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EIGENVALUE PROBLEMS OF ATKINSON, FELLER AND KREIN, AND THEIR MUTUAL RELATIONSHIP

HANS VOLKMER

Abstract. It is shown that every regular Krein-Feller eigenvalue problem can be transformed to a semidefinite Sturm-Liouville problem introduced by Atkinson. This makes it possible to transfer results between the corresponding theories. In particular, Prüfer angle methods become available for Krein-Feller problems.

1. Introduction

The classical regular Sturm-Liouville eigenvalue problem consists of the quasi-differential equation

$$-(p(x)y')' + q(x)y = \lambda r(x)y, \quad x \in [c,d] \text{ a.e.} \tag{1.1}$$

and the boundary conditions

$$y(c)\cos\alpha = (py')(c)\sin\alpha, \quad y(d)\cos\beta = (py')(d)\sin\beta. \tag{1.2}$$

It is assumed that $1/p$, q, r are real-valued Lebesgue integrable functions on $[c,d]$, $p(x) > 0$, $r(x) > 0$ a.e. and α, β are real. If (1.1), (1.2) admit a nontrivial solution y for some λ, then λ is called an eigenvalue and y is a corresponding eigenfunction. If the eigenfunction y has exactly n zeros in (c,d) then λ is called an eigenvalue with oscillation count n. It is well known that, for every nonnegative integer n, there is a unique eigenvalue λ_n with oscillation count n, and these eigenvalues form an increasing sequence $\lambda_0 < \lambda_1 < \lambda_2 < \dots$ converging to infinity. More precisely, the eigenvalues grow according to

$$\lim_{n\to\infty} n^{-2}\lambda_n = \frac{\pi^2}{L^2}, \quad L := \int_c^d \Big(\frac{r(x)}{p(x)}\Big)^{1/2}\,dx. \tag{1.3}$$

These results can be proved conveniently with the help of the Prüfer angle.

In this paper we will consider generalizations of the classical Sturm-Liouville eigenvalue problem introduced by Atkinson, Feller and Krein, and explore their relationship.

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Setting $s = 1/p$, $u = y$, $v = pu'$, Atkinson [1, Chapter 8] writes equation (1.1) as a Carathéodory system

$$u' = s(x)v, \quad v' = (q(x) - \lambda r(x))u, \tag{1.4}$$

and the boundary conditions (1.2) as

$$u(c)\cos\alpha = v(c)\sin\alpha, \quad u(d)\cos\beta = v(d)\sin\beta. \tag{1.5}$$

He assumes that s, q, r are real-valued integrable functions with $r(x) \geq 0$, $s(x) \geq 0$ for all $x \in [c, d]$. Under some additional assumptions Atkinson [1, Theorems 8.4.5, 8.4.6] proved existence and uniqueness of eigenvalues with prescribed oscillation count provided the latter is appropriately defined. We explain this and related results on Atkinson's problem in Section 4 in more detail. Here we just mention that the asymptotic formula (1.3) still holds [3].

Krein [12, 13, 15, 16] introduced the eigenvalue problem for the vibrating string with finite mass. If we set $p = 1$, $q = 0$ in the classical Sturm-Liouville problem then we obtain the eigenvalue problem of a vibrating string. The part of the string lying over the interval $[c, x]$ has mass $\int_c^x r(t)\,dt$. In Krein's eigenvalue problem the mass of the string occupying $[c, x]$ is given by $\omega([c, x])$, where ω is a measure on the Borel subsets of $[c, d]$. Equations (1.4) have now to be interpreted as integral equations

$$u(x) - u(c) = \int_{[c,x)} v(t)\,dt, \quad v(x) - v(c) = -\lambda \int_{[c,x)} u(t)\,d\omega(t).$$

These equations can be combined into one equation

$$u(x) - u(c) - xu'(c) = -\lambda \int_{[c,x)} (x - t)u(t)\,d\omega(t);$$

see [1, page 24]. For Krein's problem we also have a theorem on the existence of eigenvalues with prescribed oscillation count, and there is also an asymptotic result similar to (1.3). Feller [6, 7] gave an alternative treatment of the vibrating string problem based on generalized second derivatives. Krein-Feller problems will be considered in Section 5.

Comparing the various eigenvalues problems, we note that Atkinson's problem contains the classical Sturm-Liouville problem as a special case. However, the relationship between problems of Krein-Feller and Atkinson is less obvious. Atkinson [1, page 202] points out that the vibrating string with beads (a string whose mass is concentrated at finitely many points) can be treated within his framework.

We show in this paper that every regular Krein-Feller eigenvalue problem can be transformed to an equivalent problem of Atkinson's type. The construction of the transformation depends on the Radon-Nikodym theorem. Therefore, Atkinson's problem contains all the other problems as special cases. This makes it possible to create a unified theory where all results are first proved in Atkinson's setting and then are specialized as needed. In Section 5 we apply this idea to obtain existence of eigenvalues of Krein-Feller problems with prescribed oscillation counts and their asymptotics from the corresponding results for Atkinson's problems. We use these results as examples. In fact, it may be expected that all results for Krein-Feller problems can be obtained from corresponding ones for Atkinson's problem. This should also apply to the operator-theoretic aspect including expansion theorems as well as to singular problems. For the theory of Krein-Feller operators see Fleige [8].

In Section 5, we introduce a measure Sturm-Liouville problem in which all three functions s, q, r are replaced by measures. These problems are more general than those of Krein and Feller. We go on to show that under a condition on the location of the atoms of these measures the measure Sturm-Liouville problem is also a special case of Atkinson's problem.

In Sections 2 and 3, we deal with linear integral equations that generalize Carathéodory systems of differential equations and their transformations. The eigenvalue problems of Krein, Feller and our measure Sturm-Liouville problem will involve such systems.

2. Measure integral equations

Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of the interval $[a, b]$. For each $i, j = 1, \dots, n$, let $\omega_{ij} : \mathcal{B} \to \mathbb{C}$ be a complex-valued measure. Consider the linear system

$$y_i(t) = y_i(a) + \sum_{j=1}^{n} \int_{[a,t)} y_j(s) d\omega_{ij}(s), \quad t \in [a, b],\, i = 1, \dots, n. \tag{2.1}$$

A solution is a bounded Borel measurable function $y : [a, b] \to \mathbb{C}^n$, with $y(t) = (y_1(t), \dots, y_n(t))$, that satisfies (2.1). Note that since y is bounded and Borel measurable, the integrals on the right-hand side of (2.1) are well-defined. Atkinson [1, Section 11.8] considers such systems but he uses Stieltjes integrals; see also [17]. We prefer to work with integrals over measure spaces.

Lemma 2.1. *Let y be a solution of (2.1). Then y is of bounded variation so that the one-sided limits $y(t+)$, $t \in [a, b)$ and $y(t-)$, $t \in (a, b]$ exist. Moreover, y is left-continuous at every $t \in (a, b]$, and*

$$y_i(t+) - y_i(t) = \sum_{j=1}^{n} \omega_{ij}(\{t\}) y_j(t), \quad t \in [a, b). \tag{2.2}$$

Proof. If f is integrable with respect to a measure $\omega : \mathcal{B} \to \mathbb{C}$ then

$$F(t) := \int_{[a,t)} f(s)\, d\omega(s)$$

defines a function $F : [a, b] \to \mathbb{C}$. For any partition

$$a = t_0 < t_1 < t_2 < \dots < t_m = b,$$

we have

$$\sum_{k=1}^{m} |F(t_k) - F(t_{k-1})| \le \int_{[a,b)} |f(t)|\, d|\omega|(t),$$

where $|\omega|$ denotes the total variation of ω. This shows that F is of bounded variation. Moreover, F is left-continuous and

$$F(t+) - F(t) = f(t)\omega(\{t\}).$$

The statements of the lemma follow. □

By definition, atoms at $t = b$ do not enter the linear system (2.1). However, we may use equation (2.2) to define $y(b+)$. Then $y(b+)$ depends on atoms at $t = b$. Note that if the measures ω_{ij}, $j = 1, \dots, n$, have no atoms at t then y_i will be continuous at t.

Choose a measure $\omega : \mathcal{B} \to [0,\infty)$ such that each ω_{ij} is absolutely continuous with respect to ω, for instance,

$$\omega = \sum_{i=1}^n \sum_{j=1}^n |\omega_{ij}|.$$

By the Radon-Nikodym theorem, there exist functions w_{ij} (Radon-Nikodym derivatives), integrable with respect to ω, such that

$$\omega_{ij}(E) = \int_E w_{ij}(t)\, d\omega(t), \quad E \in \mathcal{B}.$$

In particular, for every bounded Borel measurable function $f : [a,b] \to \mathbb{C}$,

$$\int_E f(t)\, d\omega_{ij}(t) = \int_E f(t) w_{ij}(t)\, d\omega(t).$$

Therefore, we may write (2.1) in the form

$$y(t) = y(a) + \int_{[a,t)} W(s)y(s)\, d\omega(s), \quad t \in [a,b], \tag{2.3}$$

where $W(s)$ is the $n \times n$ matrix with entries $w_{ij}(s)$. If ω is the Lebesgue measure then (2.3) is a standard Carathéodory system of linear differential equations.

The following theorem is well known [1, Section 11.8]. We prove it for the sake of completeness.

Theorem 2.2. *For every $c \in \mathbb{C}^n$, there is a unique solution $y : [a,b] \to \mathbb{C}^n$ of (2.1) with initial value $y(a) = c$.*

Proof. We take the system in the form (2.3). Consider the space Y of all bounded Borel measurable functions $y : [a,b] \to \mathbb{C}^n$ equipped with the sup norm

$$\|y\| := \sup_{t\in[a,b]} |y(t)|.$$

We may use here any norm $|\cdot|$ in $\mathbb{C}^n$. Then Y is a Banach space. We define the operator $T : Y \to Y$ by

$$(Ty)(t) = c + \int_{[a,t)} W(s)y(s)\, d\omega(s).$$

Clearly, T is well-defined. Let $y, z \in Y$. We estimate

$$|(Ty)(t) - (Tz)(t)| \le \int_{[a,t)} |W(s)||y(s) - z(s)|\, d\omega(s).$$

Hence

$$\|Ty - Tz\| \le L\|y - z\|,$$

where

$$L := \int_{[a,b)} |W(s)|\, d\omega(s).$$

If $L < 1$, application of Banach's fixed point theorem completes the proof. If $L \ge 1$, we choose a partition $a = t_0 < t_1 < \dots < t_m = b$ such that

$$\int_{(t_{k-1},t_k)} |W(s)|\, d\omega(s) < \frac{1}{2} \quad \text{for every } k = 1, 2, \dots, m.$$

We set $y(a) = c$ and then define $y(a+)$ by (2.2). Then we remove the atom at $t = a$ if necessary and apply Banach's fixed point theorem to T with $y(a+)$ in place of

c and $[a, t_1]$ in place of $[a, b]$. This way we get a solution on $[a, t_1]$. After finitely many steps we obtain the desired solution of (2.1). □

Remark 2.3. It should be noted that equation (2.1) is solved from left to right. In general, we cannot solve initial value problems at points different from the left end point a. For example, take $n = 1$, $[a, b] = [0, 2]$, and $\omega_{11}(E) = -1$ if $1 \in E$ and $\omega_{11}(E) = 0$ if $1 \notin E$. The solution with $y(0) = c$ is given by $y(t) = c$ for $0 \le t \le 1$ and $y(t) = 0$ for $1 < t \le 2$. There is no solution with $y(2) \neq 0$.

3. Transformation of measure integral equations

Let $\omega : \mathcal{B} \to [0, \infty)$ be any nonnegative measure such that

$$\omega((a', b')) > 0 \quad \text{whenever } a \le a' < b' \le b. \tag{3.1}$$

Consider its distribution function $h : [a, b] \to [0, \infty)$ defined by

$$h(t) := \omega([a, t)). \tag{3.2}$$

This function is strictly increasing and left-continuous. For all $e \in [a, b]$

$$h(e+) - h(e) = \omega(\{e\}).$$

For $e = b$ we take this is as the definition of $h(b+)$. Let

$$d := h(b+) = \omega([a, b]). \tag{3.3}$$

The function $h : [a, b] \to [0, d]$ is one-to-one but usually not onto. We define a function $H : [0, d] \to [a, b]$ that is left-inverse to h by

$$H(x) := \max\{t \in [a, b] : h(t) \le x\}.$$

Hence $t = H(x)$ is the unique $t \in [a, b]$ such that $h(t) \le x \le h(t+)$. The function H is nondecreasing and continuous. We use H to transform integrals as follows; compare with [19, pages 194–195].

Lemma 3.1. *Let $f : [a, b] \to \mathbb{C}$ be integrable with respect to ω. Then $f(H(x))$ is Lebesgue integrable and*

$$\int_{[a,t)} f(s)\, d\omega(s) = \int_0^{h(t)} f(H(x))\, dx \quad \textit{for } t \in [a, b]. \tag{3.4}$$

Proof. Equation (3.4) holds for the characteristic function $f = \chi_K$ with $K = (a', b')$ since

$$\{x \in [0, d] : H(x) \in (a', b')\} = (h(a'+), h(b')).$$

In a similar way we see that (3.4) holds for characteristic functions of any subinterval of $[a, b]$. If we set $f = \chi_B$, $B \in \mathcal{B}$, then both sides of (3.4) define finite measures on $\mathcal{B}$. Since these measures agree on intervals they also agree on all Borel sets. It follows that (3.4) holds for all nonnegative Borel measurable simple functions. Since every nonnegative Borel measurable functions is the pointwise limit of a nondecreasing sequence of nonnegative simple functions, the monotone convergence theorem shows that (3.4) holds for all nonnegative Borel measurable functions. The statement of the lemma follows. □

Consider the linear system (2.1) in the form (2.3). Since we may add the Lebesgue measure to ω, we may assume that (3.1) holds without loss of generality. We define d, h and H relative to ω.

We associate with (2.3) the linear system

$$z'(x) = W(H(x))z(x), \quad x \in [0, d]\, a.e. \tag{3.5}$$

By Lemma 3.1, the entries of $W(H(x))$ are Lebesgue integrable on $[0, d]$. Hence (3.5) is a linear system in the sense of Carathéodory. Solutions of equations (2.3) and (3.5) are connected as follows.

Theorem 3.2. *Assume that*

$$W(t)^2 = 0 \quad \textit{whenever } \omega(\{t\}) > 0. \tag{3.6}$$

Let $y : [a, b] \to \mathbb{C}^n$ be the solution of (2.3) with $y(a) = c$, and let $z : [0, d] \to \mathbb{C}^n$ be the solution of (3.5) with $z(0) = c$. Then

$$y(t) = z(h(t)),\, y(t+) = z(h(t+)) \quad \textit{for all } t \in [a, b].$$

Proof. We claim that

$$W(H(x))z(x) = W(H(x))z(h(H(x)) \quad \text{for all } x \in [0, d]. \tag{3.7}$$

If x lies in the range of h then $x = h(H(x))$ and (3.7) holds. If x does not lie in the range of h, then $h(t) < x \le h(t+)$ and $\omega(\{t\}) > 0$ where $t := H(x)$. On the interval $[h(t), x]$ system (3.5) reads $z' = W(t)z$ with constant coefficient $W(t)$. Thus

$$z(x) = e^{(x-h(t))W(t)} z(h(t)).$$

Therefore,

$$W(t)z(x) = W(t)e^{(x-h(t))W(t)} z(h(t)).$$

Using assumption (3.6) and the power series expansion of the exponential function of matrices, we see that

$$W(t)e^{(x-h(t))W(t)} = W(t).$$

This completes the proof of (3.7).

We have

$$z(x) = c + \int_0^x W(H(r))z(r)\, dr \quad \text{for all } x \in [0, d].$$

Setting $x = h(t)$ we obtain

$$z(h(t)) = c + \int_0^{h(t)} W(H(r))z(r)\, dr \quad \text{for all } t \in [a, b].$$

By (3.7), we can write this as

$$z(h(t)) = c + \int_0^{h(t)} W(H(r))z(h(H(r)))\, dr \quad \text{for all } t \in [a, b].$$

Transforming the integral on the right-hand side using Lemma 3.1, we find

$$z(h(t)) = c + \int_{[a,t)} W(s)z(h(s))\, d\omega(s) \quad \text{for all } t \in [a, b].$$

This implies $z(h(t)) = y(t)$. □

If the measure ω has no atoms then condition (3.6) is satisfied. Thus all systems (2.1) which involve only measures without atoms can be transformed to Carathéodory systems. However, the equation mentioned in Remark 2.3 cannot be transformed to a Carathéodory equation.

4. Atkinson's eigenvalue problem

We consider system (1.4) subject to the boundary conditions (1.5). We assume that $s, q, r : [c, d] \to \mathbb{R}$ are integrable functions, and $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$. Note that we do not assume definiteness conditions unless stated explicitly. The complex number λ is called an *eigenvalue* if there exists a nontrivial absolutely continuous solution (u, v) of (1.4) satisfying (1.5).

Let $(u(x, \lambda), v(x, \lambda))$ be the unique solution of (1.4) with initial values $u(c, \lambda) = \sin\alpha$, $v(c, \lambda) = \cos\alpha$. A complex number λ is an eigenvalue if and only if

$$\Delta(\lambda) := \cos\beta\, u(d, \lambda) - \sin\beta\, v(d, \lambda) = 0. \tag{4.1}$$

Since Δ is an entire function, the set of eigenvalues is discrete or equals the entire complex plane. In the latter case we call the eigenvalue problem *degenerate*.

The Prüfer angle for (1.4), (1.5) is the absolutely continuous function $\theta(x) = \theta(x, \lambda)$ defined by

$$\theta(x, \lambda) = \arg(v(x, \lambda) + iu(x, \lambda)), \quad \theta(c, \lambda) = \alpha.$$

It satisfies the first order differential equation

$$\frac{d\theta}{dx} = s(x)\cos^2\theta + (\lambda r(x) - q(x))\sin^2\theta. \tag{4.2}$$

The real number λ is an eigenvalue of (1.4), (1.5) if and only if there is an integer n such that

$$\theta(d, \lambda) = \beta + n\pi. \tag{4.3}$$

Note that $\theta(d, \lambda)$ is an entire function of λ.

We are mainly interested in results on the eigenvalue problem that permit s to vanish on sets of positive measure. Results of this nature can be carried over to measure Sturm-Liouville problems in Section 5. The following asymptotic formula of the Prüfer angle $\theta(d, \lambda)$ is proved in [3].

Lemma 4.1. *We have*

$$\lim_{\lambda\to+\infty} \lambda^{-1/2}\theta(d, \lambda) = \int_c^d \sqrt{r^+s^+} - \int_c^d \sqrt{r^-s^-} =: L^+, \tag{4.4}$$

$$\lim_{\lambda\to-\infty} |\lambda|^{-1/2}\theta(d, \lambda) = \int_c^d \sqrt{r^-s^+} - \int_c^d \sqrt{r^+s^-} =: L^-, \tag{4.5}$$

where $r^+(x) := \max(0, r(x))$, $r^-(x) = \max(0, -r(x))$.

From this lemma and the intermediate value theorem, we obtain existence of real eigenvalues and their asymptotics without requiring any sign condition on r or s.

Theorem 4.2. *Assume $L^+ > 0$. For every sufficiently large integer n, there exists a real positive eigenvalue λ_n that satisfies $\theta(d, \lambda_n) = \beta + n\pi$. For any choice of λ_n, we have*

$$\lim_{n\to\infty} n^{-2}\lambda_n = \pi^2(L^+)^{-2}.$$

There are similar theorems when $L^+ < 0$, $L^- > 0$, or $L^- < 0$. Under additional assumptions the statement of this theorem can be refined as follows.

Lemma 4.3. *The derivative of $\theta(d,\lambda)$ with respect to λ is given by*

$$(u(d)^2 + v(d)^2)\frac{d}{d\lambda}\theta(d,\lambda) = \int_c^d r(x)u(x)^2\,dx, \tag{4.6}$$

where $u(x) = u(x,\lambda)$, $v(x) = v(x,\lambda)$.

For the proof of the above lemma, see [1, Theorem 8.4.2]. We conclude the following uniqueness theorem.

Theorem 4.4. *If the eigenvalue problem is not degenerate and $r \ge 0$, then all eigenvalues are real and equation (4.3) has at most one solution for every integer n.*

Proof. If λ_0 is a nonreal eigenvalue then one shows as in [1, Section 8.3] that $\int_c^d r(x)|u(x,\lambda_0)|^2\,dx = 0$. Since $r \ge 0$ we obtain that $r(x)u(x,\lambda_0) = 0$ a.e. on $[c,d]$. This shows that $(u(x,\lambda_0), v(x,\lambda_0))$ solves the system (1.4), (1.5) for all λ. Thus the problem is degenerate. By Lemma 4.3, $\theta(d,\lambda)$ is increasing or constant. If $\theta(d,\lambda)$ is increasing the second part of the statement is obvious. If $\theta(d,\lambda)$ is constant then either the problem is degenerate or has no eigenvalues. □

Lemma 4.5. *If $s \ge 0$ and $\theta(x_0,\lambda)$ is an integer multiple of π, then $\theta(x_1,\lambda) \le \theta(x_0,\lambda) \le \theta(x_2,\lambda)$ for $c \le x_1 < x_0 < x_2 \le d$.*

The proof of the above lemma follows from an analysis of equation (4.2); see [1, Theorem 8.4.1].

The zeros of $u(x) = u(x,\lambda)$ form a closed set that can be decomposed into (closed) components. Lemma 4.5 yields the following theorem.

Theorem 4.6. *Suppose $s \ge 0$. If λ_n is a solution of (4.3) then the set of zeros of $u(x,\lambda_n)$ within $[c,d]$ has exactly n components if $\alpha \ne 0$ and $\beta \ne \pi$, $n+1$ components if either $\alpha = 0$ or $\beta = \pi$, and $n+2$ components if $\alpha = 0$ and $\beta = \pi$.*

Based on this theorem, we may call a solution λ_n of (4.3) an eigenvalue with *oscillation count* n provided that $s \ge 0$.

In the rest of this section we will assume with Atkinson that $r \ge 0$ and $s \ge 0$. By Lemmas 4.3 and 4.5, the function $\theta(d,\cdot)$ is nonnegative, and increasing or constant. We write

$$\ell^\pm := \lim_{\lambda\to\pm\infty}\theta(d,\lambda). \tag{4.7}$$

If $\ell^- < \ell^+$ then the eigenvalue λ_n with oscillation count n exists if and only if $\ell^- < \beta + n\pi < \ell^+$. Moreover, there are infinitely many eigenvalues if and only if $\ell^+ = +\infty$. Formulas for $\ell^\pm$ are given in [4]. They lead to the following results.

Let $\mathcal{I}$ be the collection of closed intervals $I \subset [c,d]$ with positive length for which $\int_I r = 0$. Let $\mathcal{J}$ be the subcollection of $\mathcal{I}$ consisting of those intervals $J \in \mathcal{I}$ with the property that $I \in \mathcal{I}$, $J \subset I$ implies $I = J$ (maximal elements of $\mathcal{I}$). It is easy to show that the intervals in $\mathcal{J}$ are mutually disjoint, and that every interval in $\mathcal{I}$ is contained in an interval in $\mathcal{J}$. For $J = [c',d'] \in \mathcal{J}$ let (u_J, v_J) be the solution of (1.4), that is,

$$u' = s(x)v, \quad v' = q(x)u$$

with initial values

$$\begin{aligned} u(c') = 0, v(c') = 1 \quad & \text{if } c < c' < d' < d, \\ u(c) = \sin\alpha, v(c) = \cos\alpha \quad & \text{if } c = c', \\ u(d) = \sin\beta, v(d) = \cos\beta \quad & \text{if } c < c' < d' = d. \end{aligned}$$

Let $\tilde{n}_J$ be the number of components of the set of zeros of u_J within J. Then we set $n_J := \tilde{n}_J$ if $c = c'$, $\alpha \neq 0$ or if $c < c' < d' = d$, $\beta \neq \pi$. In all other cases we define $n_J := \tilde{n}_J - 1$.

Theorem 4.7. *Suppose that $r, s \geq 0$, the problem is not degenerate and admits at least one eigenvalue. Then the minimal oscillation count n^- of eigenvalues is given by*

$$n^- = \sum_{J \in \mathcal{J}} n_J. \tag{4.8}$$

For the proof of the above theorem, see [4, Section 3].

Under the additional assumption that $s(x) > 0$ a.e. Theorem 4.7 is due to Everitt, Kwong and Zettl [5] and, in an operator theoretic setting, to Binding and Browne [2]. Atkinson [1, Theorem 8.4.5] gave the following sufficient condition for $n^- = 0$. It is a consequence of Theorem 4.7

Corollary 4.8. *In addition to the assumptions of Theorem 4.7 suppose that*

$$\int_c^e r(x)\,dx > 0, \int_e^b r(x)\,dx > 0 \quad \text{for all } e \in (c, d)$$

and

$$\int_{c'}^{d'} r(x)\,dx = 0 \text{ implies } \int_{c'}^{d'} q(x)\,dx = 0 \text{ for all } [c', d'] \subset [c, d].$$

Then $n^- = 0$.

The following theorem singles out the problems with finite spectrum.

Theorem 4.9. *Suppose $r, s \geq 0$. The following statements are equivalent. 1. The eigenvalue problem is degenerate or has only finitely many eigenvalues;*
2. ℓ^+ is finite;
3. there is a partition

$$c = \xi_0 < \xi_1 < \cdots < \xi_m = d$$

such that, for all $j = 1, 2, \ldots, m$,

$$\int_{\xi_{j-1}}^{\xi_j} r(x)\,dx \times \int_{\xi_{j-1}}^{\xi_j} s(x)\,dx = 0.$$

For the proof of the above theorem, see [4, Section 4]. We mention that, if ℓ^+ is finite, one can give a formula for the maximal oscillation count n^+; see [4, Section 4].

5. Measure Sturm-Liouville problems

Let $\sigma, \chi, \rho : \mathcal{B} \to \mathbb{R}$ be (finite signed) measures, where again $\mathcal{B}$ denotes the σ-algebra of Borel sets in $[a, b]$. Consider the measure integral equations

$$U(t) - U(a) = \int_{[a,t)} V(s)\, d\sigma(s), \quad t \in [a, b], \tag{5.1}$$

$$V(t) - V(a) = \int_{[a,t)} U(s)\, d\chi(s) - \lambda \int_{[a,t)} U(s)\, d\rho(s), \quad t \in [a, b], \tag{5.2}$$

and the boundary conditions

$$\cos\alpha\, U(a) = \sin\alpha\, V(a), \quad \cos\beta\, U(b+) = \sin\beta\, V(b+), \tag{5.3}$$

where $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$. In Section 2 we defined the notion of solution of the system, and we also explained the meaning of $U(b+)$, $V(b+)$. A complex number λ is called an *eigenvalue* if there exists a nontrivial solution (U, V) of system (5.1), (5.2) satisfying the boundary conditions (5.3).

Note that the measure Sturm-Liouville problem includes the one considered in Section 4 when we set

$$\sigma(E) = \int_E s(x)\, dx, \quad \chi(E) = \int_E q(x)\, dx, \quad \rho(E) = \int_E r(x)\, dx.$$

Let S, Q, R denote the set of atoms for σ, χ, ρ, respectively. We now transform the measure Sturm-Liouville problem under the condition

$$S \cap (Q \cup R) = \emptyset. \tag{5.4}$$

This condition permits the measures σ, χ, ρ to have atoms but σ and ρ as well as σ and χ may not have common atoms. Choose a measure $\omega : \mathcal{B} \to [0, \infty)$ such that each measure σ, χ, ρ is absolutely continuous with respect to ω, and $\omega((a', b')) > 0$ for all $a \le a' < b' \le b$. For instance, we may take

$$\omega = |\sigma| + |\chi| + |\rho| + \nu,$$

where ν denotes Lebesgue measure. Let f_σ, f_χ, f_ρ denote the Radon-Nikodym derivatives of σ, χ, ρ with respect to ω. Consider the distribution function $h(t) = \omega([a, t))$ of ω, and let $d := h(b+) := \omega([a, b])$. As in Section 3 we define the function H which is left-inverse to h. Then we define Lebesgue integrable functions $s, q, r : [0, d] \to \mathbb{R}$ by

$$s(x) := f_\sigma(H(x)),\ q(x) := f_\chi(H(x)),\ r(x) := f_\rho(H(x)).$$

Employing these functions we call (1.4), (1.5) (with $c = 0$) the Sturm-Liouville problem associated with the given measure Sturm-Liouville problem. The numbers α, β appearing in (1.5) are the same as those in (5.3).

Let $(U(t, \lambda), V(t, \lambda))$ be the unique solution of (5.1), (5.2) with initial values $U(a, \lambda) = \sin\alpha$, $V(a, \lambda) = \cos\alpha$ according to Theorem 2.2. Then λ is an eigenvalue of the measure Sturm-Liouville problem if and only if

$$\cos\beta\, U(b+, \lambda) - \sin\beta\, V(b+, \lambda) = 0. \tag{5.5}$$

Theorem 5.1. *If condition* (5.4) *holds then*

$$u(h(t), \lambda) = U(t, \lambda), \quad v(h(t), \lambda) = V(t, \lambda) \quad \text{for } t \in [a, b]$$

and

$$u(d, \lambda) = U(b+, \lambda), \quad v(d, \lambda) = V(b+, \lambda).$$

In particular, λ is an eigenvalue of the measure Sturm-Liouville problem if and only if λ is an eigenvalue of the associated Sturm-Liouville problem.

Proof. The first part of the statement follows at once from Theorem 3.2 provided condition (3.6) holds for

$$W(t) = \begin{pmatrix} 0 & f_\sigma(t) \\ f_\chi(t) - \lambda f_\rho(t) & 0 \end{pmatrix}.$$

If $\{t\}$ is an atom for ω then, by assumption (5.4), either $f_\sigma(t) = 0$ or $f_\chi(t) = f_\rho(t) = 0$. In both cases $W(t)$ satisfies $W(t)^2 = 0$.

The second part of the statement follows by noting that the characteristic equations (4.1) and (5.5) agree. □

We now express the numbers L^+ and L^- from (4.4), (4.5) involving r and s directly in terms of the given measures. To this end let us introduce the geometric mean $GM(\omega_1, \omega_2)$ for two finite nonnegative measures $\omega_1, \omega_2 : \mathcal{B} \to [0, \infty)$. We choose a nonnegative measure ω such that ω_1 and ω_2 are absolutely continuous with respect to ω. Let f_j be the Radon-Nikodym derivative of ω_j with respect to ω. Then we define

$$GM(\omega_1, \omega_2) := \int_{[a,b]} \sqrt{f_1 f_2}\, d\omega.$$

Note that $\sqrt{f_1 f_2}$ is integrable with respect to ω. It is easy to see that the definition of the geometric mean does not depend on the choice of ω. If we use the Lebesgue decomposition $\omega_2 = \omega_3 + \omega_4$ with ω_3 absolutely continuous with respect to ω_1 and ω_4 singular with respect to ω_1, and let g denote the Radon-Nikodym derivative of ω_3 with respect to ω_1, then we may write

$$GM(\omega_1, \omega_2) = \int_{[a,b]} \sqrt{g}\, d\omega_1.$$

This shows that the geometric mean depends only on the absolutely continuous component of ω_2 with respect to ω_1 (or the absolutely continuous component of ω_1 with respect to ω_2.)

Using the geometric mean, we can write L^+, L^- as follows

$$L^+ = GM(\sigma^+, \rho^+) - GM(\sigma^-, \rho^-), \tag{5.6}$$

$$L^- = GM(\sigma^+, \rho^-) - GM(\sigma^-, \rho^+), \tag{5.7}$$

where σ^+ and σ^- are the positive and negative variations of σ according to the Jordan decomposition of σ.

Now Theorems 4.2 and 5.1 yield the following theorem.

Theorem 5.2. *Assume* (5.4) *holds, and the number L^+ in* (5.6) *is positive. For sufficiently large n, there exists a real positive eigenvalue λ_n of the measure Sturm-Liouville problem that satisfies $\theta(d, \lambda_n) = \beta + n\pi$. For any choice of λ_n, we have*

$$\lim_{n\to\infty} n^{-2}\lambda_n = \pi^2 (L^+)^{-2}.$$

We have similar results when $L^+ < 0$, $L^- > 0$, $L^- < 0$. As in Section 4 we can say more under additional assumptions.

Theorem 5.3. *Assume the measure Sturm-Liouville problem is not degenerate, satisfies* (5.4) *and $\rho \geq 0$. Then all of its eigenvalues are real and equation* (4.3) *has at most one solution for every integer n.*

Proof. If $\rho \geq 0$ then $r \geq 0$, so the statement follows from Theorem 4.4 applied to the associated problem. □

We call $t \in [a,b]$ a *generalized zero* of $U(t) = U(t,\lambda)$ if $U(t)U(t+) \leq 0$. Generalized zeros form a closed set that can be decomposed into (closed) components.

Theorem 5.4. *Suppose* (5.4)*. For $\lambda \in \mathbb{R}$, the collection of components of the set of zeros of the function $u(x,\lambda)$ within $[0,d]$ has the same cardinality as the collection of components of the set of generalized zeros of $U(t,\lambda)$ within $[a,b]$. If $\sigma \geq 0$ and λ_n is a solution of* (4.3) *then the number of components of the set of generalized zeros of $U(t,\lambda_n)$ is equal to n if $\alpha \neq 0$ and $\beta \neq \pi$, equal to $n+1$ if either $\alpha = 0$ or $\beta = \pi$, and equal to $n+2$ if $\alpha = 0$ and $\beta = \pi$.*

Proof. Let X be the set of zeros of $u(x,\lambda)$, and let T be the set of generalized zeros of $U(t,\lambda)$ for some real λ. Then H defines a continuous map from X onto T which has the additional property that the inverse image $H^{-1}(K) \subset X$ is connected whenever K is connected. It is an easy exercise in topology to show that the map $J \mapsto H(J)$ establishes a one-to-one correspondence between the components of X and T. The second part of the statement now follows from Theorem 4.6 applied to the associated problem. □

Based on this theorem we may call a solution λ_n of (4.3) an eigenvalue with *oscillation count* n provided $\sigma \geq 0$.

As a simple example, consider $[a,b] = [0,1]$, ρ Lebesgue measure, $\chi = 0$ and $\sigma \geq 0$ with $\sigma(\{0\}) = 1$, $\sigma((0,1]) = 0$, $\alpha = \frac{5}{6}\pi$, $\beta = \frac{1}{2}\pi$. Then

$$U(0,\lambda) = \frac{1}{2},\ U(t,\lambda) = \frac{1}{2}(1-\sqrt{3}) \quad \text{for } t > 0,$$
$$V(t,\lambda) = -\frac{1}{2}\sqrt{3} - \frac{1}{2}\lambda(1-\sqrt{3})t.$$

The problem has only one eigenvalue $\lambda_1 = \sqrt{3}/(\sqrt{3}-1)$. According to Theorem 5.4 its oscillation count is 1. When we set $\omega = \rho + \sigma$ then the associated Sturm-Liouville problem has $[0,d] = [0,2]$, $q = 0$, $r(x) = 0$ for $x \in [0,1]$, $r(x) = 1$ for $x \in (1,2]$ and $s = 1 - r$. One may verify that $\theta(2,\lambda_1) = \beta + \pi$. In this example, $U(t,\lambda_1)$ has no zero at $t = 0$ but it has a generalized zero there. This is the reason why we avoid talking about "interior" generalized zeros in order to determine the oscillation count.

Suppose that $\rho \geq 0$. Let $\mathcal{K}$ be the collection of all maximal subintervals K of $[a,b]$ with $\rho(K) = 0$. The intervals in K may be open, closed or half-open. We only consider intervals of positive length in $\mathcal{K}$ except possibly $\{a\}$ and $\{b\}$. We declare $\{a\} \in \mathcal{K}$ if $\rho(\{a\}) = 0$, $\rho((a,e)) > 0$ for all $e \in (a,b)$ and $\sigma(\{a\}) \neq 0$. Similarly, we define the meaning of $\{b\} \in \mathcal{K}$. Let $\overline{K} = [a',b']$. We define (U_K, V_K) as the solution of (5.1), (5.2) with $\lambda = 0$ determined by the initial values

$$U(a') = 0, V(a') = 1 \quad \text{if } a,b \notin K,$$
$$U(a) = \sin\alpha, V(a) = \cos\alpha \quad \text{if } a \in K,$$
$$U(b+) = \sin\beta, V(b+) = \cos\beta \quad \text{if } a \notin K,\ b \in K.$$

Let $\tilde{m}_K$ be the number of components of the set of generalized zeros of U_K within $\overline{K}$. We define $m_K := \tilde{m}_K$ if $a \in K$, $\alpha \neq 0$ or if $a \notin K$, $b \in K$, $\beta \neq \pi$. In all other cases, let $m_K := \tilde{m}_K - 1$.

Theorem 5.5. *Assume* (5.4) *and* $\sigma, \rho \geq 0$. *Suppose the measure Sturm-Liouville problem is not degenerate and admits at least one eigenvalue. Then the minimal oscillation count* n^- *of eigenvalues is*

$$n^- = \sum_{K\in\mathcal{K}} m_K.$$

Proof. The minimal oscillation count of eigenvalues of the given measure Sturm-Liouville problem agrees with that for the eigenvalues of the associated Sturm-Liouville problem. Therefore, by Theorem 4.7, we have

$$n^- = \sum_{J\in\mathcal{J}} n_J. \tag{5.8}$$

Let K be any subinterval of $[a, b]$, and let J be the closure of the interval $\{x \in [0, d] : H(x) \in K\}$. By Lemma 3.1, we have $\rho(K) = \int_J r$. If $K \in \mathcal{K}$ then $J \in \mathcal{J}$, and if $J \in \mathcal{J}$ then $K \in \mathcal{K}$ or $K = \{t_0\}$ is a singleton with $h(t_0) \neq h(t_0+)$. Therefore, Theorem 5.5 will follow from (5.8) once we have shown that $m_K = n_J$ for all $K \in \mathcal{K}$, and $n_J = 0$ if K is a singleton different from $\{a\}$ and $\{b\}$. If K is such a singleton and $J \in \mathcal{J}$ then s, q, r are constant on J and either $r = q = 0$ or $s = 0$ by (5.4). In both cases u_J is affine linear on J and thus $n_J = 0$.

Now consider an interval $K = (a', b') \in \mathcal{K}$ with corresponding $J = [h(a'+), h(b')]$ in $\mathcal{J}$. Then $\rho(\{a'\}) \neq 0$ and $\rho(\{b'\}) \neq 0$. In particular, $c, d \notin J$, and thus (u_J, v_J) solves $u' = sv$, $v' = qu$ with initial conditions $u_J(h(a'+)) = 0$, $v_J(h(a'+)) = 1$. By (5.4), $s = 0$ on $[h(a'), h(a'+)]$ and on $[h(b'), h(b'+)]$. Hence u_J is constant on these intervals. In particular, $u_J = 0$ on $[h(a'), h(a'+)]$. Therefore, by Theorem 5.1, $U_K(t)$ is a constant multiple of $u_J(h(t))$. By Theorem 5.4, the number of components of the set of generalized zeros of U_K within $\overline{K}$ agrees with the number of components of zeros of u_J within $[h(a'), h(b'+)]$ and then also within J. Therefore, $m_K = n_J$. If K is closed or half-open, we see in a similar way that also $m_K = n_J$. □

The example after Theorem 5.4 shows why we have to allow singletons $\{a\}$ or $\{b\}$ in the collection $\mathcal{K}$. Analogously to Corollary 4.8 we obtain the following sufficient condition for $n^- = 0$.

Corollary 5.6. *In addition to the assumption of Theorem 5.5 suppose that*

$$\begin{gathered}\sigma(\{a\}) = \sigma(\{b\}) = 0,\\ \rho([a, e))\rho((e, b]) > 0 \text{ for all } e \in (a, b),\\ \rho(I) = 0 \text{ implies } |\chi|(I) = 0 \text{ for all subintervals } I \text{ of } [a, b].\end{gathered}$$

Then $n^- = 0$.

Theorem 5.7. *Suppose* (5.4) *and* $\rho, \sigma \geq 0$. *Then the measure Sturm-Liouville problem is degenerate or has only finitely many eigenvalues if and only there is a partition*

$$a = \tau_0 < \tau_1 < \cdots < \tau_m = b$$

such that

$$\rho((\tau_{i-1}, \tau_i))\sigma((\tau_{i-1}, \tau_i)) = 0 \quad \text{for all } i = 1, 2, \ldots, m.$$

Proof. Assume such a partition exists. Then we consider the partition of $[c,d]$ with partition points $h(\tau_i)$, and, if h is discontinuous at τ_i, $h(\tau_i+)$. Then, with $J=[h(\tau_{i-1}+),h(\tau_i)]$,

$$\int_J r \times \int_J s = \rho((\tau_{i-1},\tau_i))\sigma((\tau_{i-1},\tau_i)) = 0.$$

This is also true for the intervals $J=[h(\tau_i),h(\tau_i+)]$ by virtue of our assumption (5.4). Therefore, by Theorem 4.9, the associate Sturm-Liouville and thus also the measure Sturm-Liouville problem is either degenerate or has only finitely many eigenvalues. The proof of the converse statement is reduced to Theorem 4.9 in a similar manner. □

Let us summarize our results for a given measure Sturm-Liouville problem with $\rho,\sigma\ge 0$. First we check condition (5.4). If this condition fails we have given no results and it appears that hitherto this type of problem has not been considered in the literature. So let us assume that (5.4) holds. Next we verify if a partition of the type described in Theorem 5.7 exists. If such a partition exists the problem is degenerate or has only finitely many eigenvalues. Suppose that such a partition does not exist. Then the problem is not degenerate, its eigenvalues are real and can be listed as an infinite sequence

$$\lambda_{n^-} < \lambda_{n^-+1} < \lambda_{n^-+2} < \dots$$

converging to infinity. The eigenvalue λ_n has oscillation count n and n^- is the minimal oscillation count which can be determined from Theorem 5.5. According to Theorem 5.2, the eigenvalues satisfy the asymptotic formula

$$\lim_{n\to\infty} n^{-2}\lambda_n = \Big(\frac{\pi}{GM(\sigma,\rho)}\Big)^2. \tag{5.9}$$

If $GM(\sigma,\rho)=0$ then the right hand side has to be interpreted as $+\infty$. This case has been investigated in several papers; see [9, 10, 11, 20].

The vibrating string problem [12], [13] is a special case of the measure Sturm-Liouville problem. We take $\sigma=\nu$ (Lebesgue measure), $\chi=0$ and $\rho(E)\ge 0$ is the mass of the string over $E\in\mathcal{B}$. In this case, if (U,V) is a solution of (5.1), (5.2) then U is continuous and has left-hand derivatives on $(a,b]$ and right-hand derivatives on $[a,b)$. Moreover, $V(t)$ agrees with the left-hand derivative of U at t for all $t\in(a,b]$. Since σ has no atoms assumption (5.4) holds. Choosing $\omega:=\rho+\nu$ we transform the vibrating string problem to Atkinson's eigenvalue problem. Unless the mass of the string is concentrated at finitely many points (string with beads) we have infinitely many eigenvalues by Theorem 5.7. Under the assumption that

$$\rho([a,e))>0, \rho((e,b])>0 \quad \text{for all } e\in(a,b)$$

we see from Corollary 5.6 that the minimal oscillation count is $n^-=0$. For the asymptotic formula (5.9) for the vibrating string see [15], [16], [13, (11.7)]. Gantmacher and Krein [12, Chapter 4] investigate the oscillations of vibrating strings by the method of oscillation kernels. This method is very different from the method used in this paper, namely, transformation to Atkinson's problem and use of the Prüfer angle.

If σ is Lebesgue measure and $\rho\ge 0$ the asymptotic formula (5.9) is also proved by McKean and Ray [18].

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Department of Mathematical Sciences, University of Wisconsin–Milwaukee, P. O. Box 413, Milwaukee, WI 53201 USA

E-mail address: volkmer@csd.uwm.edu