{N_3}{N})^{1/2}\rangle$$

we get the bound

$$\begin{aligned}
&c\|n_+\|_{L_t^\infty L_x^2}\|u_2\|_{L_t^2 L_x^\infty}\|n_+ D_x^{1/2}u_4\|_{L^2_{xt}} N_4^{-1/2}\langle(\frac{N_2}{N})^{1/2}\rangle\langle(\frac{N_3}{N})^{1/2}\rangle \\
&\leq c\|n_+\|_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}} N_2^{-\frac12+}\delta^{\frac12-}\|n_+\|_{X_+^{0,\frac12+}}\|u_4\|_{X^{1,\frac12}} N_4^{-\frac32+}\langle(\frac{N_2}{N})^{1/2}\rangle\langle(\frac{N_3}{N})^{1/2}\rangle \\
&\leq cN^{-\frac32+}(N_1N_2N_3N_4)^{0-}\delta^{\frac12-}\|n_+\|^2_{X_+^{0,\frac12+}}\|u_2\|_{X^{1,\frac12}}\|u_4\|_{X^{1,\frac12}}
\end{aligned}$$

The remaining cases where at least three factors have equivalent frequencies $\geq cN$ are similar or easier to handle so that (4.9) is proved in all possible situations. The proof of the proposition is complete. $\square$

5. The global existence result

Theorem 5.1. *Let $1 > s > 5/6$. The Zakharov system* (1.1)–(1.3) *has a unique global solution for data $u_0 \in H^s(\mathbb{R})$, $n_0 \in H^{s-1}(\mathbb{R})$, $A^{-1/2}n_1 \in H^{s-1}(\mathbb{R})$. More precisely, for any $T > 0$ there exists a unique solution*

$$(u, n, A^{-1/2}n_t) \in X^{s,\frac12}[0,T] \times \widetilde{X}^{s-1,\frac12+}[0,T] \times \widetilde{X}^{s-1,\frac12+}[0,T]$$

where $\widetilde{X}^{s-1,\frac12+}[0,T] := X_+^{s-1,\frac12+}[0,T] + X_-^{s-1,\frac12+}[0,T]$. This solution satisfies

$$(u, n, A^{-1/2}n_t) \in C^0([0,T], H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R}))\,,$$
$$\|u(t)\|_{H^s} + \|n(t)\|_{H^{s-1}} + \|A^{-1/2}n_t(t)\|_{H^{s-1}} \leq c(1+t)^{\frac{2(1-s)}{6s-5}+}$$

Proof. The data satisfy the estimates

$$\|Iu_0\|_{H^1} \leq cN^{1-s}\|u_0\|_{H^s}$$
$$\|In_{\pm0}\|_{L^2} \leq cN^{1-s}(\|n_0\|_{H^{s-1}} + \|A^{-1/2}n_1\|_{H^{s-1}})\,.$$

We use our local existence theorem on $[0,\delta]$, where $\delta \sim \frac{1}{N^{4(1-s)+}}$ and conclude

$$\begin{aligned}
&\|Iu\|_{X^{1,\frac12}[0,\delta]} + \|In_+\|_{X_+^{0,\frac12+}[0,\delta]} + \|In_-\|_{X_-^{0,\frac12+}[0,\delta]} \\
&\leq c(\|Iu_0\|_{H^1} + \|In_+\|_{L^2} + \|In_-\|_{L^2}) \leq c_2N^{1-s}
\end{aligned} \tag{5.1}$$

From (2.5) we get

$$E(Iu_0, In_{+0}) \leq c_0(\|Iu_0\|^2_{H^1} + \|In_{+0}\|^2_{L^2} + \|Iu_0\|^6_{L^2}) \leq \overline{c}N^{2(1-s)}$$

and from (2.4)

$$\|A^{1/2}Iu_0\|^2_{L^2} + \|In_+\|^2_{L^2} + \|In_-\|^2_{L^2} \leq \widehat{c}N^{2(1-s)}\,,\ \|Iu_0\|_{L^2} \leq M$$

with $\widehat{c} = \widehat{c}(\overline{c})$. Thus the constant in (5.1) depends only on $\overline{c}$ and M, i.e. $c_2 = c_2(\overline{c}, M)$.

