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RECTIFIABILITY OF SOLUTIONS OF THE ONE-DIMENSIONAL P-LAPLACIAN

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ABSTRACT. In the recent papers [8] and [10] a class of Carathéodory functions $f(t,\eta,\xi)$ rapidly sign-changing near the boundary point $t=a$, has been constructed so that the equation $-(|y'|^{p-2}y')'=f(t,y,y')$ in (a,b) admits continuous bounded solutions y whose graphs $G(y)$ do not possess a finite length. In this paper, the same class of functions $f(t,\eta,\xi)$ will be given, but with slightly different input data compared to those from the previous papers, such that the graph $G(y)$ of each solution y is a rectifiable curve in $\mathbb{R}^2$. Moreover, there is a positive constant which does not depend on y so that $\operatorname{length}(G(y))\le c<\infty$.

1. Introduction

Let $-\infty<a<b<\infty$ and $1<p<\infty$. Let y be a real continuous function defined on the interval $[a,b]$ and let $G(y)$ be the graph of y defined as

$$G(y)=\{(t,y(t)):a\le t\le b\}\subseteq\mathbb{R}^2.$$

The length of the graph $G(y)$ is given by

$$\operatorname{length}(G(y))=\sup\sum_{i=1}^{m}\|(t_i,y(t_i))-(t_{i-1},y(t_{i-1}))\|_2, \tag{1.1}$$

where the supremum is taken over all dissections $a=t_0<t_1<\dots<t_m=b$ of the interval $[a,b]$. Here $\|\cdot\|_2$ denotes the norm in $\mathbb{R}^2$. If $\operatorname{length}(G(y))<\infty$ then the graph $G(y)$ is said to be a *rectifiable* curve in $\mathbb{R}^2$, see [2, Chapter 5.2].

The main purpose of the paper is to consider the rectifiability of the graph $G(y)$ of solutions y to a class of the nonlinear second order differential equations:

$$\begin{gathered}-(|y'|^{p-2}y')'=f(t,y,y')\quad\text{in }(a,b),\\ y(a)=y(b)=0,\\ y\in W^{1,p}_{\rm loc}((a,b])\cap C([a,b]).\end{gathered}\tag{1.2}$$

Here $f(t,\eta,\xi)$ is a Carathéodory function, that is, measurable in t for all (η,ξ) and continuous in (η,ξ) for almost all t. Next, the condition $y\in W^{1,p}_{loc}((a,b])$ means that $y\in W^{1,p}(a+\varepsilon,b)$ for all $\varepsilon>0$.

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In the author's paper [8], a class of Carathéodory functions $f(t,\eta,\xi)$ rapidly sign-changing and singular near the boundary point $t=a$ has been constructed in such a way that all solutions of the problem (1.2) admit very singular boundary behaviour. It has been verified by means of the upper Minkowski-Bouligand dimension of the graph $G(y)$ of each solution y as well as by the order of growth for singular behaviour of the L^p norm of the derivative of solutions. This can also be sharpened by the use of notions of the upper Minkowski content and the s - dimensional density of the graph $G(y)$ of each solution y, as in the paper [10].

Taking a closer look at [8, Section 7], see also [10, Section 5], one can find that there are two hypotheses involved to the nonlinear term $f(t,\eta,\xi)$ which can reduce the singular behaviour of the graph $G(y)$ of each solution y of the problem (1.2). In the first one, it is supposed that there is a continuous positive function $\tilde\omega(t)$ such that:

$$\begin{gathered} f(t,\eta,\xi)<0, \quad t\in(a,b), \quad \eta>\tilde\omega(t) \ \text{ and } \ \xi\in\mathbb{R}, \\ f(t,\eta,\xi)>0, \quad t\in(a,b), \quad \eta<-\tilde\omega(t) \ \text{ and } \ \xi\in\mathbb{R}. \end{gathered} \tag{1.3}$$

In the second one, it is supposed that there is a decreasing sequence a_k of real numbers from interval (a,b) such that $a_k\searrow a$ and:

$$\begin{gathered} f(t,\eta,\xi)>0, \quad t\in(a_{2k},a_{2k-1}), \quad \eta\in(-\tilde\omega_0,\tilde\omega(t)) \ \text{ and } \xi\in\mathbb{R}, \\ f(t,\eta,\xi)<0, \quad t\in(a_{2k+1},a_{2k}), \quad \eta\in(-\tilde\omega(t),\tilde\omega_0) \ \text{ and } \xi\in\mathbb{R}, \end{gathered} \tag{1.4}$$

where $\tilde\omega_0$ is an arbitrarily given real number such that $\tilde\omega(t)\le\tilde\omega_0$ for all $t\in(a,b)$. Because of (1.4), we say that the nonlinear term $f(t,\eta,\xi)$ is rapidly sign-changing near $t=a$.

