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OSCILLATION OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH A DAMPING TERM

ELMETWALLY M. ELABBASY, TAHER S. HASSAN, SAMIR H. SAKER

Abstract. This paper concerns the oscillation of solutions to the differential equation
$$\big(r(t)x'(t)\big)' + p(t)x'(t) + q(t)g(x(t)) = 0,$$
where $xg(x) > 0$ for all $x \neq 0$, $r(t) > 0$ for $t \geq t_0 > 0$. No sign conditions are imposed on $p(t)$ and $q(t)$. Our results solve the open problem posed by Rogovchenko [27], complement the results in Sun [29], and improve a number of existing oscillation criteria. Our main results are illustrated with examples.

1. Introduction

This paper concerned with oscillation of the solutions to the second-order nonlinear differential equation with damping term:
$$\big(r(t)x'(t)\big)' + p(t)x'(t) + q(t)g(x(t)) = 0, \quad t \geq t_0, \tag{1.1}$$
where q and p are continuous functions defined on the interval $[t_0, \infty)$, $t_0 > 0$ and $r(t) > 0$ for $t \geq t_0 > 0$, g is a continuous function for $x \in (-\infty, \infty)$, continuously differentiable and satisfies
$$xg(x) > 0, \quad g'(x) \geq k > 0 \quad \text{for all } x \neq 0. \tag{1.2}$$
Equation (1.1) is said to be superlinear if
$$\int_{\pm\epsilon}^{\pm\infty} \frac{1}{g(u)} du < \infty \quad \text{for } \epsilon > 0, \tag{1.3}$$
and sublinear if
$$\int_{0}^{\pm\epsilon} \frac{1}{g(u)} du < \infty \quad \text{for } \epsilon > 0. \tag{1.4}$$
We restrict our attention to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > T\} > 0$ for any $T \geq t_x$. We make a standing hypothesis that (1.1) does possess such solutions. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros; otherwise it is non-oscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

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In the previous two decades, there has been increasing interest in obtaining sufficient conditions for the oscillation and non-oscillation of solutions of different classes of second order differential equations, see for example [4, 5, 6, 7, 8, 11, 12, 14, 15, 16, 17, 19, 23, 24, 25, 26, 27, 28, 29] and the references therein.

a lot of work has been done on the following particular cases of (1.1):

$$x''(t)+q(t)x(t)=0, \tag{1.5}$$

$$(r(t)x'(t))'+q(t)x(t)=0, \tag{1.6}$$

$$(r(t)x'(t))'+q(t)g(x(t))=0\,. \tag{1.7}$$

An important tool in the study of oscillatory behavior of solutions of these equations is the averaging technique which goes back as far as the classical result of Wintner [32] which proved that (1.5) is oscillatory if

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{t}q(v)dv\,ds=\infty. \tag{1.8}$$

Hartman [14] proved that that the limit in (1.8) cannot be replaced by the limit supremum and proved that the condition

$$-\infty<\lim_{t\to\infty}\inf\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(v)dv\,ds<\lim_{t\to\infty}\sup\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(v)dv\,ds\le\infty, \tag{1.9}$$

implies that every solution of (1.5) oscillates.

Kamenev [15] improved Wintner's result by proving that the condition

$$\lim_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^nq(s)ds=\infty, \tag{1.10}$$

for some integer $n>1$ is sufficient for the oscillation of (1.5).

Yan [36] proved that if

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^nq(s)ds<\infty,$$

for some integer $n>1$ and there exists a function ϕ on $[t_0,\infty)$ satisfying

$$\int t_0{}^{\infty}\phi_+^2(t)dt=\infty,$$

where $\phi_+(t)=\max\{\phi(t),0\}$ such that

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^nq(s)ds>\sup_{u\ge t_0}\phi(u),$$

then every solution of equation (1.5) oscillates.

Philos [24] further improved Kamenev's result by proving the following: Suppose there exist continuous functions $H,\ h:D\equiv\{(t,s):t\ge s\ge t_0\}\to\mathbb{R}$ such that

$$\begin{gathered}H(t,t)=0,\quad t\ge t_0,\\ H(t,s)>0,\quad t>s\ge t_0,\end{gathered} \tag{1.11}$$

and H has a continuous and nonpositive partial derivative on D with respect to the second variable and satisfies

$$-\frac{\partial H(t,s)}{\partial s}=h(t,s)\sqrt{H(t,s)}\ge 0\,. \tag{1.12}$$

Further, suppose that

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,s)q(s) - \frac{1}{4}h^2(t,s)]ds = \infty. \tag{1.13}$$

Then every solution of equation (1.5) oscillates.

We note, however, that when $q(t) = \frac{\gamma}{t^2}$, (1.5) reduces to the well known Euler equation

$$u''(t) + \frac{\gamma}{t^2}u(t) = 0, \quad t \geq 1, \tag{1.14}$$

to which none of the above mentioned oscillation criteria is applicable. In fact, the Euler equation (1.14) is oscillatory if $\gamma > \frac{1}{4}$, and non-oscillatory if $\gamma \leq \frac{1}{4}$, see [17]. For further results on the oscillation of superlinear and sublinear equations, we refer the reader to [6, 7, 8, 31].

For the oscillation of (1.6), Leighton [19] proved that if:

$$\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(t)dt = \infty, \tag{1.15}$$

then every solution of (1.6) oscillates.

Willett [30] used the transformation

$$\tau = \Big(\int_t^{\infty} \frac{ds}{r(s)}\Big)^{-1}, \quad u(t) = \tau^{-1}y(t),$$

to establish a new version of Leighton's criterion and obtained the following oscillation criteria: If

$$\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(t)\Big(t\int_t^{\infty} \frac{ds}{r(s)}\Big)^2 dt = \infty, \tag{1.16}$$

then every solution of (1.6) oscillates.

We note, however, that the oscillation criteria of Leighton and Willett are not applicable to the equation

$$(t^2u'(t))' + \gamma u(t) = 0, \quad t > 0, \tag{1.17}$$

where γ is a positive constant. Kong [17], Li [21], Li and Yeh [22], Rogovechenkov [25], and Yu [38] used the generalized Riccati technique and have given several sufficient conditions for oscillation of (1.6) which can be applied to (1.17); in fact every solution of (1.17) oscillates if $\gamma > \frac{1}{4}$; see [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38].

In the study of the differential equation (1.7), many criteria for oscillation exist which involve the behavior of the integral of q however the common restrictions that

$$q(t) > 0, \quad g'(u) > 0 \quad \text{for } u \neq 0, \quad \int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty,$$

on the functions q, g and r are required; see for example [4, 5, 11, 12].

The presence of the damping term in (1.1) calls for a modified approach to the study of the oscillatory properties of solutions, see for example the paper by Saker, Peng and Agarwal [28], Li, Wang and Yan [23] and the references therein. They cited most of the oscillation results for (1.1) when $p(t)$ and $q(t)$ are positive functions.

Recently, Rogovechenko et al. [27] considered (1.1) and established some sufficient conditions for oscillations. They posed the following open problem: It would be very important to obtain general oscillation criteria for nonlinear differential equations with damping term without requiring additional sign conditions on the coefficients $p(t)$ and $q(t)$.

In this paper, we consider (1.1) and by using a Riccati transformation technique, we establish some oscillation criteria of Kamanev and Philos type with no sign conditions on $p(t)$ and $q(t)$. Our results in this paper are the affirmative answer to the question posed by Rogovechenko et al. [27] and improve and complement the results established by Sun [29].

2. Main Results

In this section, we will use the Riccati technique to establish sufficient conditions for oscillation of (1.1). Comparisons between our results and the previously known are presented and some examples illustrate the main results.

Theorem 2.1. *Assume that* (1.2) *and* (1.3) *hold. Furthermore, suppose that there exists a function* $\rho : [t_0, \infty) \to (0, \infty)$ *such that:*

$$\rho'(t) \geq 0, \quad \big(r(t)\rho(t)\big)' \geq 0, \quad \big(r(t)\rho(t)\big)'' \leq 0, \quad \big[r(t)\rho'(t) - \rho(t)p(t)\big]' \leq 0, \tag{2.1}$$

$$\liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)q(s)ds > -\infty, \tag{2.2}$$

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \big[\int_{t_0}^{s} \rho(u)q(u)du\big]^2 ds = \infty. \tag{2.3}$$

Then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that (1.1) possesses a non-oscillatory solution x on an interval $[T, \infty)$, $T \geq t_0$. Without loss of generality, we shall assume that $x(t) > 0$ for all $t \geq T$ (the case $x(t) < 0$ can be treated similarly and will be omitted). Let $w(t)$ be defined by the Riccati transformation

$$w(t) = \frac{\rho(t)r(t)x'(t)}{g(x(t))}.$$

This and (1.1) imply for $t \geq T$ that

$$w'(t) = -\rho(t)q(t) + \frac{x'(t)}{g(x(t))}[r(t)\rho'(t) - \rho(t)p(t)] - \frac{g'(x(t))w^2(t)}{\rho(t)r(t)}. \tag{2.4}$$

We consider the following two cases:

Case 1. the integral $\int_T^\infty \frac{g'(x(s))w^2(s)}{\rho(s)r(s)}ds$ is finite. Then there exists a positive constant N such that

$$\int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)}ds \leq N \quad \text{for all } t \geq T. \tag{2.5}$$

Now, from (2.4),

$$\begin{aligned} &\int_{t_0}^{t} \rho(s)q(s)ds \\ &= \int_{t_0}^{T} \rho(s)q(s)ds + \int_{T}^{t} \rho(s)q(s)ds \\ &= -w(t) + c_1 + \int_T^t [r(s)\rho'(s) - \rho(s)p(s)]\frac{x'(s)}{g(x(s))}ds - \int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)}ds, \end{aligned} \tag{2.6}$$

where $c_1 = w(T) + \int_{t_0}^{T} \rho(s)q(s)ds$. By Bonnet's theorem since $r(t)\rho'(t) - \rho(t)p(t)$ is non-increasing, for a fixed $t \geq T$, there exists $\xi \in [T, t]$ such that

$$\begin{aligned}\int_T^t [r(s)\rho'(s) - \rho(s)p(s)] \frac{x'(s)}{g(x(s))} ds &= [r(T)\rho'(T) - \rho(T)p(T)] \int_T^{\xi} \frac{x'(s)}{g(x(s))} ds \\ &= [r(T)\rho'(T) - \rho(T)p(T)] \inf_{x(T)}^{x(\xi)} \frac{1}{g(u)} du\,.\end{aligned}$$

Since $[r(T)\rho'(T) - \rho(T)p(T)] \geq 0$ and

$$\int_{x(T)}^{x(\xi)} \frac{1}{g(u)} du < \begin{cases} 0 & \text{if } x(\xi) \leq x(T) \\ \int_{x(T)}^{\infty} \frac{1}{g(u)} du & \text{if } x(\xi) \geq x(T), \end{cases}$$

we have

$$-\infty < \int_T^t [r(s)\rho'(s) - \rho(s)p(s)] \frac{x'(s)}{g(x(s))} ds \leq k_1, \tag{2.7}$$

where

$$k_1 = [r(T)\rho'(T) - \rho(T)p(T)] \int_{x(T)}^{\infty} \frac{1}{g(u)} du.$$

Thus, for $t \geq T$,

$$\begin{aligned}&\big[\int_{t_0}^t \rho(s)q(s)ds\big]^2 \\ &= \big\{c_1 - w(t) + \int_T^t [r(s)\rho'(s) - \rho(s)p(s)] \frac{x'(s)}{g(x(s))} ds - \int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds\big\}^2 \\ &\leq 4c_1^2 + 4(w(t))^2 + 4[\int_T^t [r(s)\rho'(s) - \rho(s)p(s)] \frac{x'(s)}{g(x(s))} ds]^2 \\ &\quad + 4\big[\int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds\big]^2.\end{aligned}$$

Therefore, taking into account (2.5) and (2.7), we conclude that

$$\big[\int_{t_0}^t \rho(s)q(s)ds\big]^2 \leq c_2 + 4(w(t))^2,$$

where

$$c_2 = 4c_1^2 + 4\big[\int_T^t [r(s)\rho'(s) - \rho(s)p(s)] \frac{x'(s)}{g(x(s))} ds\big]^2 + 4\big[\int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds\big]^2.$$

Thus, for every $t \geq T$,

$$\begin{aligned}\int_{t_0}^t \big[\int_{t_0}^s \rho(u)q(u)du\big]^2 ds &= \int_{t_0}^T \big[\int_{t_0}^s \rho(u)q(u)du\big]^2 ds + \int_T^t \big[\int_{t_0}^s \rho(u)q(u)du\big]^2 ds \\ &= c_3 + \int_T^t \big[\int_{t_0}^s \rho(u)q(u)du\big] ds \\ &\leq c_3 + c_2(t - T) + 4\int_T^t w^2(s)ds \\ &\leq c_3 + c_2(t - T) + \frac{4}{k} \int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} r(s)\rho(s)ds,\end{aligned}$$

since $r(t)\rho(t)$ is positive and nondecreasing for $t \in [t_0, \infty)$, Bonnet's theorem ensures the existence of $T_1 \in [T, t]$ such that

$$\int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} r(s)\rho(s)ds = r(t)\rho(t) \int_{T_1}^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds.$$

Also, since $r(t)\rho(t)$ is positive on $[t_0, \infty)$ and $(r(t)\rho(t))'$ is nonnegative and bounded above, it follows that $r(t)\rho(t) \le \beta t$ for all large t where $\beta > 0$ is a constant and this implies that

$$\int_{t_0}^\infty \frac{1}{\rho(s)r(s)} ds = \infty,$$

Thus, we conclude that

$$\int_{t_0}^t [\int_{t_0}^s \rho(u)q(u)du]^2 ds \le c_3 + c_2(t-T) + \frac{4\beta}{k} t \int_{T_1}^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds.$$

So, we have

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^t \big[\int_{t_0}^s \rho(u)q(u)du \big]^2 ds \le c_2 + \frac{4\beta}{k} N < \infty,$$

which contradicts (2.3).

Case 2. The integral $\int_T^\infty \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds$ is infinite. From (2.4), taking into account (2.6) and (2.7), for every $t \ge T$ we obtain

$$\int_{t_0}^t \rho(s)q(s)ds \le -w(t) + A - \int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds, \tag{2.8}$$

where $A = c_1 + k_1$. By the condition (2.2), from (2.8), it follows that for some constant B,

$$-w(t) \ge B + \int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds \quad \text{for every } t \ge T. \tag{2.9}$$

We can consider a $T_2 \ge T$ such that

$$C = B + \int_T^{T_2} \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds > 0.$$

Then (2.9) ensures that $w(t)$ is negative on $[T_2, \infty)$. Now, (2.9) gives

$$\frac{g'(x(t))w^2(t)}{\rho(t)r(t)} \Big(B + \int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds \Big)^{-1} \ge \frac{g'(x(t))x'(t)}{g(x(t))}, \quad t \ge T_2,$$

and consequently for all $t \ge T_2$,

$$\log \frac{B + \int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds}{C} \ge \log \frac{g(x(T_2))}{g(x(t))}.$$

Hence,

$$B + \int_T^t \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds \ge C \frac{g(x(T_2))}{g(x(t))}, \quad t \ge T_2.$$

So, (2.9) yields

$$x'(t) \le \frac{-C'}{\rho(t)r(t)}, \quad t \ge T_2,$$

where $C' = Cg(x(T_2)) > 0$. Thus, we have

$$x(t) \le x(T_2) - C' \int_T^t \frac{1}{\rho(s)r(s)} ds \quad \text{for all } t \ge T_2$$

which leads to $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction. This completes the proof. □

Example 2.2. Consider the differential equation

$$(t^{\frac{1}{2}} x'(t))' - \frac{1}{t} x'(t) + t^{\frac{-3}{2}} (x^2(t) \operatorname{sgn} x(t) + x(t)) = 0, \quad t \ge 1. \tag{2.10}$$

Here $p(t) = \frac{-1}{t}$, $q(t) = t^{-3/2}$ and $g(x) = x^2 \operatorname{sgn} x + x$. We see that p, q and g satisfy conditions (1.2), (1.3) and (2.1). To apply Theorem 2.1. It remains to prove that (2.2) and (2.3) hold. By choosing $\rho(t) = t^{1/2}$ we see that

$$\liminf_{t\to\infty} \int_1^t \rho(s)q(s)ds = \liminf_{t\to\infty} \int_1^t s^{-1} ds > -\infty,$$
$$\limsup_{t\to\infty} \frac{1}{t} \int_1^t [\int_1^s q(u)du]^2 ds = \limsup_{t\to\infty} \frac{2}{t} \int_1^t [1 - \frac{1}{\sqrt{s2}}]^2 ds < \infty,$$
$$\limsup_{t\to\infty} \frac{1}{t} \int_1^t [\int_1^s \rho(u)q(u)du]^2 ds = \infty.$$

Thus Theorem 2.1 ensures that every solution of (2.10) oscillates. Note that that the results from Wong [33, 34], and Bhatia [4] can not be applied to (2.10).

Example 2.3. Consider the differential equation

$$\Big(\frac{1}{t} x'(t)\Big)' + (\frac{1}{2} + \sin t)x'(t) + (\frac{1}{2} + \cos t)x(t) = 0, \quad t \ge 1. \tag{2.11}$$

Here $p(t) = \frac{1}{2} + \sin t$, $q(t) = \frac{1}{2} + \cos t$ and $g(x) = x$. Note that (1.2) and (1.3) are satisfied. By choosing $\rho(t) = t$ we have

$$\liminf_{t\to\infty} \int_1^t \rho(s)q(s)ds = \infty,$$
$$\limsup_{t\to\infty} \frac{1}{t} \int_1^t [\int_1^s \rho(u)q(u)du]^2 ds = \infty.$$

Thus Theorem 2.1 ensures that every solution of (2.11) oscillates.

Theorem 2.4. *Assume that* (1.2), (1.3), (2.1), *and* (2.2) *hold. Furthermore, Suppose that*

$$\int_{\pm\epsilon}^{\pm\infty} \frac{\sqrt{g'(x)}}{g(x)} dx < \infty, \quad \epsilon > 0, \tag{2.12}$$
$$\sqrt{g'(x)}G(x) > 0 \quad \textit{for all } x \ne 0, \tag{2.13}$$

where $G(x) = \int_x^\infty \frac{\sqrt{g'(u)}}{g(u)} du$,

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t^\alpha} \int_T^t \{&(t-s)^\alpha \rho(s)q(s) \\ &- \frac{1}{4c} \int_T^t [\alpha - (t-s)\frac{\lambda(s)}{r(s)\rho(s)}]^2 (t-s)^{\alpha-2} \frac{V(s)}{V'(s)}\} ds = \infty, \end{aligned} \tag{2.14}$$

for some integer $\alpha > 1$*, where*

$$V(t) = \int_{t_0}^{t} \frac{1}{r(s)\rho(s)} ds, \quad \lim_{t\to\infty} V(t) = \infty, \quad \lambda(t) = [r(t)\rho'(t) - \rho(t)p(t)].$$

Then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that (1.1) possesses a non-oscillatory solution x on an interval $[T,\infty), T \geq t_0$. Without loss of generality, we shall assume that $x(t) > 0$ for all $t \geq T \geq t_0$ (the case $x(t) < 0$ treated similarly and will be omitted). Again we define $w(t)$ as in Theorem 2.1. and prove that (1.1) holds then we have equation (2.4). First, we claim that

$$I = \int_{T}^{\infty} \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds,$$

is infinite. Otherwise, if $I < \infty$, there exists a positive constant N such that

$$\int_{T}^{t} \frac{g'(x(s))w^2(s)}{\rho(s)r(s)} ds \leq N, \quad \text{for all } t \geq T.$$

Now, by the Schwarz's inequality, we have

$$\begin{aligned} \Big| \int_{T}^{t} \frac{w(s)}{\rho(s)r(s)} \sqrt{g'(x(s))} ds \Big|^2 &\leq \Big(\int_{T}^{t} \frac{ds}{\rho(s)r(s)} \Big) \Big(\int_{T}^{t} \frac{w^2(s)g'(x(s))}{\rho(s)r(s)} ds \Big) \\ &\leq N \Big(\int_{T}^{t} \frac{ds}{\rho(s)r(s)} \Big). \end{aligned}$$

Set

$$M_1 = \int_{x(T)}^{\infty} \frac{\sqrt{g'(u)}}{g(u)} du > 0, \quad \sqrt{g'(x)} \int_{x(t)}^{\infty} \frac{\sqrt{g'(u)}}{g(u)} du \geq M_2 > 0. \tag{2.15}$$

Then, we have

$$\begin{aligned} g'(x(t)) &\geq M_2^2 \Big[\int_{x(t)}^{\infty} \frac{\sqrt{g'(u)}}{g(u)} du \Big]^{-2} \\ &= M_2^2 \Big[M_1 - \int_{x(T)}^{x(t)} \frac{\sqrt{g'(u)}}{g(u)} du \Big]^{-2} \\ &= M_2^2 \Big[M_1 - \int_{T}^{t} \frac{x'(s)\sqrt{g'(x(s))}}{g(x(s))} ds \Big]^{-2} \\ &\geq M_2^2 \Big[M_1 + \Big| \int_{T}^{t} \frac{w(s)}{\rho(s)r(s)} \sqrt{g'(x(s))} ds \Big| \Big]^{-2} \\ &\geq M_2^2 [M_1 + \sqrt{NV(t)}]^{-2}. \end{aligned}$$

Condition (2.1) ensures (as in Theorem 2.1) that $\lim_{t\to\infty} V(t) = \infty$. Hence,

$$g'(x) \geq M_2^2 \big\{ [M_1 + \sqrt{N}] \sqrt{V(t)} \big\}^{-2} = \frac{c}{V(t)}, \tag{2.16}$$

where $c = M_2^2 [M_1 + \sqrt{N}]^{-2}$. Therefore, by (2.4), taking into account (2.16), we have

$$\rho(t)q(t) \leq -w'(t) + \frac{\lambda(t)}{r(t)\rho(t)} w(t) - c\frac{V'(t)}{V(t)} w^2(t), \tag{2.17}$$

where $\lambda(t) = [r(t)\rho'(t) - \rho(t)p(t)]$. Hence, for all $t \geq T$, we have

$$\begin{aligned}
&\int_T^t (t-s)^\alpha \rho(s)q(s)ds \\
&\leq -\int_T^t (t-s)^\alpha w'(s)ds + \int_T^t \frac{(t-s)^\alpha \lambda(s)w(s)}{r(s)\rho(s)}ds - c\int_T^t \frac{(t-s)^\alpha V'(s)w^2(s)}{V(s)}ds \\
&= (t-T)^\alpha w(T) - \alpha\int_T^t (t-s)^{\alpha-1}w(s)ds + \int_T^t \frac{(t-s)^\alpha \lambda(s)w(s)}{r(s)\rho(s)}ds \\
&\quad - c\int_T^t \frac{(t-s)^\alpha V'(s)w^2(s)}{V(s)}ds \\
&= (t-T)^\alpha w(T) \\
&\quad - \int_T^t \Big\{\sqrt{\frac{cV'(s)}{V(s)}}(t-s)^{\frac{\alpha}{2}}w(s) + \frac{[\alpha - (t-s)\frac{\lambda(s)}{r(s)\rho(s)}](t-s)^{\frac{\alpha}{2}-1}}{\sqrt{\frac{cV'(s)}{V(s)}}}\Big\}^2 ds \\
&\quad + \frac{1}{4c}\int_T^t [\alpha - (t-s)\frac{\lambda(s)}{r(s)\rho(s)}]^2 (t-s)^{\alpha-2}\frac{V(s)}{V'(s)}ds.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{t^\alpha}\int_T^t \Big\{(t-s)^\alpha \rho(s)q(s) - \frac{1}{4c}[\alpha - (t-s)\frac{\lambda(s)}{r(s)\rho(s)}]^2 (t-s)^{\alpha-2}\frac{V(s)}{V'(s)}\Big\}ds \\
&\leq (1 - \frac{T}{t})^\alpha w(T).
\end{aligned}$$

Then, taking a limit superior on both sides, we obtain a contradiction to the condition (2.14). Thus it must be the case $I = \infty$. As in the proof of Theorem 2.1 (Case 2) we arrive at the contradiction $\lim_{t\to\infty} x(t) = -\infty$. This completes the proof. □

Theorem 2.5. *Assume that* (1.2), (2.12), *and* (2.13) *hold. Furthermore, Suppose that*

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty}\int_{t_0}^t \frac{ds}{r(s)} = \infty, \tag{2.18}$$

$$\lim_{t\to\infty}\inf \int_{t_0}^t q(s)ds > -\infty, \tag{2.19}$$

there exist a function $\phi \in C^1([t_0,\infty),(0,\infty))$, $H,\ h : D \equiv \{(t,s) : t \geq s \geq t_0\} \to \mathbb{R}$ *such that*

$$\begin{gathered}
H(t,t) = 0, \quad t \geq t_0, \\
H(t,s) > 0, \quad t > s \geq t_0,
\end{gathered}$$

and H *has a continuous and nonpositive partial derivative on* D *with respect to the second variable and satisfies*

$$-\frac{\partial}{\partial u}(H(t,u)\phi(u)) + H(t,u)\phi(u)\frac{p(u)}{r(u)\xi} = h(t,u)\sqrt{H(t,u)\phi(u)DR'(u)},$$

$$\lim_{t\to\infty}\inf \frac{1}{H(t,s)}\int_s^t [H(t,u)\phi(u)q(u) - \frac{1}{4}R(u)h^2(t,u)]du \geq f(s), \tag{2.20}$$

for some constant $D > 0$, and

$$\lim_{t\to\infty}\int_{t_0}^{t}\frac{[f_+(s)]^2R'(s)}{[\phi(s)]^2R(s)}ds=\infty, \tag{2.21}$$

where $f_+(t)=\max\{f(t),0\}$. Then (1.1) is oscillatory.

Proof. Suppose to the contrary that (1.1) possesses a non-oscillatory solution x on an interval $[T,\infty)$, $T\ge t_0$. Without loss of generality, we shall assume that $x(t)>0$ for all $t\ge T$ (the case $x(t)<0$ can be treated similarly). Let

$$w(t)=\frac{r(t)x'(t)}{g(x(t))}.$$

From this definition and (1.1) it follows that

$$w'(t)=-q(t)-\frac{p(t)}{r(t)}w(t)-\frac{g'(x(t))}{r(t)}w^2(t),\quad t\ge t_0, \tag{2.22}$$

$$\int_{t_0}^{t}q(s)ds=-w(t)+w(t_0)-\int_{t_0}^{t}\frac{p(s)w(s)}{r(s)}ds-\int_{t_0}^{t}\frac{g'(x(s))w^2(s)}{r(s)}ds. \tag{2.23}$$

We distinguish two cases:

Case 1. The integral $\int_{t_0}^{\infty}\frac{g'(x(s))w^2(s)}{r(s)}ds$ is finite. Then there exists a positive constant N such that

$$\int_{T}^{t}\frac{g'(x(s))w^2(s)}{r(s)}ds\le M\quad\text{for all } t\ge t_0. \tag{2.24}$$

By the Schwarz's inequality, we have

$$\begin{aligned}\Big|\int_T^t\frac{w(s)}{r(s)}\sqrt{g'(x(s))}ds\Big|^2&\le\Big(\int_T^t\frac{ds}{r(s)}\Big)\Big(\int_T^t\frac{w^2(s)g'(x(s))}{r(s)}ds\Big)\\&\le M\Big(\int_T^t\frac{ds}{r(s)}\Big)=MR(t).\end{aligned} \tag{2.25}$$

Taking into account (2.15), (2.25) and (2.19), the procedure in Theorem 2.4 implies

$$g'(x(t))\ge\frac{D}{R(t)},\quad t\ge t_0, \tag{2.26}$$

where $D=M_2^2[M_1+\sqrt{M}]^{-2}$. Hence, for every $t\ge t_0$, (2.18) and (2.22) imply

$$q(t)\le -w'(t)-\frac{p(t)}{r(t)}w(t)-\frac{DR'(t)}{R(t)}w^2(t).$$

Then

$$\begin{aligned}
\int_s^t H(t,u)\phi(u)q(u)du &\le -\int_s^t H(t,u)\phi(u)w'(u)du - \int_s^t H(t,u)\phi(u)\frac{p(u)}{r(u)}w(u)du \\
&\quad - D\int_s^t H(t,u)\phi(u)\frac{R'(u)}{R(u)}w^2(u)du, \quad (s \ge t_0) \\
&= H(t,s)\phi(s)w(s) - D\int_s^t H(t,u)\phi(u)\frac{R'(u)}{R(u)}w^2(u)du \\
&\quad - \int_s^t [-\frac{\partial}{\partial u}(H(t,u)\phi(u)) + H(t,u)\phi(u)\frac{p(u)}{r(u)}]w(u)du \\
&= H(t,s)\phi(s)w(s) + \frac{1}{4}R(u)h^2(t,u) \\
&\quad - \int_s^t \Big[\sqrt{\frac{H(t,u)\phi(u)DR'(u)}{R(u)}}w(u) + \frac{1}{2}\sqrt{R(u)}h(t,u)\Big]^2 du.
\end{aligned}$$

Thus for all $t \ge s \ge t_0$, we have

$$\begin{aligned}
&\int_s^t [H(t,u)\phi(u)q(u) - \frac{1}{4}R(u)h^2(t,u)]du \\
&\le H(t,s)\phi(s)w(s) - \int_s^t \Big[\sqrt{\frac{H(t,u)\phi(u)DR'(u)}{R(u)}}w(u) + \frac{1}{2}\sqrt{R(u)}h(t,u)\Big]^2 du.
\end{aligned}$$

Hence,

$$\int_s^t [H(t,u)\phi(u)q(u) - \frac{1}{4}R(u)h^2(t,u)]du \le H(t,s)\phi(s)w(s). \tag{2.27}$$

Dividing (2.27) by $H(t,s)$ and then taking the lower limit as $t \to \infty$, we have

$$f(s) \le \phi(s)w(s), \quad s \ge t_0,$$

and hence

$$[f_+(s)]^2 \le [\phi(s)]^2[w(s)]^2, \quad s \ge t_0.$$

In view of (2.24) and (2.26) we have

$$\begin{aligned}
\lim_{t\to\infty}\int_{t_0}^t \frac{D[f_+(s)]^2R'(s)}{[\phi(s)]^2R(s)}ds &\le \lim_{t\to\infty}\int_{t_0}^t \frac{D[w(s)]^2R'(s)}{R(s)}ds \\
&\le \lim_{t\to\infty}\int_{t_0}^t \frac{[w(s)]^2g'(x(s))}{r(s)}ds \le M < \infty.
\end{aligned}$$

This contradicts (2.21).

Case 2. The integral $\int_{t_0}^\infty \frac{g'(x(s))w^2(s)}{r(s)}ds$ is infinite. In this case, by (2.19) and (2.23), the procedure of Theorem 2.1 (Case 2) leads to the contradiction $\lim_{t\to\infty} x(t) = -\infty$. This completes the proof. □

Corollary 2.6. *Replace* (2.20) *and* (2.21) *by*

$$\lim_{t\to\infty}\sup \frac{1}{H(t,s)}\int_s^t [H(t,u)\phi(u)q(u) - \frac{1}{4}R(u)h^2(t,u)]du = \infty. \tag{2.28}$$

Then the conclusion of Theorem 2.5 still holds.

Proof. Divide (2.27) by $H(t,s)$ and then take the upper limit as $t \to \infty$. This way, we get a contradiction to (2.28). Them the rest of the proof is similar to the proof of Theorem 2.5 (Case 2). $\square$

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Elmetwally M. Elabbasy
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt
Email: emelabbasy@mans.edu.eg

Taher S. Hassan
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt
Email: tshassan@mans.edu.eg

Samir H. Saker
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt
Email: shsaker@mans.edu.eg

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Page 4. To the assumptions in Theorem 2.1, add

$$r(t)\rho'(t) - \rho(t)p(t) \geq 0\,.$$

Page 7. The same assumption needs to added to Theorem 2.4.

Page 7. In Example 2.3 replace (2.11) with

$$\Big(\frac{1}{t}x'(t)\Big)' - \frac{1}{t^2}x'(t) + (2+\cos t)x(t) = 0, \quad t \geq 1.$$

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