To reapply the local existence result with time intervals of equal length we need a uniform bound of the solution at time $t = \delta$ and $t = 2\delta$ etc. which follows from a uniform control over the energy by (2.4). The increment of the energy is controlled by Proposition 4.1 and (5.1) as follows:

$$
\begin{aligned}
&|E(Iu(\delta), In_+(\delta)) - E(Iu_0, In_{+0})| \\
&\le c[(N^{-\frac{1}{2}+}\delta^{\frac{1}{2}-} + N^{-\frac{3}{2}+})\|In_+\|_{X_+^{0,\frac{1}{2}+}[0,\delta]}\|Iu\|^2_{X^{1,\frac{1}{2}}[0,\delta]} \\
&\quad + (N^{-3+} + N^{-1+}\delta^{\frac{1}{2}-})\|In_+\|^2_{X_+^{0,\frac{1}{2}+}[0,\delta]}\|Iu\|^2_{X^{1,\frac{1}{2}}[0,\delta]}] \\
&\le c((N^{-\frac{1}{2}+}\delta^{\frac{1}{2}-} + N^{-\frac{3}{2}+})N^{3(1-s)} + (N^{-3+} + N^{-1+}\delta^{\frac{1}{2}-})N^{4(1-s)})
\end{aligned}
$$

Using the definition of δ we arrive at

$$
\begin{aligned}
&|E(Iu(\delta), In_+(\delta)) - E(Iu_0, In_{+0})| \\
&\le c_3((N^{-\frac{1}{2}+}N^{-2(1-s)+} + N^{-\frac{3}{2}+})N^{3(1-s)} + (N^{-3+} + N^{-1+}N^{-2(1-s)+})N^{4(1-s)}) \\
&\le c_3(N^{-\frac{1}{2}+}N^{-2(1-s)+}N^{3(1-s)} + N^{-1+}N^{-2(1-s)+}N^{4(1-s)})
\end{aligned}
$$

where $c_3 = c_3(\overline{c}, M)$. This is easily seen to be bounded by $\overline{c}N^{2(1-s)}$ (for large N).

The number of iteration steps to reach the given time T is $\frac{T}{\delta} \sim TN^{4(1-s)+}$. This means that in order to give a uniform bound of the energy of the iterated solutions, namely by $2\overline{c}N^{2(1-s)}$, from the last inequality the following condition has to be fulfilled:

$$
c_3(N^{-\frac{1}{2}+}N^{-2(1-s)}N^{3(1-s)} + N^{-1+}N^{-2(1-s)+}N^{4(1-s)})TN^{4(1-s)+} < \overline{c}N^{2(1-s)}
$$

where $c_3 = c_3(2\overline{c}, 2M)$ (recall here that the initial energy is bounded by $\overline{c}N^{2(1-s)}$). This can be fulfilled for N sufficiently large provided the following conditions hold:

$$
\begin{gathered}
-\frac{1}{2} - 2(1-s) + 3(1-s) + 4(1-s) < 2(1-s) \iff s > 5/6 \\
-1 - 2(1-s) + 4(1-s) + 4(1-s) < 2(1-s) \iff s > 3/4
\end{gathered}
$$

So here is the point where the decisive bound on s appears. A uniform bound of the energy implies by (2.4) uniform control of

$$
\|A^{1/2}Iu(t)\| + \|In(t)\| + \|A^{-1/2}In_t(t)\| \le cN^{1-s}
$$

Moreover $\|Iu(t)\| \le \|u(t)\| = \|u_0\|$, thus

$$
\|u(t)\|_{H^s} + \|n(t)\|_{H^{s-1}} + \|A^{-1/2}n_t(t)\|_{H^{s-1}} \le cN^{1-s}
$$

Now, one can directly give a bound on the growth of the solution as follows. The most restrictive condition on N comes from the inequality

$$
c_3TN^{-\frac{1}{2}+}N^{-2(1-s)}N^{3(1-s)}N^{4(1-s)+} < \overline{c}N^{2(1-s)} \iff N > cT^{\frac{2}{6s-5}+}
$$

This implies

$$
\sup_{0\le t\le T}(\|u(t)\|_{H^s} + \|n(t)\|_{H^{s-1}} + \|A^{-1/2}n_t(t)\|_{H^{s-1}}) \le c(1+T)^{\frac{2(1-s)}{6s-5}+}.
$$

□

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