In [8, Corollary 7.2], see also [10, Corollary 5.9], including the hypotheses (1.3) and (1.4), a class of the functions $\tilde\omega(t)$ and a class of the sequences a_k were given in such a way that a type of the singular boundary behaviour of solutions of the problem (1.2) is valid. In particular, one can conclude that the graph $G(y)$ of each solution y of (1.2) is not a rectifiable curve in $\mathbb{R}^2$, that is to say, the graph $G(y)$ does not possess a finite length. This can be summarized this way.

Theorem 1.1. *Let a Carathéodory function $f(t,\eta,\xi)$ satisfy the hypotheses (1.3) and (1.4) where the function $\tilde\omega(t)$ and the sequence a_k are given by:*

$$\begin{gathered} \tilde\omega(t)=c(t-a)^q \quad \text{in } [a,b], \text{ where } c>0 \text{ and } q\in(0,1), \\ a_k=a+(b-a)(\frac{1}{k})^{1/\beta}, \quad \forall k\in\mathbb{N}, \text{ where } \beta>q. \end{gathered} \tag{1.5}$$

Then problem (1.2) admits the solutions $y\in C^2(a,b)$ such that the graph $G(y)$ is not a rectifiable curve in $\mathbb{R}^2$, that is to say, $G(y)$ does not possess a finite length.

In this paper, under the hypotheses (1.3) and (1.4), one gives a possibility to avoid the nonrectifiability of $G(y)$ stated in Theorem 1.1. It will be done by choosing a class of sequences a_k from the interval (a,b) which decays exponentially to a, that is to say, $a_k-a\approx ce^{-kT}$ as $k\approx\infty$, for any $c>0$ and $T>0$. Such a class of sequences a_k is quite different from the previous one given in (1.5). It concerns the following main result.

Theorem 1.2. *Let a Carathéodory function $f(t,\eta,\xi)$ satisfy the hypotheses (1.3) and (1.4), where the function $\tilde\omega(t)$ is determined as in (1.5) and the sequence a_k is given by*

$$a_k=a+(b-a)e^{-(k-1)T}, \quad \forall k\in\mathbb{N}, \quad \text{where } T>0. \tag{1.6}$$

Then the graph $G(y)$ of each solution $y \in C^2(a,b)$ of problem (1.2) *is a rectifiable curve in $\mathbb{R}^2$. Moreover, there is a positive constant c which does not depend on y so that:*

$$\operatorname{length}(G(y)) \leq c < \infty. \tag{1.7}$$

Thus, under such a class of Carathéodory functions $f(t,\eta,\xi)$, the rectifiability of the graph $G(y)$ of solutions of the problem (1.2) depends only on the appropriate choice of the sequence a_k appearing in (1.4).

The main statement (1.7) will be proved in Section 3, by the use of the upper Minkowski content $M^1(G(y))$ of the graph $G(y)$ defined in Section 2. The first step is to give a condition such that $\operatorname{length}(G(y)) \leq M^1(G(y))$. The second step is to derive an upper bound for $M^1(G(y))$ which depends only on the behaviour of the function $\tilde{\omega}(t)$ and of the sequence a_k both appearing inside the main hypotheses (1.3) and (1.4). Finally, at the end of Section 3, an example of such a class of Carathéodory functions $f(t,\eta,\xi)$ that satisfies the hypotheses (1.3) and (1.4) will be given.

Further Remarks. A. The rectifiability of the graph $G(y)$ considered in this paper contributes to the completion of some results presented in the papers [8, Section 7] and [10, Section 5].
B. Including the hypotheses (1.3) and (1.4), the existence of at least one solution of the problem (1.2) was considered in [8, Appendix].
C. The main result of the paper can be applied to the study of rectifiable oscillations of the second order differential equations. Such oscillations appears in the famous linear Euler's equation as well as in the p - Laplace equation, see [9].
D. Although the nonlinear term $f(t,\eta,\xi)$ could have very singular behaviour near $t = a$, the rectifiability of $G(y)$ can be understood as a regular property of solutions of the problem (1.2). As for some known results on the regularity of solutions for p - Laplace equations, see for instance [3], [12], [13], and [14], and references therein.

