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SOLUTIONS APPROACHING POLYNOMIALS AT INFINITY TO NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

CHRISTOS G. PHILOS, PANAGIOTIS CH. TSAMATOS

Abstract. This paper concerns the solutions approaching polynomials at ∞ to n-th order ($n > 1$) nonlinear ordinary differential equations, in which the nonlinear term depends on time t and on $x, x', \dots, x^{(N)}$, where x is the unknown function and N is an integer with $0 \le N \le n-1$. For each given integer m with $\max\{1, N\} \le m \le n-1$, conditions are given which guarantee that, for any real polynomial of degree at most m, there exists a solution that is asymptotic at ∞ to this polynomial. Sufficient conditions are also presented for every solution to be asymptotic at ∞ to a real polynomial of degree at most $n-1$. The results obtained extend those by the authors and by Purnaras [25] concerning the particular case $N = 0$.

1. Introduction

Since its invention by Isaac Newton around 1666, the theory of ordinary differential equations has occupied a central position in the development of mathematics. One reason for this is its widespread applicability in the sciences. Another is its natural connectivity with other areas of mathematics. In the theory of ordinary differential equations, the study of the asymptotic behavior of the solutions is of great importance, especially in the case of nonlinear equations. In applications of nonlinear ordinary differential equations, any information about the asymptotic behavior of the solutions is usually extremely valuable. Thus, there is every reason for studying the asymptotic theory of nonlinear ordinary differential equations.

Very recently, the authors and Purnaras [25] studied solutions, which are asymptotic at infinity to real polynomials of degree at most $n-1$, for the n-th order ($n > 1$) nonlinear ordinary differential equation

$$x^{(n)}(t) = f(t, x(t)), \quad t \ge t_0 > 0, \tag{1.1}$$

where f *is a continuous real-valued function on* $[t_0, \infty) \times \mathbb{R}$. The work in [25] is essentially motivated by the recent one by Lipovan [15] concerning the special case of the second order nonlinear ordinary differential equation

$$x''(t) = f(t, x(t)), \quad t \ge t_0 > 0. \tag{1.2}$$

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The application of the main results in [25] to the second order nonlinear ordinary differential equation (1.2) leads to improved versions of the ones given in [15] (and of other previous related results in the literature). Some closely related results for second order nonlinear differential equations involving the derivative of the unknown function have been given by Rogovchenko and Rogovchenko [29] (see, also, Mustafa and Rogovchenko [17]).

It is the purpose of the present article to extend the results in [25] to the more general case of the n-th order $(n > 1)$ nonlinear ordinary differential equation

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(N)}(t)), \quad t \geq t_0 > 0, \tag{1.3}$$

where N *is an integer with* $0 \leq N \leq n-1$, *and* f *is a continuous real-valued function on* $[t_0, \infty) \times \mathbb{R}^{N+1}$. Note that our thoughts to extend the results in [25] for the differential equation (1.3), in some future time, had been made known in this paper.

Throughout the paper, we are interested in solutions of the differential equation (1.3) which are defined for all large t, i.e., in solutions of (1.3) on an interval $[T, \infty)$, $T \geq t_0$, where T may depend on the solution. For questions about the global existence in the future of the solutions of (1.3), we refer to standard classical theorems in the literature (see, for example, Corduneanu [6], Cronin [7], and Lakshmikantham and Leela [14]).

The paper is organized as follows. In Section 2, for each given integer m with $\max\{1, N\} \leq m \leq n-1$, sufficient conditions are presented in order that, for any real polynomial of degree at most m, the differential equation (1.3) has a solution defined for all large t, which is asymptotic at ∞ to this polynomial and such that the first $n-1$ derivatives of the solution are asymptotic at ∞ to the corresponding first $n-1$ derivatives of the given polynomial. Section 3 is devoted to establishing conditions, which are sufficient for every solution defined for all large t of the differential equation (1.3) to be asymptotic at ∞ to a real polynomial of degree at most $n-1$ (depending on the solution) and the first $n-1$ derivatives of the solution to be asymptotic at ∞ to the corresponding first $n-1$ derivatives of this polynomial. Moreover, in Section 3, conditions are also given, which guarantee that every solution x defined for all large t of (1.3) satisfies $[x^{(j)}(t)/t^{n-1-j}] \to [c/(n-1-j)!]$ for $t \to \infty$ $(j = 0, 1, \dots, n-1)$, where c is some real number (depending on the solution x). Section 4 contains the application of the results to the special case of second order nonlinear ordinary differential equations. For $n = 2$ and $N = 0$, (1.3) becomes (1.2). Moreover, in the special case where $n = 2$ and $N = 1$, (1.3) can be written as

$$x''(t) = f(t, x(t), x'(t)), \quad t \geq t_0 > 0, \tag{1.4}$$

where f *is a continuous real-valued function on* $[t_0, \infty) \times \mathbb{R}^2$. Some general examples are given in the last section (Section 5), which demonstrate the applicability of the results (and, especially, of the main result in Section 2).

The asymptotic theory of n-th order $(n > 1)$ nonlinear differential equations has a very long history. A central role in this theory plays the problem of the study of solutions which have a prescribed asymptotic behavior via solutions of the equation $x^{(n)} = 0$. In the special case of second order nonlinear differential equations, a large number of papers have appeared concerning this problem; see, for example, Cohen [3], Constantin [4], Dannan [8], Hallam [9], Kamo and Usami [10], Kusano, Naito and Usami [11], Lipovan [15], Mustafa and Rogovchenko [17], Naito [18, 19, 20], Philos and Purnaras [24], Rogovchenko and Rogovchenko [29, 30], Rogovchenko [31], Rogovchenko and Villari [32], Souplet [34], Tong [36], Waltman [38], Yin [39], and Zhao [40]. For higher order differential equations (ordinary or, more generally, functional), the above mentioned problem has also been investigated by several researchers; see, for example, Kusano and Trench [12, 13], Meng [16], Philos [21, 22, 23], Philos, Sficas and Staikos [26], Philos and Staikos [27], and the references cited in these papers. We also mention the paper by Trench [37] concerning linear second order ordinary differential equations as well as the paper by Philos and Tsamatos [28] about the problem of the asymptotic equilibrium for nonlinear differential systems with retardations.

Before closing this section, we note that it is especially interesting to examine the possibility of generalizing the results of the present paper in the case of the n-th order ($n > 1$) nonlinear delay differential equation

$$x^{(n)}(t) = f(t, x(t-\tau_0(t)), x'(t-\tau_1(t)), \dots, x^{(N)}(t-\tau_N(t))), \quad t \geq t_0 > 0,$$

where τ_k ($k = 0, 1, \dots, N$) are nonnegative continuous real-valued functions on $[t_0, \infty)$ such that $\lim_{t\to\infty}[t - \tau_k(t)] = \infty$ ($k = 0, 1, \dots, N$).

2. Conditions for the Existence of Solutions that are Asymptotic to Polynomials at Infinity

Our results in this section are the theorem below and its corollary.

Theorem 2.1. *Let m be an integer with $\max\{1, N\} \leq m \leq n-1$, and assume that*

$$|f(t, z_0, z_1, \dots, z_N)| \leq \sum_{k=0}^{N} p_k(t) g_k\big(\frac{|z_k|}{t^{m-k}}\big) + q(t) \quad \text{for all } (t, z_0, z_1, ..., z_N) \in [t_0, \infty) \times \mathbb{R}^{N+1}, \tag{2.1}$$

where p_k ($k = 0, 1, \dots, N$) and q are nonnegative continuous real-valued functions on $[t_0, \infty)$ such that

$$\int_{t_0}^{\infty} t^{n-1} p_k(t) dt < \infty \quad (k = 0, 1, \dots, N), \quad \text{and} \quad \int_{t_0}^{\infty} t^{n-1} q(t) dt < \infty, \tag{2.2}$$

and g_k ($k = 0, 1, \dots, N$) are nonnegative continuous real-valued functions on $[0, \infty)$ which are not identically zero. Let $c_0, c_1, \dots, c_m$ be real numbers and T be a point with $T \geq t_0$, and suppose that there exists a positive constant K so that

$$\begin{aligned} &\max_{k=0,1,\dots,N} \Big\{ \sum_{\ell=0}^{N} \Big[\int_T^{\infty} \frac{(s-T)^{n-1-k}}{(n-1-k)!} p_\ell(s) ds \Big] \Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K) \\ &+ \int_T^{\infty} \frac{(s-T)^{n-1-k}}{(n-1-k)!} q(s) ds \Big\} \leq K, \end{aligned} \tag{2.3}$$

where

$$\Theta_0(c_0, c_1, \dots, c_m; T; K) = \sup\big\{ g_0(z) : \ 0 \leq z \leq \frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}} \big\} \tag{2.4}$$

and, provided that $N > 0$,

$$\Theta_\ell(c_\ell, c_{\ell+1}, ..., c_m; T; K)$$

$$= \sup\big\{g_\ell(z) : 0 \le z \le \frac{K}{T^{m-\ell}} + \sum_{i=\ell}^{m} \frac{i(i-1)...(i-\ell+1)\,|c_i|}{T^{m-i}}\big\} \quad (\ell = 1, ..., N). \tag{2.5}$$

Then the differential equation (1.3) *has a solution x on the interval $[T, \infty)$, which is asymptotic to the polynomial $c_0 + c_1 t + \dots + c_m t^m$ as $t \to \infty$; i.e.,*

$$x(t) = c_0 + c_1 t + \dots + c_m t^m + o(1) \quad \text{as } t \to \infty, \tag{2.6}$$

and satisfies

$$x^{(j)}(t) = \sum_{i=j}^{m} i(i-1)\dots(i-j+1)c_i t^{i-j} + o(1) \quad \text{as } t \to \infty \quad (j = 1, \dots, m) \tag{2.7}$$

and, provided that $m < n - 1$,

$$x^{(\lambda)}(t) = o(1) \quad \text{as } t \to \infty \quad (\lambda = m+1, \dots, n-1). \tag{2.8}$$

Corollary 2.2. *Let m be an integer with $\max\{1, N\} \le m \le n-1$, and assume that* (2.1) *is satisfied, where p_k $(k = 0, 1, \dots, N)$ and q, and g_k $(k = 0, 1, \dots, N)$ are as in Theorem* 2.1. *Then, for any real numbers $c_0, c_1, \dots, c_m$, the differential equation* (1.3) *has a solution x on an interval $[T, \infty)$ (where $T \ge \max\{t_0, 1\}$ depends on $c_0, c_1, \dots, c_m$), which is asymptotic to the polynomial $c_0 + c_1 t + \dots + c_m t^m$ as $t \to \infty$; i.e.,* (2.6) *holds, and satisfies* (2.7) *and* (2.8) (*provided that $m < n - 1$*).

The method which will be applied in the proof of Theorem 2.1 is based on the use of the well-known Schauder fixed point theorem (Schauder [33]). This theorem can be found in several books on functional analysis (see, for example, Conway [5]).

Theorem 2.3 (Schauder theorem)**.** *Let E be a Banach space and X be any non-empty convex and closed subset of E. If S is a continuous mapping of X into itself and SX is relatively compact, then the mapping S has at least one fixed point* (*i.e., there exists an $x \in X$ with $x = Sx$*).

We need to consider the Banach space $BC([T, \infty), \mathbb{R})$ of all bounded continuous real-valued functions on the given interval $[T, \infty)$, endowed with the sup-norm $\|\cdot\|$:

$$\|h\| = \sup_{t \ge T} |h(t)| \quad \text{for } h \in BC([T, \infty), \mathbb{R}).$$

In the proof of Theorem 2.1, we will use the set $(BC)^N([T, \infty), \mathbb{R})$ defined as follows: $(BC)^0([T, \infty), \mathbb{R})$ coincides with $BC([T, \infty), \mathbb{R})$; for $N > 0$, $(BC)^N([T, \infty), \mathbb{R})$ is the set of all bounded continuous real-valued functions on the interval $[T, \infty)$, which have bounded continuous k-order derivatives on $[T, \infty)$ for each $k = 1, \dots, N$. Clearly, $(BC)^N([T, \infty), \mathbb{R})$ is a Banach space endowed with the norm $\|\cdot\|_N$ defined by

$$\|h\|_N = \max_{k=0,1,\dots,N} \|h^{(k)}\| \quad \text{for } h \in (BC)^N([T, \infty), \mathbb{R}).$$

To present a compactness criterion for subsets of the space $(BC)^N([T, \infty), \mathbb{R})$, we first give some well-known definitions of notions referred to sets of real-valued functions. Let U be a set of real-valued functions defined on the interval $[T, \infty)$. The set U is called *uniformly bounded* if there exists a positive constant M such that, for all functions u in U,

$$|u(t)| \le M \quad \text{for every } t \ge T.$$

Also, U is said to be *equicontinuous* if, for each $\epsilon > 0$, there exists a $\delta \equiv \delta(\epsilon) > 0$ such that, for all functions u in U,

$$|u(t_1) - u(t_2)| < \epsilon \quad \text{for every } t_1, t_2 \geq T \text{ with } |t_1 - t_2| < \delta.$$

Moreover, U will be called *equiconvergent at* ∞ if all functions in U are convergent in $\mathbb{R}$ at the point ∞ and, for each $\epsilon > 0$, there exists a $T_\epsilon \geq T$ such that, for all functions u in U,

$$|u(t) - \lim_{s\to\infty} u(s)| < \epsilon \quad \text{for every } t \geq T_\epsilon.$$

We have the following compactness criterion for subsets of $(BC)^N([T,\infty),\mathbb{R})$.

Lemma 2.4 (Compactness criterion)**.** *Let H be a subset of the Banach space $(BC)^N([T,\infty),\mathbb{R})$ endowed with the norm $\|\cdot\|_N$. Define $H^{(0)} = H$ and, provided that $N > 0$, $H^{(k)} = \{h^{(k)} : h \in H\}$ for $k = 1,\dots,N$. If $H^{(k)}$ $(k = 0,1,\dots,N)$ are uniformly bounded, equicontinuous and equiconvergent at ∞, then H is relatively compact.*

In the special case $N = 0$, i.e., in the case of the Banach space $BC([T,\infty),\mathbb{R})$, the above compactness criterion is well-known (see Avramescu [1], Staikos [35]). The method used in the proof of our compactness criterion is a generalization of the one applied in proving this criterion in the special case of the Banach space $BC([T,\infty),\mathbb{R})$.

Proof of Lemma 2.4. First, we notice that the sets $H^{(k)}$ $(k = 0,1,\dots,N)$ are uniformly bounded if and only if the set H is uniformly bounded in $(BC)^N([T,\infty),\mathbb{R})$ in the sense that there exists a positive constant M such that, for all functions h in H,

$$|h^{(k)}(t)| \leq M \quad \text{for every } t \geq T \quad (k = 0,1,\dots,N).$$

Also, we observe that $H^{(k)}$ $(k = 0,1,\dots,N)$ are equicontinuous if and only if H is equicontinuous in $(BC)^N([T,\infty),\mathbb{R})$, that is, for each $\epsilon > 0$, there exists a $\delta \equiv \delta(\epsilon) > 0$ such that, for all functions h in H,

$$|h^{(k)}(t_1) - h^{(k)}(t_2)| < \epsilon \quad \text{for every } t_1, t_2 \geq T \text{ with } |t_1 - t_2| < \delta \quad (k = 0,1,\dots,N).$$

Moreover, $H^{(k)}$ $(k = 0,1,\dots,N)$ are equiconvergent at ∞ if and only if H is equiconvergent at ∞ in $(BC)^N([T,\infty),\mathbb{R})$ in the sense that all functions in H are convergent in $\mathbb{R}$ at the point ