Electronic Journal of Differential Equations, Vol. 2005(2005), No. 63, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

QUASI-GEOSTROPHIC EQUATIONS WITH INITIAL DATA IN BANACH SPACES OF LOCAL MEASURES

SADEK GALA

Abstract. This paper studies the well posedness of the initial value problem for the quasi-geostrophic type equations

$$\partial_t \theta + u\nabla\theta + (-\Delta)^\gamma \theta = 0 \quad \text{on } \mathbb{R}^d \times]0,+\infty[$$

$$\theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}^d$$

where $0 < \gamma \leq 1$ is a fixed parameter and the velocity field $u = (u_1, u_2, \dots, u_d)$ is divergence free; i.e., $\nabla u = 0$). The initial data θ_0 is taken in Banach spaces of local measures (see text for the definition), such as Multipliers, Lorentz and Morrey-Campanato spaces. We will focus on the subcritical case $1/2 < \gamma \leq 1$ and we analyse the well-posedness of the system in three basic spaces: $L^{d/r,\infty}$, $\dot{X}_r$ and $\dot{M}^{p,d/r}$, when the solution is global for sufficiently small initial data. Furtheremore, we prove that the solution is actually smooth. Mild solutions are obtained in several spaces with the right homogeneity to allow the existence of self-similar solutions.

1. Introduction

We analyse the well-posedness of initial value problems for the quasi-geostrophic equations in the subcritical case. Mild solutions are obtained in several spaces with the right homogeneity to allow the existence of self-similar solutions. Singularities, global existence and long time behavior for models in fluid mechanics have become an important topics in the mathematical community in the last decades. Understanding these features in incompressible Navier-Stockes (NS) and Euler equations in 3D in yet unsolved. Lower dimentional models have been deduced, not only for the simplification that might bring up in the mathematical approaches, but also because of the great resemblance that some of these models have with respect to the original incompressible NS equations and the mathematical insight their study may produce. The dissipative 2DQG equations are derived from general quasi-geostrophic equations (see [14]) in the special case of constant potential vorticity and constant buoyancy frequency (see [4]). Quasi-geostrophic equations are not only important as a simplification of 3DNS but also since they appear as a natural reduction for vertically stratified flows (see [14]). Well posedness in several spaces and long time behavior of the solutions of the dissipative QG equations in different

2000 *Mathematics Subject Classification.* 35Q35, 35A07.
Key words and phrases. Quasi-geostrophic equation; local spaces; mild solutions; self-similar solutions.

Submitted April 24, 2005. Published June 15, 2005.

cases have been studied in ([3], [4], [5], [10], [17], [18]). The objective of this paper is to study the well posedness of the initial value problem for 2-dimensional of the quasi-geostrophic type equations

$$\partial_t \theta + u\nabla\theta + (-\Delta)^\gamma \theta = 0 \quad \text{on } \mathbb{R}^d \times]0, +\infty[, \tag{1.1}$$

$$\theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}^d \tag{1.2}$$

where $0 < \gamma \leq 1$ is a fixed parameter. The velocity field $u = (u_1, u_2, \dots, u_d)$ is divergence free; i.e., $\nabla u = 0$) and determined from the potential temperature θ by linear combinations of Riesz transforms; i.e.,.

$$u_k = \sum_{j=1}^{d} a_{jk} \mathcal{R}_j \theta, \quad 1 \leq k \leq d$$

where $\mathcal{R}_j = \partial_j (-\Delta)^{-\frac{1}{2}}$. Let us remark that suitable choices of a_{jk} assures that the velocity field is divergence free, that we will assume throughout this paper. The Riesz potential operator $(-\Delta)^\gamma$ is defined as usual through the Fourier transform as

$$\widehat{(-\Delta)^\gamma f}(\xi) = |\xi|^{2\gamma} \widehat{f}(\xi)$$

It is well-know that QG equation is very similar to the three dimensional Navier-Stokes equations (see [4]). Besides its similarity to the three dimensional fluid equations, (see [14]). The case $\gamma > \frac{1}{2}$ is called sub-critical, and the case $\gamma = \frac{1}{2}$ is critical, while the case $0 \leq \gamma < \frac{1}{2}$ is super-critical, respectively. Our aim is to show existence and uniqueness and regularity of solutions for the quasi-geostrophic equations in the sub-critical case (1.1)-(1.2) when the initial data is taken in Banach spaces of local measures, such as Multipliers, Lorentz and Morrey-Campanato spaces.

2. Shift-invariant spaces of local measures

We consider in this paper a special class of shift-invariant spaces of distributions was introduced by Lemarié-Rieusset in his work [12]. The spaces which are invariant under pointwise multiplication with bounded continuous functions.

Definition 2.1. (A) *A shift-invariant Banach spaces of test functions is a Banach space E such that we have the continuous embeddings $\mathcal{D}(\mathbb{R}^d) \subset E \subset \mathcal{D}'(\mathbb{R}^d)$ and so that*

(a) *for all $x_0 \in \mathbb{R}^d$ and for all $f \in E$, $f(x - x_0) \in E$ and*

$$\|f\|_E = \|f(x - x_0)\|_E$$

(b) *for all $\lambda > 0$, there exists $C_\lambda > 0$ so that for all $f \in E$, $f(\lambda x) \in E$ and*

$$\|f(\lambda x)\|_E \leq C_\lambda \|f\|_E$$

(c) *$\mathcal{D}(\mathbb{R}^d)$ is dense in E.*

(B) *A shift-invariant Banach spaces of distributions is a Banach space E which is the topological dual of a shift-invariant Banach of test functions space $E^{(0)}$ of smooth elements of E is defined as the closure of $\mathcal{D}(\mathbb{R}^d)$ in E.*

(C) *A shift-invariant Banach space of local measures is a shift-invariant Banach space of distributions E so that for all $f \in E$ and all $g \in \mathcal{S}(\mathbb{R}^d)$, we have $fg \in E$ and*

$$\|fg\|_E \leq C_E \|f\|_E \|g\|_{L^\infty}$$

where C_E is a positive constant which depends neither on f nor on g.

Remark 2.2. An easy consequence of hypothesis (a) is that a shift-invariant Banach space of test functions E satisfies $\mathcal{S}(\mathbb{R}^d) \subset E$ and a consequence of hypothesis (b) is that $E \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$. Similarly, we have for a shift-invariant Banach space of distributions E that $\mathcal{S}(\mathbb{R}^d) \subset E^{(0)} \subset E \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$. In particular, $E^{(0)}$ is a shift-invariant Banach space of test functions.

A shift-invariant Banach spaces of distributions are adapted to convolution with integrable kernels.

Lemma 2.3 (Convolution in shift-invariant spaces of distributions)**.** *If E is a shift-invariant Banach space of test functions or of distributions and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then for all $f \in E$, we have $f * \varphi \in E$ and*

$$\|f * \varphi\|_E \leq \|f\|_E \|\varphi\|_{L^1} \tag{2.1}$$

Moreover, the convolution may be extended into a bounded bilinear operator from $E \times L^1$ to E and we have for all $f \in E$ and for all $g \in L^1$, the inequality

$$\|f * g\|_E \leq \|f\|_E \|g\|_{L^1}$$

The proof of the above lemma can be found in [12]. Now, we introduce the functional spaces relevant to our study of solutions of the Cauchy problem for system (1.1)-(1.2), we list some facts about convolution and we discuss the notion of solution in these spaces.

2.1. **Pointwise multipliers $\dot{X}_r$.** In this section, we give a description of the multiplier space $\dot{X}^r$ introduced recently by Lemarié-Rieusset in his work [12]. The space $\dot{X}^r$of pointwise multipliers which map L^2 into $\dot{H}^{-r}$ is defined as follows:

Definition 2.4. *For $0 \leq r < d/2$, we define the homogeneous space $\dot{X}^r$ by*

$$\dot{X}^r = \{f \in L^2_{\rm loc} : \forall g \in \dot{H}^r \quad fg \in L^2\}$$

where we denote by $\dot{H}^r(\mathbb{R}^d)$ the completion of the space $\mathcal{D}(\mathbb{R}^d)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{r/2} u\|_{L^2}$.

The norm in $\dot{X}^r$ is given by the operator norm of pointwise multiplication

$$\|f\|_{\dot{X}^r} = \sup_{\|g\|_{\dot{H}^r} \leq 1} \|fg\|_{L^2}$$

Similarly, we define the nonhomogeneous space X^r for $0 \leq r < d/2$ equipped with the norm

$$\|f\|_{X^r} = \sup_{\|g\|_{H^r} \leq 1} \|fg\|_{L^2}$$

We have the following homogeneity properties: For all $x_0 \in \mathbb{R}^d$,

$$\begin{gathered}
\|f(x + x_0)\|_{X^r} = \|f\|_{X^r} \\
\|f(x + x_0)\|_{\dot{X}^r} = \|f\|_{\dot{X}^r} \\
\|f(\lambda x)\|_{X^r} \leq \frac{1}{\lambda^r} \|f\|_{X^r}, \quad 0 < \lambda \leq 1 \\
\|f(\lambda x)\|_{\dot{X}^r} \leq \frac{1}{\lambda^r} \|f\|_{\dot{X}^r}, \quad \lambda > 0.
\end{gathered}$$

The following imbeddings hold

$$L^{\frac{d}{t}} \subset X^r, \quad 0 \le r < \frac{d}{2},\ 0 \le t \le r.$$
$$L^{\frac{d}{r}} \subset \dot{X}^r, \quad 0 \le r < \frac{d}{2}\,.$$

We now turn to another way of introducing capacity.

Capacitary measures and capacitary potentials. In this section, we give the notion of capacity which will develop in term of Riesz and Bessel potentials.

Definition 2.5. *Let $d \ge 2$ and $x \in \mathbb{R}^d$. The Riesz kernel is defined by*

$$k_\alpha(x) = C(d,\alpha)|x|^{\alpha-d}, \quad 0 < \alpha < d,$$

where the constant $C(d,\alpha)$ is chosen so that $k_\alpha * k_\beta = k_{\alpha+\beta}$.

The natural question arises: When is $k_\alpha * k_\beta$ defined? There are obvious problems at infinity. Assume $|x| \ll R$, where R is a constant and $x \neq 0$. We have then

$$\begin{aligned}\int_{|y|>R} \frac{dy}{|y|^{d-\alpha}|x-y|^{d-\beta}} &\approx \int_{|y|>R} \frac{dy}{|y|^{2d-\alpha-\beta}} \\ &\approx \int_R^\infty t^{d-1-2d+\alpha+\beta} dt = \int_R^\infty t^{\alpha+\beta-d-1} dt,\end{aligned}$$

which is convergent if and only if $\alpha+\beta < d$.The Bessel kernel G_r, $r > 0$ is defined as that function whose Fourier transform is

$$\widehat{G_r}(x) = (2\pi)^{-d/2}(1+|x|^2)^{-r/2}$$

It is known that G_r is a positive, integrable function which is analytic except at $x = 0$. Similar to the Riesz kernel, we have

$$G_r * G_s = G_{r+s}, \quad r, s \ge 0\,.$$

Remark 2.6. The Riesz potential leads to many important applications, but for the purpose of investigating Sobolev functions, the Bessel potential is more suitable. For an analysis of the Bessel kernel, we refer the reader to [15].

The Riesz capacity $\operatorname{cap}(e; \dot{H}^r)$ of a compact set $e \subset \mathbb{R}^d$ is defined by (see [1])

$$\operatorname{cap}(e; \dot{H}^r) = \inf\{\|u\|^2_{\dot{H}^r} : u \in C_0^\infty(\mathbb{R}^d),\ u \ge 1 \text{ sur } e\}$$

The Bessel capacity $\operatorname{cap}(e; H^r)$ of a compact set $e \subset \mathbb{R}^d$ is defined in a similar way, with the kernel k_r replaced by G_r. Since $G_r(x) \le k_r(x)$ $(x \in \mathbb{R}^d)$, it follows immediately from definitions that for $0 < r < d$, there exists a constant $C(d,r)$ such that

$$\operatorname{cap}(e; \dot{H}^r) \le C(d,r)\operatorname{cap}(e; H^r).$$

A brief outline of the theory of A_p weights is given in this section. A complete expositions can be found in the monographs by Garcia-Cuerva and Rubio de Francia [8].

A_p weights. The class of A_p weights was introduced by Muckenhoupt in [13], where he showed that the A_p weights are precisely those weights w for which the Hardy-Littlewood maximal operator is bounded from $L^p_w(\mathbb{R}^d)$ to $L^p_w(\mathbb{R}^d)$, where $1 < p < \infty$ and from $L^1_w(\mathbb{R}^d)$ to weak-$L^1_w(\mathbb{R}^d)$, when $p = 1$. We begin by defining the class of A_p weights. Let $1 \le p < \infty$. A weight w is said to be an A_p weight, if there exists a positive constant C_p such that, for every ball $B \subset \mathbb{R}^d$,

$$\Big(\frac{1}{|B|}\int_B w(x)dx\Big)\Big(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}dx\Big)^{p-1} \le C_p, \tag{2.2}$$

if $p > 1$, or

$$\frac{1}{|B|}\int_B w(x)dx \le K \operatorname*{ess\,inf}_{x\in B} w(x)), \tag{2.3}$$

if $p = 1$. The infinimum over all such constants C is called the A_p constant of w. We denote by A_p, $1 \le p < \infty$, the set of all A_p weights. Below we list some simple, but useful properties of A_p weights.

Proposition 2.7. (1) *If $w \in A_p$, $1 \le p < \infty$, then since $w(x)^{-\frac{1}{p-1}}$ is locally integrable, when $p > 1$, and $\frac{1}{w}$ is locally bounded, when $p = 1$, we have $L^p(w(x)dx) \subset L^1_{\text{loc}}(\mathbb{R}^d)$.*

(2) *Note that if w is a weight, then, by writing $1 = w^{\frac{1}{p}}w^{-\frac{1}{p}}$, Hölder's inequality implies that, for every ball B*

$$1 \le \Big(\frac{1}{|B|}\int_B w(x)dx\Big)\Big(\frac{1}{|B|}\int_B w(x)^{\frac{-1}{p-1}}dx\Big)^{p-1}$$

when $p > 1$ and similarly for the expression that gives the A_1 condition.It follows that if $w \in A_p$, then the constant of w is ≥ 1.

(3) *If $w \in A_p$, where $1 < p < \infty$, then $w^{-\frac{1}{p-1}} \in A_{p'}$, and conversely.*

(4) *It is not so difficult to see that a weight $w \in A_1$ if and only if $Mw(x) \le A_1 w(x)$ a.e.*

(5) *It follows that if $w \in A_1$, then there is a constant C such that*

$$w(x) \ge \frac{C}{(1+|x|)^d}$$

for a.e. $x \in \mathbb{R}^d$. In fact, if $x \in \mathbb{R}^d$ and $R = 2\max(1, |x|)$, then

$$\frac{1}{R^d}\int_{B(R,x)} w(y)dy \ge \frac{2^{-d}}{(1+|x|)^d}\int_{B(1,0)} w(y)dy$$

so $Mw(x) \ge C(1+|x|)^{-d}$ a.e.

(6) *If w is a weight and there exist two positive constants C and D such that $C \le w(x) \le D$, for a.e. $x \in \mathbb{R}^d$, then obviously $w \in A_p$ for $1 \le p < \infty$.*

We will need the following theorem, which shows that many operators of classical analysis are bounded in the space of multipliers.

Theorem 2.8. *Let $0 \le r < d/2$. Suppose that a function $h \in L^2_{\text{loc}}$ satisfies*

$$\int_e |h(x)|^2 dx \le C\,\text{cap}(e) \tag{2.4}$$

for all compact sets e with $\operatorname{cap}(e) = \operatorname{cap}(e; H^r)$. *Suppose that, for all weights $\rho \in A_1$,*

$$\int_{\mathbb{R}^d} |g(x)|^2 \rho dx \leq K \int_{\mathbb{R}^d} |h(x)|^2 \rho dx \tag{2.5}$$

with a constant K depending only on d and the constant A in the Muckenhoupt condition. Then

$$\int_e |g(x)|^2 dx \leq C \operatorname{cap}(e)$$

for all compact sets e with $C = C(d, r, K)$.

Similar results are obtained for the homogeneous multipliers space $\dot{X}^r$. To show this theorem, we need some facts from the equilibrium potential of a compact set e of positive capacity [1]. The equilibrium potential of a measure $\mu \in M^+$ is defined by

$$P = P_e = G_r(G_r \mu).$$

Lemma 2.9 ([1]). *For any compact set $e \subset \mathbb{R}^d$, there exists a measure $\mu = \mu_e$ such that*

(i) $\operatorname{supp} \mu \subset e$
(ii) $\mu(e) = \operatorname{cap}(e, H^r)$
(iii) $\|G_r \mu\|_{L^2}^2 = \operatorname{cap}(e, H^r)$
(iv) $P_e(x) \geq 1$ *quasi-everywhere on e*
(v) $P_e(x) \leq K = K(d, r)$ *on* $\mathbb{R}^d$
(vi) $\operatorname{cap}\{P_e \geq t\} \leq A t^{-1} \operatorname{cap}(e, H^r)$ *for all $t > 0$ and the constant is independent of e.*

The measure μ_e associated with e is called the capacitary (equilibrium) measure of e. We will also need the asymptotic (see [1])

$$\begin{gathered} G_r(x) \simeq |x|^{r-d}, \quad \text{if } d \geq 3,\ |x| \to 0; \\ G_r(x) \simeq |x|^{\frac{r-d}{2}} e^{-|x|}, \quad \text{if } d \geq 2,\ |x| \to +\infty\,. \end{gathered} \tag{2.6}$$

Sometimes, it will be more convenient to use a modified kernel

$$\widetilde{G}_r(x) = \max(G_r(x), 1)$$

which does not have the exponential decay at ∞. Obviously, both G_r and $\widetilde{G}_r$ are positive non-increasing radial kernels. Moreover, $\widetilde{G}_r$ has the doubling property

$$\widetilde{G}_r(2s) \leq \widetilde{G}_r(s) \leq c(d) \widetilde{G}_r(2s)$$

The corresponding modified potential is

$$\widetilde{P}(x) = \widetilde{G}_r * \mu(x)$$

The rest of the proof of theorem 2.8 is based on the following proposition.

Proposition 2.10. *Let $d \geq 2$ and let $0 < \delta < \frac{d}{d-2r}$. Then $\widetilde{P}$ lies in the Muckenhoupt class A_1 on $\mathbb{R}^d$, i.e.,*

$$M\big(\widetilde{P}^{\delta}(x)\big) \leq C(\delta, d) \widetilde{P}^{\delta}(x), \ dx \ p.p$$

where M denotes the Hardy-Littlewood maximal operator on $\mathbb{R}^d$, and the corresponding A_1-bound $C(\delta, d)$ depends only on d and δ.

Proof. Let $k : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function which satisfies the doubling condition
$$k(2s) \le ck(s), \quad s > 0\,.$$
It is easy to see that the radial weight $k(|x|) \in A_1$ if and only if
$$\int_0^R k^\delta(t)t^{d-1}dt \le cR^d k(R), \quad R > 0 \tag{2.7}$$
Moreover, the A_1-bound of k is bounded by a constant which depends only on C in the preceding estimate and the doubling constant c (see [16]). It follows that
$$\widetilde{G}_r(s) \simeq |s|^{r-d} \quad \text{if } d \ge 3 \text{ for } 0 < s < 1$$
and
$$\widetilde{G}_r(s) \simeq 1 \quad \text{for } s \ge 1\,.$$
Hence, $k(|s|) = \widetilde{G}_r^{\delta}(s)$ is a radial non-increasing kernel with the doubling property. By Jensen' inequality, we have
$$\widetilde{G}_r^{\delta_1} \in A_1 \quad \text{implies} \quad \widetilde{G}_r^{\delta_2} \in A_1 \text{ if } \delta_1 \ge \delta_2$$
Clearly (2.7) holds if and only if $0 < \delta < \frac{d}{d-2r}$. Hence, without loss of generality, we assume $1 \le \delta < \frac{d}{d-2r}$. Then by Minkowski's inequality and the A_1-estimate for $\widetilde{G}_r^{\delta}$ established above, it follows
$$M(\widetilde{P}^{\delta}(x)) \le M\big((\widetilde{G}_r^{\delta})^{\frac{1}{\delta}} * \mu(x)\big)^\delta \le C(\delta,d)(\widetilde{G}_r * \mu)^\delta(x) = C(\delta,d)\widetilde{P}^{\delta}(x).$$
□

We are now in a position to prove theorem 2.8.

Proof of Theorem 2.8. Suppose v_e is the capacitary measure of $e \subset \mathbb{R}^d$ and let $\varphi = P$ is its potential. Then, by lemma 2.9, we have

- (i) $\varphi(x) \ge 1$ quasi-everywhere on e
- (ii) $\varphi(x) \le B = B(d,r)$ for all $x \in \mathbb{R}^d$
- (iii) $\operatorname{cap}\{\varphi \ge t\} \le Ct^{-1}\operatorname{cap}(e)$ for all $t > 0$ with the constant C is independent of e.

Now, it follows from a proposition 2.10 that $\varphi^\delta \in A_1$. Hence, by (2.5),
$$\int_{\mathbb{R}^d} |g(x)|^2\varphi^\delta dx \le K\int_{\mathbb{R}^d} |h(x)|^2\varphi^\delta dx$$
Applying this inequality with (i) and (ii), we get
$$\int_e |g(x)|^2dx \le \int_{\mathbb{R}^d} |g(x)|^2\varphi^\delta dx \le C\int_{\mathbb{R}^d} |h(x)|^2\varphi^\delta dx = C\int_0^B\int_{\varphi\ge t} |h(x)|^2dxt^{\delta-1}dt\,.$$
By (2.4) and (iii),
$$\int_{\varphi\ge t} |h(x)|^2dx \le Ccap\,\{\varphi \ge t\} \le \frac{C}{t}\operatorname{cap}(e)\,.$$
Hence,
$$\int_e |g(x)|^2dx \le C\int_0^B t^{-1}\operatorname{cap}(e)t^{\delta-1}dt = C\operatorname{cap}(e)\int_0^B t^{\delta-2}dt\,.$$

Clearly, for all $0 \le r < d/2$, we can choose $\delta > 1$ so that $0 < \delta < \frac{d}{d-2r}$. Then

$$\int_0^B t^{\delta-2} dt = \frac{B^{\delta-1}}{\delta-1} < \infty$$

which concludes $\int_e |g(x)|^2 dx \le C \operatorname{cap}(e)$. $\square$

We will need the boundedness of the Riesz transforms $\mathcal{R}_j f = f * \frac{x_j}{|x|^{d+1}}$ $(j = 1, 2, \dots)$ in the spaces of functions defined by the capacitary condition (2.4).

Corollary 2.11. *Let $0 \le r < d/2$. Then*

$$\sup_e \frac{\int_e |\mathcal{R}_j f|^2 dx}{\operatorname{cap}(e, \dot{H}^r)} \le C \sup_e \frac{\int_e |f|^2 dx}{\operatorname{cap}(e, \dot{H}^r)}, \quad (j = 1, 2, \dots),$$

where the suprema are taken over all compact sets in $\mathbb{R}^d$.

Proposition 2.12. *If $f \in \dot{X}^r$, then $e^{-t(-\Delta)^\gamma} f \in \dot{X}^r$.*

Proof. Let $g \in \dot{H}^r$. Then

$$\begin{aligned} \|e^{-t(-\Delta)^\gamma} f g\|_{L^2}^2 &= \int \Big| \int f(x-y) g(x) k_\gamma(y) dy \Big|^2 dx \\ &\le \int \int |f(x-y) g(x)|^2 k_\gamma(y) dy\, dx \\ &\le \int k_\gamma(y) \int |f(u) g(u+y)|^2 dy du \le C \|g\|_{\dot{H}^r} \end{aligned}$$

since $\dot{H}^r$ is invariant under translation.

$$x \to \varphi\big(\frac{x - x_0}{\sqrt{t}}\big) \in \dot{H}^r$$

with a norm $= t^{\frac{d}{4} - \frac{r}{2}} \|\varphi\|_{\dot{H}^r}$. $\square$

2.2. **Morrey-Campanato spaces.** We first recall the definition [11]: For $1 < p \le q \le +\infty$, the Morrey-Campanato space is the set

$$M_{p,q} = \big\{ f \in L^p_{\rm loc} : \|f\|_{M_{p,q}} = \sup_{x \in I\!R^d} \sup_{0 < R \le 1} R^{d/q - d/p} \|f(y) 1_{B(x,R)}(y)\|_{L^p(dy)} < \infty \big\} \tag{2.8}$$

Let us define the homogeneous Morrey-Campanato spaces $\dot{M}_{p,q}$ for $1 < p \le q \le +\infty$ by

$$\|f\|_{\dot{M}_{p,q}} = \sup_{x \in \mathbb{R}^d} \sup_{R > 0} R^{d/q - d/p} \Big(\int_{B(x,R)} |f(y)|^p dy \Big)^{1/p} \tag{2.9}$$

It is easy to check the following properties

$$\|f(\lambda x)\|_{M_{p,q}} = \frac{1}{\lambda^{\frac{d}{q}}} \|f\|_{M_{p,q}}, \quad 0 < \lambda \le 1.$$

$$\|f(\lambda x)\|_{\dot{M}_{p,q}} = \frac{1}{\lambda^{\frac{d}{q}}} \|f\|_{\dot{M}_{p,q}}, \ \lambda > 0$$

We shall assume the following classical results [11].

(a) For $1 \le p \le p'$, $p \le q \le +\infty$ and for all functions f in $\dot{M}_{p,q} \cap L^\infty$,

$$\|f\|_{\dot{M}_{p', q\frac{p'}{p}}} \le \|f\|_{L^\infty}^{1 - \frac{p}{p'}} \|f\|_{\dot{M}_{p,q}}^{\frac{p}{p'}}.$$

(b) For p,q,p',q' so that $\frac{1}{p}+\frac{1}{p'}\le 1$, $\frac{1}{q}+\frac{1}{q'}\le 1$, $f\in \dot M_{p,q}$, $g\in \dot M_{p',q'}$. Then $fg\in \dot M_{p",q"}$ with $\frac{1}{p}+\frac{1}{p'}=\frac{1}{p"}$, $\frac{1}{q}+\frac{1}{q'}=\frac{1}{q"}$.
(c) For $1\le p\le d$ and all $\lambda>0$, we have $\lambda f(\lambda x)\|_{\dot M_{p,d}}=\|f\|_{\dot M_{p,d}}$.
(d) If $p'<p$, then $\dot M_{p,q}\subset M_{p,q}$ and $\dot M_{p,q}\subset M_{p',q}$
(e) If $q_2<q_1$, then $M_{p,q_1}\subset M_{p,q_2}$ and $L^q=\dot M_{q,q}\subset \dot M_{p,q}$, $p\le q$

We have the following comparison between and Morrey-Campanato spaces multiplier spaces.

Proposition 2.13. *For $0\le r<d/2$, we have*
$$X^r\subseteq M_{2,\frac{d}{r}}\quad \textit{and}\quad \dot X^r\subseteq \dot M_{2,\frac{d}{r}}\,.$$

Proof. Let $f\in X^r$, $0<R\le 1$, $x_0\in\mathbb{R}^d$ and $\phi\in\mathcal{D}$, $\phi\equiv 1$ on $B(\frac{x_0}{R},1)$. We have
$$\begin{aligned}
R^{r-\frac{d}{2}}\Big(\int_{|x-x_0|\le R}|f(x)|^2dx\Big)^{1/2}&=R^r\Big(\int_{|y-\frac{x_0}{R}|\le 1}|f(Ry)|^2dy\Big)^{1/2}\\
&\le R^r\Big(\int_{y\in\mathbb{R}^d}|f(Ry)\phi(y)|^2dy\Big)^{1/2}\\
&\le R^r\|f(Ry)\|_{X^r}\|\phi\|_{H^r}\\
&\le \|f(y)\|_{X^r}\|\phi\|_{H^r}\\
&\le C\|f(y)\|_{X^r}.
\end{aligned}$$
We observe that the same proof is valid for homogeneous spaces. □

We will need a result concerning the Calderon-Zygmund-type integral operators on Morrey-Campanto spaces. The Riesz transform is a particular example of these types of singular integral operators.

Lemma 2.14. *The Riesz transform $\mathcal{R}_j=\partial_j(-\Delta)^{-1/2}$, $j=1,2,\dots,d$ is continuous in $\dot M_{p,q}$ for $1<p<\infty$ and $1<q<\infty$.*

A proof for this lemma can be found in [11].

2.3. **Lorentz spaces: $L^{p,q}$.** Associated with a function f, we define its distribution function
$$\lambda_f(s)=|\left\{x\in\mathbb{R}^d:|f(x)|>s\right\}|$$
where $s>0$. Given a real function $\lambda_f(s)$, we define its rearrangement $f^*(t)$ as
$$f^*(t)=\inf\left\{s>0:\lambda_f(s)\le t\right\},\, t>0$$
It is easy to check that f^* and $\lambda_f(s)$ are non-negative and non-increasing functions. Moreover, if λ_f is strictly decreasing and continuous, then f^* is the inverse function of λ_f and both f^* and f have the same distribution function. From this fact, we deduce that
$$\begin{aligned}
\int_{\mathbb{R}^d}|f(x)|^pdx=\int_{\mathbb{R}^d}\int_0^{|f(x)|}pt^{p-1}dtdx&=\int_0^\infty\int_{\mathbb{R}^d}\kappa_{\lambda_f(t)}pt^{p-1}dtdx\\
&=\int_0^\infty pt^{p-1}\lambda_f(t)dt=\int_0^\infty pt^{p-1}\lambda_{f^*}(t)dt\\
&=\int_0^\infty(t^{\frac{1}{p}}f^*(t))^p\frac{dt}{t}
\end{aligned}\tag{2.10}$$
We may now introduce the Lorentz spaces.

The Lorentz spaces $L^{p,q}(\mathbb{R}^d)$ is defined as the set of all functions f such that $\|f\|^*_{L^{p,q}} < +\infty$, with

$$\|f\|^*_{L^{p,q}} = \begin{cases} \Big(\frac{q}{p}\int_0^\infty \big[t^{\frac{1}{p}} f^*(t)\big]^q \frac{dt}{t}\Big)^{\frac{1}{q}} & \text{if } 0 < p < +\infty,\ 0 < q < +\infty \\ \sup_{t>0} \big[t^{\frac{1}{p}} f^*(t)\big] & \text{if } 0 < p < +\infty,\ q = +\infty \end{cases}$$

We observe that $L^{p,p} = L^p$, $L^{p,\infty}$ are called the Marcinkiewicz spaces or weak-L^p spaces. Moreover, $L^{p,q_1} \subset L^{p,q_2}$ for $0 < q_1 \le q_2 \le +\infty$. The quantity $\|f\|^*_{L^{p,q}}$ give a natural topology for $L^{p,q}(\mathbb{R}^d)$ such that $L^{p,q}(\mathbb{R}^d)$ is a topological vector space. However, the triangle inequality is not true for $\|f\|^*_{L^{p,q}}$. As a natural way of metrizing the space $L^{p,q}(\mathbb{R}^d)$ is to define

$$f^{**}(t) = \frac{1}{t}\int_0^\infty f^*(s)ds \quad \text{for } t > 0$$

which can be computed [H] as

$$f^{**}(t) = \sup_{m(E)\ge t} \Big[\frac{1}{m(E)}\int_E |f(x)|dx\ \Big].$$

Hence, we define the norm

$$\|f\|_{L^{p,q}} = \begin{cases} \Big(\frac{q}{p}\int_0^\infty \big[t^{\frac{1}{p}} f^{**}(t)\big]^q \frac{dt}{t})^{1/q}, & \text{if } 1 < p < +\infty,\ 1 \le q < +\infty \\ \sup_{t>0} \big[t^{\frac{1}{p}} f^{**}(t)\big], & \text{if } 1 < p < +\infty,\ q = +\infty \end{cases}$$

The spaces $L^{p,q}$ endowed with the norm $\|f\|_{L^{p,q}}$ are Banach spaces and

$$\|f\|^*_{L^{p,q}} \le \|f\|_{L^{p,q}} \le \frac{p}{p-1}\|f\|^*_{L^{p,q}}.$$

An alternative definition of the norm $\|f\|_{L^{p,\infty}}$ is

$$\|f\|_{L^{p,\infty}} = \sup_{t>0} t|\{x \in \mathbb{R}^d : |f(x)| > t\}|^{1/p}$$

Lorentz spaces have the same scaling relation as L^p spaces, i.e., for all $\lambda > 0$, we have

$$\|f(\lambda x)\|_{L^{p,q}} = \lambda^{-d/p}\|f\|_{L^{p,q}}$$

where $1 \le p < +\infty$, $1 \le q \le +\infty$. We first need an interpolation result in Lorentz spaces [12].

Lemma 2.15 (Interpolation of linear operator)**.** *Let p_0, $p_1 \in [1, +\infty]$ with $p_0 \ne p_1$. Let $\theta \in]0, 1[$ and let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. If T is a linear operator defined from $L^{p_0} + L^{p_1}$ to Banach space E such that $T \in \mathcal{L}(L^{p_0}, E)$ (with the operator norm M_0) and $T \in \mathcal{L}(L^{p_1}, E)$ (with the operator norm M_1), then T is bounded from $L^{p,\infty}$ to E and the operator norm M is controlled by*

$$M \le C(p_0, p_1,\ \theta)M_0^{1-\theta}M_1^{\theta}.$$

Proof. We just write $f = g + h$ with

$$\big(\frac{M_0}{M_1}\big)^{\theta}\|g\|_{L^{p_0}} + \big(\frac{M_0}{M_1}\big)^{\theta-1}\|h\|_{L^{p_1}} \le C\|f\|_{L^{p,\infty}}$$

Then, we have $Tf = Tg + Th$ where

$$\begin{aligned}\|Tf\|_E &\le \|Tg\|_E + \|Th\|_E \\ &\le C\|f\|_{L^{p,\infty}}(M_0(\frac{M_0}{M_1})^{-\theta} + M_1\big(\frac{M_0}{M_1}\big)^{1-\theta}) \\ &= 2CM_0^{1-\theta}M_1^{\theta}\|f\|_{L^{p,\infty}}\end{aligned}$$

which proves the lemma. □

Now, we deal with the continuity of operators in Lorentz spaces and in particular continuity of the Riesz transform in $L^{p,\infty}$, $1 < p < +\infty$.

Corollary 2.16. *The Riesz transform $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$, $j = 1, 2, \dots d$ is continuous in $L^{p,\infty}$ for $p > 1$.*

Proof. We know that the Riesz transform is continuous in L^p, $p > 1$. Now, take $1 < p_0$, $p_1 < \infty$ and using that $L^{p_0} = L^{p_0,p_0}$, $L^{p_1} = L^{p_1,p_1}$ and lemma 2.15, we deduce that

$$\|R_j f\|_{L^{p,q}} \le C\|f\|_{L^{p,q}}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $1 \le q \le +\infty$. Taking $q = +\infty$, the proof is complete. □

Finally, let us prove a proposition which will be useful in the study of the QG equations in lorentz spaces.

Proposition 2.17. *Let $\varphi \in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \varphi(y)dy = 1$. For each $\delta > 0$, we define $\varphi_\delta(x) = \frac{1}{\delta^d}\varphi(\frac{x}{\delta})$. If $1 < p < \infty$, $1 \le q < \infty$ and $f \in L^{p,q}$, then*

$$\lim_{\delta\to 0} \|\varphi_\delta * f - f\|_{L^{p,q}} = 0$$

Proof. Using the change of variable $\delta t = y$ and the fact that $\int_{\mathbb{R}^d} \varphi_\delta(y)dy = 1$, for all $\delta > 0$, we have

$$\begin{aligned}(\varphi_\delta * f - f)(x) &= \int_{\mathbb{R}^d} \varphi_\delta(y)[f(x-y) - f(x)]dy \\ &= \int_{\mathbb{R}^d} \varphi(t)[f(x-t\delta) - f(x)]dt\,.\end{aligned}$$

Next, taking the norm $\|.\|_{L^{p,q}}$, we obtain

$$\begin{aligned}\|\varphi_\delta * f - f\|_{L^{p,q}} &\le \int_{\mathbb{R}^d} \|\varphi(t)\,[f(x-t\delta) - f(x)]\,\|_{L^{p,q}}dt \\ &\le \|f(x-t\delta) - f(x)\|_{L^{p,q}}\,.\end{aligned}$$

Note that $\|f(x-t\delta) - f(x)\|_{L^{p,q}} \le 2\|f\|_{L^{p,q}}$. Since $1 \le q < \infty$, then

$$\lim_{\delta\to 0} \|f(x-t\delta) - f(x)\|_{L^{p,q}} = 0$$

and we can use the Lebesgue dominated convergence theorem to conclude the proof. □

3. The solution operator for the linear equation

Consider the solution operator for the linear equation

$$\partial_t\theta + (-\Delta)^\gamma\theta = 0, \quad \text{on } \mathbb{R}^d\times]0,+\infty[$$

where $\gamma \in [0,1]$. For a given initial data θ_0, the solution of this equation is given by

$$\theta(t) = K_\gamma(t)\theta_0 = e^{-t(-\Delta)^\gamma}\theta_0,$$

where $K_\gamma(t) = e^{-t(-\Delta)^\gamma}$ is a convolution operator with its kernel k_γ being defined through the Fourier transform

$$\widehat{k}_\gamma(x,t) = e^{-t|\xi|^{2\gamma}}$$

In particular, k_γ is the heat kernel for $\gamma = 1$ and the Poisson kernel for $\gamma = 1/2$. The kernel k_γ possesses similar properties as the heat kernel does. For example, for $\gamma \in [0,1]$ and $t > 0$, $k_\gamma(x,t)$ is a nonnegative and non-increasing radial function and satisfies the following homogeneity properties

$$\begin{aligned} k_\gamma(x,t) &= t^{-\frac{d}{2\gamma}} k_\gamma(t^{-\frac{1}{2\gamma}}x, 1) \\ (\nabla_x k_\gamma)(x,t) &= t^{-\frac{(d+1)}{2\gamma}} (\nabla_x k_\gamma)(t^{-\frac{1}{2\gamma}}x, 1) \end{aligned} \tag{3.1}$$

This remarks lead to the following result.

Lemma 3.1. *For all $t > 0$ and $\gamma \in [0,1]$, $K_\gamma(t) = e^{-t(-\Delta)^\gamma}$ is a convolution operator with its kernel $k_\gamma \in L^1$.*

Proof. We have $\widehat{k}_\gamma(x,t) = e^{-t|\xi|^{2\gamma}} \in L^1$; thus k_γ is a continuous bounded function. Hence

$$\|k_\gamma(.,t)\|_{L^1} = \widehat{k}_\gamma(0,\xi) = 1$$

□

Furthermore, the operators K_γ and ∇K_γ are bounded on L^∞. To prove this fact, we need the following lemma.

Lemma 3.2. *For any $t > 0$, the operators K_γ and ∇K_γ are bounded operators from L^∞ to L^∞. Furthermore, we have for any $u \in L^\infty$,*

$$\|K_\gamma(t)u\|_{L^\infty} \le \|u\|_{L^\infty}, \tag{3.2}$$

$$\|\nabla K_\gamma(t)u\|_{L^\infty} \le Ct^{-\frac{1}{2\gamma}}\|u\|_{L^\infty}, \tag{3.3}$$

where C is a constant depending on γ.

Proof. To prove (3.2), we have

$$\|K_\gamma(t)u\|_{L^\infty} \le \|k_\gamma(.,t)\|_{L^1}\|u\|_{L^\infty}\,.$$

Estimate (3.3) can be proved similarly by using the identity

$$\partial_x k_\gamma(x,t) = t^{-\frac{1}{2\gamma}}\widetilde{k}_\gamma(x,t)$$

where $\widetilde{k}_\gamma$ is another radial function enjoying the same properties as k_γ does. □

Now, we introduce a basic tool for this papier: The Besov spaces $\dot{B}_\infty^{-r,\infty}$ for $r > 0$.

3.1. **Littlewood-Paley decomposition of tempered distribution. Definition of Dyadics blocks** Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be a non negative function so that $|\xi| \leq \frac{1}{2} \Rightarrow \varphi(\xi) = 1$ and $|\xi| \geq 1 \Rightarrow \varphi(\xi) = 0$. Let $\psi(\xi) = \varphi(\frac{\xi}{2}) - \varphi(\xi)$. Let Δ_j and S_j be defined as the Fourier multipliers

$$\mathcal{F}(S_j f) = \varphi(\frac{\xi}{2^j})\mathcal{F}(f) \quad \text{and} \quad \mathcal{F}(\Delta_j f) = \psi(\frac{\xi}{2^j})\mathcal{F}(f)$$

The distribution $\Delta_j f$ is called the j-th dyadic block of the Littlewood-Paley decomposition of f.

For all $k \in \mathbb{Z}$ and for all $f \in \mathcal{S}'(\mathbb{R}^d)$, we have

$$f = S_k f + \sum_{j \geq k} \Delta_j f$$

in $\mathcal{S}$'$(\mathbb{R}^d)$. This equality is called the Littlewood-Paley decomposition of the distribution f. If, moreover, $\lim_{k \to -\infty} S_k f = 0$ in $\mathcal{S}'$, then the equality

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f.$$

is called the homogeneous Littlewood-Paley decomposition of the distribution f.

Distributions vanishing at infinity. We define the space of tempered distributions vanishing at infinity as the space

$$\mathcal{S}_0'(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) \ \text{ so that } \ f = \sum_{j \in \mathbb{Z}} \Delta_j f \ \text{ in } \mathcal{S}'\}$$

We may now define homogeneous Besov spaces in the following way:

For $s \in]-\infty, +\infty[$, p, $q \in [1, +\infty]$, the homogeneous Besov spaces $\dot{B_p}^{s,q}$is defined as

$$\dot{B_p}^{s,q} = \{f \in \mathcal{S}'/\mathbb{C}[X] : 2^{js}\|\Delta_j f\|_{L^p} \in l^q(\mathbb{Z})\}.$$

equipped with the norm

$$\|f\|_{\dot{B_p}^{s,q}} = \Big(\sum_{j \in \mathbb{Z}} (2^{js}\|\Delta_j f\|_{L^p})^q\Big)^{1/q}$$

and $\mathbb{C}[X]$ denotes the set of all multinomials.

Similarly, the inhomogeneous Besov spaces $B_p^{s,q}$ are defined by

$$B_p^{s,q}(\mathbb{R}^d) = \{f \in \mathcal{S}' : \|f\|_{B_p^{s,q}} < \infty\},$$

where for $q \neq \infty$,

$$\|f\|_{B_p^{s,q}} = \Big(\sum_{j \in \mathbb{Z}} (2^{js}\|\Delta_j f\|_{L^p})^q\Big)^{1/q};$$

and for $q = \infty$,

$$\|f\|_{B_p^{s,\infty}} = \sup j \geq 0 2^{js}\|\Delta_j f\|_{L^p}.$$

If $s < 0$ and if f_j satisfies $\operatorname{supp} f_j \subset \{\xi : \frac{2^j}{2} \leq |\xi| \leq 2.2^j\}$ and $(2^{js}\|f_j\|_{L^p})_{j \in \mathbb{Z}} \in l^\infty(\mathbb{Z})$, then $\sum_{j \in \mathbb{Z}} f_j$ converges in $\mathcal{S}'$. Indeed, we have $\sum_{j \leq 0} \|f_j\|_{L^p} < +\infty$, but for all $g \in \mathcal{S}$ and all $k \in \mathbb{N}$: $\sum_{j \geq 0} 2^{jk}\|\Delta_j g\|_{L^p} < +\infty$. We write for some $M \in \mathbb{N}$,

$$\|g\|_{L^{p'}} \leq \sum_{|\alpha| \leq M} \sum_{|\beta| \leq M} \big\|\xi^\alpha . \frac{\partial^\beta}{\partial \xi^\beta} \hat{g}\big\|_{L^\infty} .$$

We use the semi-group $e^{-t(-\Delta)^\gamma}$ operating on shift-invariant Banach spaces of distributions, it will be very useful to characterize the action of the kernel $k_\gamma(t)$ on Besov spaces associated with such spaces [12, Theorem 5.3, p.44-45].

Proposition 3.3. *Let $s<0$ and $T>0$. There exists a constant $C_T>0$ such that*

$$\dot{B}_p^{s,q} = \Big\{ f\in\mathcal{S}' : e^{-t(-\Delta)^\gamma} f \in L^p, \Big(\int_0^T (t^{-\frac{s}{2}} \|e^{-t(-\Delta)^\gamma} f\|_{L^p})^q \frac{dt}{t}\Big)^{1/q} \le C_T \Big\} \quad (3.4)$$

for p, $q\in[1,+\infty]$. Moreover, the norm

$$\|e^{-t(-\Delta)^\gamma} f\|_{L^p} + \|t^{-\frac{s}{2}} (e^{-t(-\Delta)^\gamma} f)\|_{L^q(]0,\infty[,\frac{dt}{t},L^p)}$$

and the norm $\|f\|_{\dot{B}_p}^{s,q}$ are equivalent.

Let us return to the condition $E\hookrightarrow \dot{B}_\infty^{-r,\infty}$. One of the purposes of this section is, if possible, to show under a characterization (3.4) that

$$f\in \dot{B}_\infty^{-r,\infty} \Leftrightarrow t^{\frac{r}{2\gamma}} e^{-t(-\Delta)^\gamma} f \in L^\infty(]0,T[,L^\infty), \quad \text{for all } T>0$$

That is, we want to know whether

$$\|f\|_{\dot{B}_\infty^{-r,\infty}} \sim \|e^{-t(-\Delta)^\gamma} f\|_{L^\infty} + \|t^{-\frac{s}{2}} (e^{-t(-\Delta)^\gamma} f)\|_{L^\infty(]0,\infty[,L^\infty)}$$

for every function $f\in \dot{B}_\infty^{-r,\infty}$ and all $t>0$. The proof uses exactly the same ideas as in [12, of theorem 5.4 in] and we omit the details here.

Lemma 3.4. *Let E a Banach space satisfying $\mathcal{S}\hookrightarrow E\hookrightarrow \mathcal{S}'$, for all $x_0\in\mathbb{R}^d$; $\|f\|_E = \|f(x-x_0)\|_E$ and so that*

$$\sup_{0<\lambda\le 1} \lambda^r \|f(\lambda x)\|_E \le C\|f\|_E \quad \textit{with } r\in\mathbb{R}\,.$$

Then

$$E\hookrightarrow B_\infty^{-r,\infty}\,.$$

Similarly, if

$$\sup_{0<\lambda} \lambda^r \|f(\lambda x)\|_E \le C\|f\|_E \quad \textit{with } r\in\mathbb{R},$$

then

$$E\hookrightarrow \dot{B}_\infty^{-r,\infty}\,.$$

Proof. We shall prove only the first statement; the proof of the second is similar. We may assume that

$$\sup_{0<\lambda\le 1} \lambda^r \|f(\lambda x)\|_E \le C\|f\|_E \quad \text{with } r\in\mathbb{R}$$

Using the Littlewood-Paley decomposition, we have by noting $S_0 f = \varphi * f$

$$\|S_0 f\|_{L^\infty} = \sup_{x\in\mathbb{R}^d} \Big|\int_{\mathbb{R}^d} \varphi(y) f(x-y)dy\Big| \le \sup_{x\in\mathbb{R}^d} \|\varphi\|_{L^1}\|f\|_E$$

Moreover, for all $j\ge 0$, we have

$$\begin{aligned} 2^{-jr}\|\Delta_j f\|_{L^\infty} &= 2^{-jr}\|\Delta_0 f(\tfrac{\cdot}{2^j})(2^j x)\|_{L^\infty} \\ &= 2^{-jr}\|\Delta_0 f(\tfrac{\cdot}{2^j})\|_{L^\infty} \\ &\le C 2^{-jr}\|f(\tfrac{\cdot}{2^j})\|_E \le C\|f\|_E \end{aligned}$$

□

We now establish estimates for the operator $K_\gamma(t)$ in Banach spaces of local measures.

Proposition 3.5. *For any $t > 0$, the operators $K_\gamma(t)$ and $\nabla K_\gamma(t)$ are bounded operators in Banach spaces of local measures and depend on t continuously, where ∇ denotes the space derivative. Furthermore, we have for $u \in E$,*

$$\|K_\gamma(t)u\|_E \leq \|u\|_E,$$
$$\|\nabla K_\gamma(t)u\|_E \leq Ct^{-\frac{1}{2\gamma}}\|u\|_E$$

The statement of the above proposition is easy to check.

4. Well Posedness in Banach spaces of local measures

The result of Wu's theorem may be generalized in a direct way to the setting of spaces of local measures, such as Lorentz spaces, Multipliers spaces or Morrey-Campanato spaces. Let us introduce suitable functional spaces to analyse the Cauchy problem for system (1.1)-(1.2) based on the spaces of measures local.
Definition Let $\mathcal{E}_E$ be the space of all function $f(x,t)$, with $t > 0$ and $x \in \mathbb{R}^2$, such that

$$f(x,t) \in L^\infty((0,\infty), E)$$

and therefore the norm in $\mathcal{E}_E$ is defined by

$$\|f\|_{\mathcal{E}_E} = \sup_{t>0} \|f(.,t)\|_E$$

Let us also define by $\mathcal{E}_\infty$ the space of all functions $f(x,t)$, with $t > 0$ and $x \in \mathbb{R}^2$, such that

$$f(x,t) \in L^\infty((0,\infty), E),$$
$$t^{\frac{r}{2\gamma}} f(x,t) \in L^\infty((0,\infty), L^\infty),$$

where $r = 2\gamma - 1 < 1$. The norm in $\mathcal{E}_\infty$ is

$$\|f\|_{\mathcal{E}_\infty} = \sup_{t>0} \|f(.,t)\|_E + \sup_{t>0} t^{\frac{r}{2\gamma}} \|f(.,t)\|_{L^\infty}.$$

Set

$$\mathcal{B}_\infty = \{\theta / \ t^{\frac{r}{2\gamma}}\theta \in L^\infty((0,\infty), L^\infty)\}.$$

Let us make precise the notion of mild solution.

Definition 4.1. A global mild solution of the system (1.1)-(1.2) in $\mathcal{E}_E$ and $\mathcal{E}_\infty$ is a function $\theta(t)$ in the corresponding space satisfying

$$\theta(t) = K_\gamma(t)\theta_0(t) - B(\theta,\theta)(t) = K_\gamma(t)\theta_0(t) - \int_0^t K_\gamma(t-s)\nabla(\theta u)(s)ds \tag{4.1}$$

and $\theta(t) \to \theta_0$ as $t \to 0^+$, where the limit is taken in the weak-* topology of E.

We prove the following well-posedness results for mild solutions.

Theorem 4.2 (well-posedness). *Let $\theta_0 \in E$ and $\gamma < 1$ for $d = 2$. There exists $\delta > 0$ such that if $\|\theta_0\|_E < \delta$, then the initial value problem for (1.1)-(1.2) has a global mild solution $\theta(x,t) \in \mathcal{E}_E$. Moreover, if $\|\theta\|_{\mathcal{E}_E} < 2\delta$, then the solution is unique in $\mathcal{E}_E$.*

Well posedness theorem will be a consequence of the following lemma for generic Banach spaces [12]:

Lemma 4.3. *Let F be a Banach space with norm $\|.\|_F$ and $B : F \times F \to F$ be a continuous bilinear operator, i.e., there exists $K > 0$ such that for all $x, y \in F$, we have*

$$\|B(x,y)\|_F \leq K\|x\|_F\|y\|_F$$

Given $0 < \delta < \frac{1}{4K}$ and $y \in F$, $y \neq 0$, such that $\|y\|_F < \delta$, there exists a solution $x \in F$ for the equation $x = y + B(x,x)$ such that $\|x\|_F < 2\delta$. The solution is unique in the ball $\overline{B}(0, 2\delta)$. Moreover, the solution depends continuously on y in the following sense : if $\|\widetilde{y}\|_F < \delta$, $\widetilde{x} = \widetilde{y} + B(\widetilde{x}, \widetilde{x})$, and $\|\widetilde{x}\|_F < 2\delta$, then

$$\|x - \widetilde{x}\|_F \leq \frac{1}{1 - 4K\delta}\|y - \widetilde{y}\|_F$$

As a consequence of the previous lemma, one needs to verify the continuity of the bilinear terms in the integral form of the QG equation to obtain the well-posedness theorem for the solutions of the integral equation. The weak-* continuity at 0^+ will finally end the proof of well-posedness of mild solutions.

Proof of theorem 4.2. To proceed, we write the QG equation into the integral form

$$\theta(t) = K_\gamma(t)\theta_0 - \int_0^t K_\gamma(t-s)(u\nabla\theta)(s)ds \tag{4.2}$$

We observe that $(u\nabla\theta) = \nabla.(u\theta)$ because $\nabla.u = 0$. The nonlinear term can then alternatively written as

$$B(u,\theta)(t) = \int_0^t \nabla K_\gamma(t-s)(u\theta)(s)ds \tag{4.3}$$

We start by a series of lemmas in order to prove the continuity of the bilinear term $B(\theta, u)$ defined by (4.3). We will solve (4.3) in $\mathcal{E}_\infty$ and the following estimates for the operator B acting on this type of spaces will be used. The three properties of $K_\gamma(t)$ that we shall use are the following obvious remarks.

Lemma 4.4. *Given $h \in E$. We have*

$$\|K_\gamma(t-s)h\|_E \leq \|h\|_E,$$
$$\|K_\gamma(t-s)h\|_{L^\infty} \leq C(t-s)^{-\frac{r}{2\gamma}}\|h\|_{\dot{B}_\infty^{-r,\infty}},$$
$$\|K_\gamma(t-s)\nabla.\|_{L^1} \leq C(t-s)^{-\frac{1}{2\gamma}}$$

The proof of the above lemma is obvious.

Proposition 4.5. *Let $\frac{1}{2} < \gamma \leq 1$ and $t > 0$. Assume that u and θ are in $\mathcal{E}_\infty$. Then the operator B is bounded in $\mathcal{E}_\infty$ with*

$$\|B(u,\theta)\|_{\mathcal{E}_\infty} \leq C_\gamma\|u\|_{\mathcal{E}_\infty}\|\theta\|_{\mathcal{E}_\infty}$$

Proof. Applying (3.3) from proposition 3.2, we estimate

$$\begin{aligned}\|B(u,\theta)\|_{L^\infty} &\le \int_0^t \|K_\gamma(t-s)\nabla(\theta u)(s)\|_{L^\infty} ds \\ &\le C\int_0^t (t-s)^{-\frac{1}{2\gamma}}\|(\theta u)(s)\|_{L^\infty} ds \\ &\le C\int_0^t (t-s)^{-\frac{1}{2\gamma}} s^{-\frac{r}{\gamma}}(s^{\frac{r}{2\gamma}}\|\theta(s)\|_{L^\infty})(s^{\frac{r}{2\gamma}}\|u(s)\|_{L^\infty}) ds \\ &\le C\|\theta\|_{\mathcal{B}_\infty}\|u\|_{\mathcal{B}_\infty}\int_0^t (t-s)^{-\frac{1}{2\gamma}} s^{-\frac{r}{\gamma}} ds\end{aligned}$$

where in the last line we used the continuity of the Riesz transform. Therefore,

$$\|B(u,\theta)\|_{L^\infty} \le CI(t)\|\theta\|_{\mathcal{B}_\infty}\|u\|_{\mathcal{B}_\infty}$$

where the integral $I(t)$ in the right-hand side can be computed as

$$\begin{aligned}I(t) &= \int_0^t (t-s)^{-\frac{1}{2\gamma}} s^{-\frac{r}{\gamma}} ds = t^{-\frac{1}{2\gamma}+1-\frac{r}{\gamma}}\int_0^1 (1-z)^{-\frac{1}{2\gamma}} z^{-\frac{r}{\gamma}} dz \\ &= t^{-\frac{r}{2\gamma}}\Gamma(\frac{r}{2\gamma})\Gamma(\frac{1-r}{2\gamma}) \quad \text{with } 0<r=2\gamma-1\le 1.\end{aligned}$$

□

Remark 4.6. A sufficient condition for E to be continuously embedded in the Besov space $\dot{B}_\infty^{-r,\infty}$ for some $r>0$ is the existence of constant $C\ge 0$ so that for all $f\in E$, we have

$$\sup_{\lambda>0}\lambda\|f(\lambda x)\|_E \le C\|f\|_E\,.$$

Proposition 4.7. *If θ and u belongs to $\mathcal{E}_\infty$, then there exists $C=C(d,\gamma)$ such that*

$$\|B(\theta,u)\|_E \le C\sup_{t>0}\|\theta(t)\|_E(t^{\frac{r}{2\gamma}}\sup_{t>0}\|u(t)\|_{L^\infty}).$$

Proof. The boundedness of B from $\mathcal{E}_E\times\mathcal{B}_\infty$ to $\mathcal{E}_E$ and from $\mathcal{B}_\infty\times\mathcal{E}_E$ to $\mathcal{E}_E$ is obvious, since

$$\|K_\gamma(t-s)\nabla.\|_{\mathcal{L}(E\times E,E)} \le C(t-s)^{-\frac{1}{2\gamma}}$$

and the invariance of E under convolution with L^1 kernels. Thus, we get

$$\begin{aligned}\|B(\theta,u)\|_E &\le C\int_0^t \|K_\gamma(t-s)\nabla(\theta u)(s)\|_E ds \\ &\le C\int_0^t \|\nabla k_\gamma(t-s)\|_{L^1}\|(\theta u)(s)\|_E ds \\ &\le C\int_0^t (t-s)^{-\frac{1}{2\gamma}}\|u(s)\|_E\|\theta(s)\|_{L^\infty} ds \\ &\le C\int_0^t (t-s)^{-\frac{1}{2\gamma}} s^{-\frac{r}{2\gamma}}\|u(s)\|_E(s^{\frac{r}{2\gamma}}\|\theta(s)\|_{L^\infty}) ds \\ &\le C\|u\|_{\mathcal{E}_E}\|\theta\|_{\mathcal{B}_\infty},\end{aligned}$$

where the continuity of the Riesz transform in E has been used, which gives the result, since

$$\int_0^t (t-s)^{-\frac{1}{2\gamma}} s^{-\frac{r}{2\gamma}} ds = \Gamma(\frac{r}{2\gamma})\Gamma(\frac{1}{2\gamma})$$

where Γ being Euler's Gamma function. □

A direct application of Lemma 4.3 in E completes the proof of the well-posedness of the integral equation. To finish the proof of the well-posedness of mild solutions in E by applying Lemma 4.3, it remains to check that the linear part of the integral equation can be bounded in terms of the initial data and that solutions of the integral equation are indeed mild solutions by definition 4.1. Following the arguments in [2, lemma 6], it can be proved in a completely analogous way that $B(u,\theta)(t) \to 0$ when $t \to 0^+$ weakly-star in the topology of E. Now, it remains to verify that the linear part takes the initial data and is bounded in the corresponding space.

Proposition 4.8. *If $\theta_0 \in E$, then $K_\gamma(t)\theta_0 \in \mathcal{E}_E$,*

$$\|K_\gamma(t)\theta_0\|_{\mathcal{E}_E} \le C\|\theta_0\|_E,$$
$$K_\gamma(t)\theta_0 \to \theta_0 \text{ when } t \to 0^+,$$

where the limit is taken in the weak-star topology of E. Moreover, if $\theta_0 \in E \cap L^\infty$, then

$$\|K_\gamma(t)\theta_0\|_{\widetilde{E}} \le C(\|\theta_0\|_E + \|\theta_0\|_{L^\infty})$$

where $\widetilde{E} = L^\infty((0,\infty), E \cap L^\infty)$.

Proof. Using (2.1), we deduce

$$\|K_\gamma(t)\theta_0\|_E \le C\|k_\gamma(t)\|_{L^1}\|\theta_0\|_E = C\|\theta_0\|_E,$$
$$\|K_\gamma(t)\theta_0\|_{L^\infty} \le C\|k_\gamma(t)\|_{L^1}\|\theta_0\|_{L^\infty} = C\|\theta_0\|_{L^\infty}.$$

To prove the weak-star continuity, note that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}|\langle K_\gamma(t)\theta_0 - \theta_0, \varphi\rangle| &= \Big|\int (e^{-t|\xi|^{2\gamma}} - 1)\widehat{\theta}_0\widehat{\varphi} d\xi\Big| \\ &\le t \sup_{\xi\in\mathbb{R}^d} \frac{(1-e^{-t|\xi|^{2\gamma}})}{t|\xi|^{2\gamma}} \|\theta_0\|_E \||\xi|^{2\gamma}\widehat{\varphi}\|_{L^1(\mathbb{R}^d)}\end{aligned}$$

which converges to 0 as $t \to 0^+$, because

$$\sup_{\xi\in\mathbb{R}^d} \frac{(1-e^{-t|\xi|^{2\gamma}})}{t|\xi|^{2\gamma}} \le \sup_{z>0} e^{-z} = 1,.$$

Since $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$ and $R > 0$,

$$\||\xi|^{2\gamma}\widehat{\varphi}\|_{L^1(B(0,R))} < \infty, \quad \text{when } \gamma > 0.$$

□

A direct application of lemma 4.3 in $\mathcal{E}_E$ completes the proof of the well-posedness of mild solution in spaces of local measures. We generalize Wu's [17] theorem to shift-invariant Banach spaces of local measures by using the regularizing of the kernel $K_\gamma(t)$.

Theorem 4.9. *Let E be a shift-invariant Banach space of local measures.*

(a) *Let* $\mathcal{E}_E = L^\infty((0,\infty),E)$ *and* $\mathcal{B}_\infty = \{\theta/t^{\frac{r}{2\gamma}}\theta \in L^\infty((0,\infty),L^\infty)\}$. *The bilinear operator* B *defined by*

$$B(\theta,u)(t) = \int_0^t K_\gamma(t-s)\nabla(\theta u)(s)ds$$

is bounded from $\mathcal{E}_E \times \mathcal{B}_\infty$ *to* $\mathcal{E}_E$ *and from* $\mathcal{B}_\infty \times \mathcal{E}_E$ *to* $\mathcal{E}_E$. *Moreover,* $B(\theta,u) \in C((0,\infty),E)$ *and converges** *-weakly to* 0 *as* t *goes to* 0. *If* $\lim_{t\to 0} t^{\frac{r}{2\gamma}}\|\theta\|_{L^\infty} = 0$, *if* $\theta \in \mathcal{B}_\infty$ *and* $u \in \mathcal{E}_E$, *then the convergence is strong.*

(b) *If* E *is continuously embedded in the Besov space* $\dot{B}_\infty^{-r,\infty}$ *for some* $r \le 1$, *then the bilinear operator* B *is bounded as well from* $(\mathcal{E}_E \cap \mathcal{B}_\infty)\times\mathcal{B}_\infty$ *to* $\mathcal{B}_\infty$ *and from* $\mathcal{B}_\infty \times (\mathcal{E}_E\cap\mathcal{B}_\infty)$ *to* $\mathcal{B}_\infty$. *Hence, there exists a constant* $\delta_E > 0$ *so that for all* $\theta_0 \in E$ *(with* $\nabla.\theta_0 = 0$*) with* $\|\theta_0\|_E < \delta_E$, *then the initial value problem for* (1.1)-(1.2) *with initial data* θ_0 *has a global mild solution* $\theta(x,t) \in \mathcal{E}_E \cap \mathcal{B}_\infty$.

For the proof of the above theorem, we use the same ideas as in [12, Theorem 17.2]. Then we can get the desired result, but we omit the details here.

Theorem 4.10 (Regularization). *Under the assumptions of previous theorem, there exists* $0 < \epsilon < \delta$ *such that if* $\|\theta_0\|_E < \epsilon$, *then the solution of theorem 4.2,* $\theta(x,t) \in \mathcal{E}_\infty$. *Moreover, if* $\|\theta\|_{\mathcal{E}_\infty} < 2\epsilon$, *then the solution is unique in this class.*

Proof. We just need to show the continuity of the bilinear form in the regularizing norm $\sup\limits_{t>0} t^{\frac{r}{2\gamma}}\|.\|_\infty$. For this, we have by proposition 4.5,

$$\|B(u,\theta)\|_{L^\infty} \le Ct^{-\frac{r}{2\gamma}}\|\theta\|_{\mathcal{B}_\infty}\|u\|_{\mathcal{B}_\infty}\,;$$

therefore,

$$\sup_{t>0} t^{\frac{r}{2\gamma}}\|B(u,\theta)\|_{L^\infty} \le C\|\theta\|_{\mathcal{B}_\infty}\|u\|_{\mathcal{B}_\infty}\,.$$

This completest the proof of continuity of the bilinear form. □

A direct application of lemma 4.3 in $\mathcal{E}_\infty$ completes the proof of the well-posedness of the integral equation. Initial data are taken in the same sense as in theorem 4.2 and therefore, we have completed the proof of regularization in the solutions of theorem 4.2. We now show that the solution obtained in the previous theorem is actually smooth. We adapt the arguments of Kato in [11] for proving that the constructed solutions are C^∞-smooth instantly.

Proposition 4.11. *Let* $\theta(x,t) \in \mathcal{E}_\infty$ *be the unique global mild-solution in theorem 4.10. Then*

$$\partial_t^k\partial_x^m\theta(x,t) \in C((0,\infty),\widetilde{E}) \tag{4.4}$$

As a consequence, the solution $\theta(x,t)$ *is infinitely smooth in space and time.*

Proof. We will just give the main steps of the proof since this is a small variation of arguments found in [11]. The proof is done by induction. We will focus in the case $k = 0$ since the same induction argument one can derive the estimates for the temporal derivatives (see [11] for details). Let θ be the solution of theorem 4.10. Denote $\beta = \theta(\tau)$ for any $\tau > 0$. Let us remark that we have local existence, $T - \tau$ small enough, for the mild formulation

$$\theta(t) = K_\gamma(t-\tau)\beta - \int_\tau^t \nabla K_\gamma(t-s)(\theta R[\theta])(s)ds \tag{4.5}$$

in the space $L^\infty\big((\tau,T),\widetilde{E}\big)$. Denote

$$B_\tau(\theta,\theta)=\int_\tau^t \nabla K_\gamma(t-s)(\theta R[\theta])(s)ds\,.$$

Let us perform an induction argument on m. The assertion (4.4) for $k=m=0$ is true by theorem 4.10. Let us assume that (4.4) is true for all $m\le N-1$, $k=0$. Our aim is to show that the mild formulation (4.5) has a solution in the space $\mathcal{X}_N$ defined by

$$\begin{gathered}\partial_x^j\theta(x,t)\in L^\infty([\tau,T[,\widetilde{E}),\quad j=0,1,2,\dots N-1,\\ (t-\tau)^{\frac{1}{2\gamma}}\partial_x^N\theta(x,t)\in L^\infty([\tau,T[,\widetilde{E})\end{gathered}$$

endowed with the norm

$$\|\theta\|_{\mathcal{X}_N}=\sup_{t\in(\tau,T)}\big[(t-\tau)^{\frac{1}{2\gamma}}(\|\partial_x^N\theta\|_{L^\infty}+\|\partial_x^N\theta\|_E)+\sum_{j=0}^d(\|\partial_x^j\theta\|_{L^\infty}+\|\partial_x^j\theta\|_E)\big]$$

Note that $K_\gamma(t-\tau)\beta\in\mathcal{X}_N$ since by induction hypothesis, we have $\partial_x^j\theta(\tau)\in\widetilde{E}$ for $j=0,1,2,\dots N-1$ and by convolution

$$\begin{aligned}\|\partial_x^N K_\gamma(t-\tau)\beta\|_{L^\infty}&=\|(\partial_x k_\gamma(t-\tau))*\partial_x^{N-1}\beta\|_{L^\infty}\\&\le C\|(\partial_x k_\gamma(t-\tau))\|_{L^1}\|\partial_x^{N-1}\beta\|_{L^\infty}\\&\le C(t-\tau)^{\frac{1}{2\gamma}}\|\partial_x^{N-1}\beta\|_{L^\infty}\end{aligned}$$

and

$$\begin{aligned}\|\partial_x^N K_\gamma(t-\tau)\beta\|_E&=\|(\partial_x k_\gamma(t-\tau))*\partial_x^{N-1}\beta\|_E\\&\le C\|(\partial_x k_\gamma(t-\tau))\|_{L^1}\|\partial_x^{N-1}\beta\|_E\\&\le C(t-\tau)^{\frac{1}{2\gamma}}\|\partial_x^{N-1}\beta\|_{L^\infty}\end{aligned}$$

Now, given $t\in(\tau,T)$ and θ, $u\in\mathcal{X}_N$, we have

$$\begin{gathered}\|B_\tau(\theta,u)(t)\|_{L^\infty}\le C(T-\tau)^{\frac{r}{2\gamma}}\sup_{t\in[\tau,T[}\|\theta(t)\|_{L^\infty}\sup_{t\in[\tau,T[}\|u(t)\|_{L^\infty},\\ \|B_\tau(\theta,u)(t)\|_E\le C(T-\tau)^{\frac{r}{2\gamma}}\sup_{t\in[\tau,T[}\|\theta(t)\|_{L^\infty}\sup_{t\in[\tau,T[}\|u(t)\|_E\end{gathered}$$

Therefore, given θ, $u\in\mathcal{X}_N$, we conclude that

$$\begin{gathered}\|\partial_x^j B_\tau(\theta,u)(t)\|_{L^\infty}\le C(T-\tau)^{\frac{r}{2\gamma}}\sup_{t\in[\tau,T[}\|\theta(t)\|_{L^\infty}\sup_{t\in[\tau,T[}\|\partial_x^j u(t)\|_{L^\infty},\\ \|\partial_x^j B_\tau(\theta,u)(t)\|_E\le C(T-\tau)^{\frac{r}{2\gamma}}\sup_{t\in[\tau,T[}\|\theta(t)\|_{L^\infty}\sup_{t\in[\tau,T[}\|\partial_x^j u(t)\|_E\end{gathered}$$

for all $j=0,1,2,\dots N-1$. Now, we observe that $\partial_x^N B_\tau(\theta,u)$ is a linear combination of terms of the form

$$\partial_{j,i,x}^N B_\tau(\theta,u)(t)=C\int_\tau^t \partial_x\nabla K_\gamma(t-s)(\partial^j\theta R[\partial^i\theta])(s)ds$$

with $j+i=N-1$. Thus, we can get the estimate

$$\|\partial_x^N B_\tau(\theta,u)(t)\|_{L^\infty} \le C(T-\tau)^{\frac{r}{2\gamma}-\frac{1}{2\gamma}} \sup_{t\in[\tau,T[} \|\partial_x^j \theta(t)\|_{L^\infty} \sup_{t\in[\tau,T[} \|\partial_x^j u(t)\|_{L^\infty},$$
$$\|\partial_x^N B_\tau(\theta,u)(t)\|_{E} \le C(T-\tau)^{\frac{r}{2\gamma}-\frac{1}{2\gamma}} \sup_{t\in[\tau,T[} \|\partial_x^j \theta(t)\|_{L^\infty} \sup_{t\in[\tau,T[} \|\partial_x^j u(t)\|_{E}\,.$$

Finally we conclude that

$$\|B_\tau(\theta,u)(t)\|_{\mathcal{X}_N} \le C(T-\tau)^{\frac{r}{2\gamma}} \|\theta\|_{\mathcal{X}_N} \|u\|_{\mathcal{X}_N}\,.$$

By choosing $T-\tau$ small enough, we are allowed to use lemma 4.3 to obtain a solution $\widetilde{\theta}(x,t)$ in $\mathcal{X}_N$ of the mild formulation (4.5). Owing to the uniqueness of the mild formulation (4.5) in $L^\infty([\tau,T[,\widetilde{E})$, we have finally shown that $\widetilde{\theta}(x,t)=\theta(x,t)$ in (τ,T). Now, it is easy to perform all these arguments in a recursive finite collection of equal-length time intervals A_i such that $(0,\infty)=\cup_i A_i$ where $A_i=(\tau_i,T_i)$ concluding that

$$\partial_x^j\theta \in C((0,\infty),\widetilde{E}) \quad \text{for all } j=0,1,2,\dots N.$$

As said above, the induction on temporal derivatives is done analogously and thus $\theta(x,t)$ has derivatives of all orders in $\widetilde{E}$. □

5. Self-Similar solutions in E.

Assuming that $\theta(x,t)$ is a smooth solution of the QG equation (1.1)-(1.2), it is straightforward to check that

$$\theta_\lambda(x,t)=\lambda^{2\gamma-1}\theta(\lambda^{2\gamma}x,\lambda t)$$

is also a solution of (1.1)-(1.2). In fact, we can look for particular solutions of the system (1.1)-(1.2) satisfying

$$\theta_\lambda(x,t)=\theta(x,t) \tag{5.1}$$

for any $t>0$, $x\in\mathbb{R}^2$ and $\lambda>0$. These particular solutions are called self-similar solutions of the system and it is clear that taking $t\to 0^+$ formally in (5.1), $\theta(x,0)$ should be a homogeneous function of degree $(1-2\gamma)$. This remark gives the hint that a suitable space to find self-similar solutions should be one containing homogeneous functions with that exponent. Moreover, in case such a self-similar solution exists, its norm is invariant by scaling transformation, i.e.,

$$\theta(x,t)\to\theta_\lambda(x,t)=\lambda^{2\gamma-1}\theta(\lambda^{2\gamma}x,\lambda t)$$

Now we show the existence and uniqueness of global in time non trivial self-similar solutions in Banach spaces of local measures with the right homogeneity.

Theorem 5.1. *Let $\theta_0\in E$. Assume that θ_0 is a homogeneous function of degree $(1-2\gamma)$. Then, if $\|\theta_0\|_E<\delta$, the solution $\theta(x,t)$ provided by theorem 4.2 is self-similar. The solution is unique if $\|\theta\|_{\mathcal{E}_E}<2\delta$. Moreover, if the initial data is small, $\|\theta_0\|_E<\epsilon$, the previous unique self-similar solution becomes regularized.*

Proof. All well-posedness theorem has been proved by ultimately using Lemma 4.3. One can check (see [12]) that the solution is obtained by a successive approximation method. In fact, defining the recursive sequence

$$\theta_1(x,t)=K_\gamma(t)\theta_0(x)$$
$$\theta_{k+1}(x,t)=\theta_1(x,t)-B(\theta_k,\theta_k)$$

when $k = 1, 2, \dots$. It is easy to verify that $\theta_1(x,t)$ satisfies the self-similarity property

$$\theta_1(x,t) = \lambda^{2\gamma-1}\theta_1(\lambda^{2\gamma}x, \lambda t)\,.$$

A simple induction argument proves that θ_k has this property for all k. Therefore, as the mild solution $\theta(x,t)$ is obtained as the limit of the sequence $\{\theta_k\}$, we have that $\theta(x,t)$ must verify

$$\theta(x,t) = \lambda^{2\gamma-1}\theta(\lambda^{2\gamma}x, \lambda t)$$

for all $\lambda > 0$, all $t > 0$ and $x \in \mathbb{R}^2$. The assertion of uniqueness results from well-posedness theorem. $\square$

Acknowledgement. The author wants to thank the anonymous referee for his/her helpful comments, also to Professor P. G. Lemarié-Rieusset for his constant encouragement and suggestions.

References

[1] Adams, D. R., and Hedberg, L. I.; *Function spaces and potentiel theory, springer-verlag*, Berlin-Heidelberg-New york, 1996.
[2] Barraza, O.; *Self-similar solutions in the weak L^p -spaces of the Navier-Stokes equations*, Rev. Matem. Iberoam. 12(2) (1996) 411-439.
[3] Chae, D., Lee, J,; *Global well-posedness in the super-critical dissipative quasi-geostrophic equations*, Comm. Math. Phys. 233 (2003) 297-311.
[4] Constantin, P., Majda, A., Tabak, E.; *Formation of strong fronts in th quasi-geostrophic thermal activ scalar*, Nonlinearity 7(1994) 1459-1533.
[5] Constantin, P., Cordoba, D., Wu, J.; *On the critical dissipative quasi-geostrophic equation*, Indiana Univ. Math. J.50 (2001) 97-107.
[6] Córdoba, A., Córdoba, D.; *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys. 249 (2004) 511-528.
[7] Gala, S.; *Opérateurs de multiplication entre espaces de Sobolev*, thèse de Doctorat, Université d'Evry, 2005.
[8] Garcia-Cuerva, J. and Rubio de Francia, J. L.; *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.
[9] Hunt, R.; *On $L^{p,q}$ spaces*, L'Enseignement Mathématiques, t. (12) (4) (1966), 249-276.
[10] Ju, N.; *Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space*, Comm. Math. Phys. 251 (2004) 365-376.
[11] Kato, T.; *Strong L^p solutions of the Navier-Stokes equations in Morrey spaces*, Bol. Soc. Bras. Mat. 22,2 (1992) 127-155.
[12] Lemarié-Rieusset, P.G.; *Developments in the Navier-Stokes problem.* Chapman & Hall/ CRC Press, Boca Raton, 2002.
[13] Muckenhoupt, B.; *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207-226.
[14] Pedlosky, J.; *Geophsical fluid dynamics.* Springer-Verlag, New-York, 1987.
[15] Stein, E. M.; *Singular Intgrals and Differntiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.
[16] Stein, E. M.; *Harmonic Analysis : Real-Varible Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, New Jersey, 1993.
[17] Wu, J.; *Quasi-geostrophic type equations with initial data in Morrey spaces*, Nonlinearity 10 (1997) 1409-1420.
[18] Wu, J.; *Dissipative quasi-geostrophic equations with L^p data*, Electronic. J. Differential Equations 2001 (2001), N56, 1-13.

Sadek Gala
Université d'Evry Val d'Essonne, Département de Mathématiques
Bd F. Mitterrand. 91025 Evry Cedex. France Sadek.Gala@maths.univ-evry.fr

Addendum: April 4, 2006.

After publication, the managing editors were informed that key results in this manuscript were "borrowed" from the article below, without the proper citation.

The reader should be aware of this claim and decide on her/his own about the originality of this article.

[1] Jose A. Carrillo, Lucas C. F. Ferreira; *Self-similar solutions and large time asymptotics for the dissipative quasi-geostrophic equations*, HYKE Preprints, 2005-10, http://www.hyke.org/preprint/index.php