2. How to measure the length of $G(y)$?

Let y be a real continuous function defined on the interval $[a,b]$ and let the length of the graph $G(y)$, denoted by $\operatorname{length}(G(y))$, be defined as in (1.1).

Since we are going to work with the solutions y of a nonlinear differential equation, from computational point of view, the formula (1.1) does not seem to be convenient. For the purpose of proving the rectifiability of the graph $G(y)$ and in order to prove the main statement (1.7), we will calculate the length of the graph $G(y)$ by means of the upper Minkowski content $M^1(G(y))$ as follows.

Let $G_\varepsilon(y)$ denote the $\varepsilon-$ neighborhood of the graph $G(y)$ and let $|G_\varepsilon(y)|$ denote the Lebesgue measure of $G_\varepsilon(y)$. The upper Minkowski content of $G(y)$, denoted by $M^1(G(y))$, is defined by

$$M^1(G(y)) = \limsup_{\varepsilon \to 0} (2\varepsilon)^{-1} |G_\varepsilon(y)|.$$

Referring to [2, Chapter 5.2], [6, Chapter 5.5] and [15, Chapter 9], it holds that

$$\operatorname{length}(G(y)) = M^1(G(y)), \tag{2.1}$$

provided the graph $G(y)$ is a rectifiable curve in $\mathbb{R}^2$. However, this result cannot be immediately used here because the condition of rectifiability of the graph

$G(y)$ is not yet presumed but it must be proved. Hence, we will use the following complementary claim.

Lemma 2.1. *Let y be a real continuous function defined on the interval $[a,b]$ and let $y|_{[a+\varepsilon,b]}$ denote the function restriction of y on the interval $[a+\varepsilon,b]$. We assume that the graph $G(y|_{[a+\varepsilon,b]})$ is a rectifiable curve in $\mathbb{R}^2$ for all $\varepsilon>0$. If $M^1(G(y))<\infty$ then the graph $G(y)$ is also a rectifiable curve in $\mathbb{R}^2$. Moreover, there holds true*

$$\lim_{\varepsilon\to 0}\operatorname{length}(G(y|_{[a+\varepsilon,b]}))=\operatorname{length}(G(y))\le M^1(G(y)).$$

Proof. By means of the rectifiability of the graphs $G(y|_{[a+\varepsilon,b]})$, $\varepsilon>0$, and by the use of (2.1) we observe that

$$\operatorname{length}(G(y|_{[a+\varepsilon,b]}))=M^1(G(y|_{[a+\varepsilon,b]}))\le M^1(G(y))<\infty,\quad \forall\varepsilon>0.$$

Since the sequence $\operatorname{length}(G(y|_{[a+\varepsilon,b]}))$ is increasing, while ε is decreasing, it implies the existence of its limit which satisfies

$$\lim_{\varepsilon\to 0}\operatorname{length}(G(y|_{[a+\varepsilon,b]}))\le M^1(G(y))<\infty. \tag{2.2}$$

Next, it easy to check that:

$$\operatorname{length}(G(y))\le\lim_{\varepsilon\to 0}\operatorname{length}(G(y|_{[a+\varepsilon,b]})). \tag{2.3}$$

Indeed, for any given dissection $a=t_0<t_1<\dots<t_m=b$ of the interval $[a,b]$ we can set a dissection $a+\varepsilon<t_1<\dots<t_m=b$ of the interval $[a+\varepsilon,b]$, for all $\varepsilon\in(0,t_1-a)$. Since y is a continuous function on $[a,b]$, by means of the definition (1.1) we derive

$$\begin{aligned}
&\sum_{i=1}^{m}\|(t_i,y(t_i))-(t_{i-1},y(t_{i-1}))\|_2\\
&=\lim_{\varepsilon\to 0}\Big[\|(t_1,y(t_1))-(a+\varepsilon,y(a+\varepsilon))\|_2+\sum_{i=2}^{m}\|(t_i,y(t_i))-(t_{i-1},y(t_{i-1}))\|_2\Big]\\
&\le\lim_{\varepsilon\to 0}\operatorname{length}(G(y|_{[a+\varepsilon,b]})).
\end{aligned}$$

Taking the supremum over all dissections of the interval $[a,b]$ from the previous inequality follows (2.3).

We complete the proof of this lemma by noting that $\operatorname{length}(G(y|_{[a+\varepsilon,b]}))\le\operatorname{length}(G(y))$ for any $\varepsilon>0$, and combining this inequality with (2.2) and (2.3). $\square$

With the help of Lemma 2.1 applied to solutions of the problem (1.2) we obtain the following consequence.

Corollary 2.2. *For all solutions y of the problem* (1.2) *such that $M^1(G(y))<\infty$ we have*

$$\operatorname{length}(G(y))=M^1(G(y)). \tag{2.4}$$

Proof. Let y be a solution of (1.2). Let us mention that $y\in W^{1,p}_{loc}((a,b])$, that is to say, $y\in W^{1,p}(a+\varepsilon,b)$ for all $\varepsilon>0$. It implies that y is an absolutely continuous function (of bounded variation) on $[a+\varepsilon,b]$. Therefore, the graph $G(y|_{[a+\varepsilon,b]})$ is a rectifiable curve in $\mathbb{R}^2$ for all $\varepsilon>0$. Since $M^1(G(y))<\infty$, we may apply Lemma 2.1 to obtain $\operatorname{length}(G(y))\le M^1(G(y))<\infty$. Now, it is possible to use (2.1) and thus, (2.4) is shown. $\square$

Thus, according to Corollary 2.2, in order to prove the rectifiability of $G(y)$ it is enough to show that $M^1(G(y)) < \infty$. For this we need the following general result.

Lemma 2.3. *Let $\tilde{\omega}(t)$ be a continuously increasing and concave function such that $\tilde{\omega}(a) = 0$. Next, let $a_1 = b$ and let a_k be a decreasing sequence of real numbers from interval (a, b) satisfying:*

$$\begin{aligned} &a_k \searrow a \text{ and there is an } \varepsilon_1 > 0 \text{ such that for all } \varepsilon \in (0, \varepsilon_1) \\ &\text{there is an } m(\varepsilon) \in \mathbb{N} \text{ such that } a_{j-1} - a_j > 4\varepsilon, \text{ for each } j \le m(\varepsilon), \\ &\text{and } m(\varepsilon) \to \infty \text{ as } \varepsilon \to 0. \end{aligned} \tag{2.5}$$

Let a Carathéodory function $f(t, \eta, \xi)$ satisfy hypotheses (1.3) *and* (1.4)*. Then the graph $G(y)$ of each solution $y \in C^2(a, b)$ of problem* (1.2) *satisfies*

$$\begin{aligned} M^1(G(y)) \le (\pi + 6)(b - a) + (\pi + 6) \sum_{j=1}^{\infty} \tilde{\omega}(a_j) \\ + \limsup_{\varepsilon \to 0} [\frac{1}{\varepsilon}(a_{m(\varepsilon)} - a)\tilde{\omega}(a_{m(\varepsilon)}) + (\pi + 4)\varepsilon\, m(\varepsilon)]. \end{aligned} \tag{2.6}$$

Proof. With the help of hypothesis (1.3) and [8, Lemma 3.1], we have that each solution $y \in C^2(a, b)$ of problem (1.2) satisfies

$$-\tilde{\omega}(t) \le y(t) \le \tilde{\omega}(t), \quad \text{for all } t \in [a, b]. \tag{2.7}$$

Next, taking into account hypothesis (1.4) and [8, Remark A in Appendix], we have also that

$$y \text{ is concave in } (a_{2k}, a_{2k-1}) \quad \text{and} \quad y \text{ is convex in } (a_{2k+1}, a_{2k}), \tag{2.8}$$

for each $k \ge 1$. Now, according to assertions (2.5), (2.7) and (2.8) it is clear the all assumptions of [10, Definition 5.1] and [10, Lemma 5.3] are fulfiled, where $\tilde{\tilde{\omega}}(t) = \tilde{\omega}(t)$ and $\tilde{\theta}(t) = -\tilde{\omega}(t)$. So, by means of [10, Lemma 5.3], we obtain

$$\begin{aligned} |G_\varepsilon(y|_{[a,a_1]})| &\le 2(a_{m(\varepsilon)} - a + 2\varepsilon)(\tilde{\omega}(a_{m(\varepsilon)}) + \varepsilon) \\ &\quad + 2(\pi + 6)\varepsilon \sum_{j=2}^{m(\varepsilon)} [\tilde{\omega}(a_{j-1}) + (a_{j-1} - a_j)] + 2(\pi + 4)\varepsilon^2 m(\varepsilon), \end{aligned}$$

where $a_1 = b$. Since $m(\varepsilon) \to \infty$, $a_{m(\varepsilon)} \to a$, and $\tilde{\omega}(0) = 0$, the right hand side in the previous inequality can be simplified and so, the statement (2.6) is proved. $\square$

As a consequence of the preceding lemma we have the following statement.

Corollary 2.4. *Let the hypotheses of Lemma 2.3 be still fulfilled. Let $\tilde{\omega}(t)$ and a_k be such that there are three positive constants c_1, c_2 and c_3 satisfying*

$$\sum_{j=1}^{\infty} \tilde{\omega}(a_j) \le c_1, \quad (a_{m(\varepsilon)} - a)\tilde{\omega}(a_{m(\varepsilon)}) \le c_2\varepsilon, \quad \text{and} \quad \varepsilon\, m(\varepsilon) \le c_3, \tag{2.9}$$

for sufficiently small ε. Then there is a positive constant c such that each solution $y \in C^2(a, b)$ of the problem (1.2) *satisfies $M^1(G(y)) \le c < \infty$.*

Thus, according to Corollary 2.2, in order to prove (1.7), we need to ensure that a class of the functions $\tilde{\omega}(t)$ given in (1.5) and a class of the sequences a_k given in (1.6) satisfy all the assumptions of Corollary 2.4.

3. The proofs of the main results and an example

Sketch of the proof of Theorem 1.1. Including some additional assumptions on the Caratheodory function $f(t,\eta,\xi)$ which are completely balanced with the main hypotheses (1.3) and (1.4), one can show (see in [8, Corollary 7.2] and [10, Corollary 5.9]) that the graph $G(y)$ of each solution $y \in C^2(a,b)$ of the problem (1.2) satisfies:

$$\dim_M G(y) = s \quad \text{and} \quad 0 < M^s(G(y) < \infty, \tag{3.1}$$

where s is a prescribed real number, $s \in (1,2)$. Here $\dim_M G(y)$ denotes the upper Minkowski-Bouligand dimension and $M^s(G(y)$ denotes the s-dimensional upper Minkowski content of $G(y)$ defined by

$$\dim_M G(y) = \limsup_{\varepsilon\to 0} \big(2 - \frac{\log |G_\varepsilon(y)|}{\log \varepsilon}\big) \quad \text{and} \quad M^s(G(y)) = \limsup_{\varepsilon\to 0} (2\varepsilon)^{s-2}|G_\varepsilon(y)|.$$

The existence of at least one solution y of the problem (1.2) satisfying the preceding properties was given in [8, Appendix]. Let us mention a very well known relation between $\dim_M G(y)$ and $M^s(G(y)$:

$$\dim_M G(y) = \inf\{s \ge 0 : M^s(G(y)) = 0\} = \sup\{s \ge 0 : M^s(G(y)) = \infty\}, \tag{3.2}$$

Since $s > 1$ and combining (3.1) with (3.2) we obtain that $M^1(G(y)) = \infty$. With the help of (2.1), it follows that the graph $G(y)$ does not possess a finite length. □

Next, we give the proof of the main result of the paper.

Proof of Theorem 1.2. Let us mention that the function $\tilde\omega(t)$ and the sequence a_k are given by

$$\begin{aligned} &\tilde\omega(t) = c(t-a)^q \quad \text{in } [a,b], \quad \text{where } c > 0 \text{ and } q \in (0,1),\\ &a_k = a + (b-a)e^{-(k-1)T}, \quad \forall k \in \mathbb{N}, \quad \text{where } T > 0. \end{aligned} \tag{3.3}$$

According to Corollary 2.4, it is sufficient to show that such $\tilde\omega(t)$ and a_k satisfy the all conditions of Lemma 2.3 as well as condition (2.9).

It is clear that the function $\tilde\omega(t)$ is continuous and concave in (a,b), and that $\tilde\omega(0) = 0$. Also, the sequence a_k is decreasing, $a_1 = b$ and $a_k \searrow a$.

Let $m(\varepsilon)$ be a sequence of integers determined by the following inequality:

$$\frac{1}{T}\ln\frac{(e^T-1)(b-a)}{4\varepsilon} \le m(\varepsilon) \le \frac{1}{T}\ln\frac{(e^{2T}-e^T)(b-a)}{4\varepsilon}, \quad \varepsilon > 0. \tag{3.4}$$

With the help of (3.3) and (3.4) it is easy to check the following calculations. Firstly, we derive that

$$\frac{1}{T}\ln\frac{(e^{2T}-e^T)(b-a)}{4\varepsilon} - \frac{1}{T}\ln\frac{(e^T-1)(b-a)}{4\varepsilon} = \frac{1}{T}\ln e^T = 1. \tag{3.5}$$

Also, for each $j \le m(\varepsilon)$, we obtain

$$\begin{aligned} a_{j-1} - a_j &= (b-a)(e^{2T}-e^T)e^{-jT} \ge (b-a)(e^{2T}-e^T)e^{-m(\varepsilon)T}\\ &\ge (b-a)(e^{2T}-e^T)e^{-\frac{1}{T}T\ln\frac{(e^{2T}-e^T)(b-a)}{4\varepsilon}} = 4\varepsilon. \end{aligned} \tag{3.6}$$

Next, there is a positive constant c_1 such that for all $\varepsilon > 0$ we obtain:

$$\sum_{j=1}^\infty \tilde\omega(a_j) = c\sum_{j=1}^\infty (a_j-a)^q = c(b-a)^q e^{qT}\sum_{j=1}^\infty e^{-jqT} \le c_1. \tag{3.7}$$

Furthermore, there is a positive constant c_2 such that for all $\varepsilon > 0$ we derive

$$\begin{aligned}(a_{m(\varepsilon)} - a)\tilde{\omega}(a_{m(\varepsilon)}) &= c[(b-a)e^T]^{q+1}e^{-(q+1)Tm(\varepsilon)}\\ &\le c[(b-a)e^T]^{q+1}e^{-Tm(\varepsilon)}\\ &\le c[(b-a)e^T]^{q+1}e^{-T\frac{1}{T}\ln\frac{(e^T-1)(b-a)}{4\varepsilon}} = c_2\varepsilon.\end{aligned} \tag{3.8}$$

Since $\lim_{\varepsilon\to 0}[\varepsilon \ln(1/\varepsilon)] = 0$, there is a positive constant c_3 such that for all $\varepsilon > 0$ we have:

$$\varepsilon\, m(\varepsilon) \le \frac{\varepsilon}{T}\ln\frac{(e^{2T}-e^T)(b-a)}{4\varepsilon} \le c_3. \tag{3.9}$$

We now conclude with the help of (3.5) that it is possible to choose a $m(\varepsilon) \in \mathbb{N}$ in the statement (3.4). Also, according to (3.6) we obtain that the sequence a_k satisfies the condition (2.5), where ε_1 can be taken arbitrarily.

Next, by means of (3.7), (3.8) and (3.9) we obtain that the function $\tilde{\omega}(t)$ and the sequence a_k satisfy all the conditions from (2.9). Therefore, we may apply Corollary 2.4 which immediately gives the desired statement (1.7). ☐

Example 3.1. Let $g_k(t)$ be an arbitrarily given sequence of measurable functions such that $g_k(t) > 0$ in (a,b), for each $k = 0,1,2,\dots$. Let $f(t,\eta,\xi)$ be a nonlinear function defined by:

$$\begin{aligned}f(t,\eta,\xi) =& -g_0(t)(\eta-\tilde{\omega}(t))^+ + g_1(t)(\eta+\tilde{\omega}(t))^-\\ &+\sum_{k=1}^{\infty}\big[g_{2k}(t)(\eta-\tilde{\omega}(t))^-\sin\big(\frac{\pi}{a_{2k-1}-a_{2k}}(t-a_{2k})\big)K_{[a_{2k},a_{2k-1}]}(t)\\ &-g_{2k+1}(t)(\eta+\tilde{\omega}(t))^+\sin\big(\frac{\pi}{a_{2k}-a_{2k+1}}(t-a_{2k+1})\big)K_{[a_{2k+1},a_{2k}]}(t)\big],\end{aligned}$$

where $t \in [a,b]$, $\eta \in \mathbb{R}$ and $\xi \in \mathbb{R}$. Here $K_A(t)$ denotes, as usual, the characteristic function of a set A and also, $\eta^- = \max\{0,-\eta\}$ and $\eta^+ = \max\{0,\eta\}$. It is easy to check that such a function $f(t,\eta,\xi)$ is of Carathéodory type and that it satisfies the desired hypotheses (1.3) and (1.4).

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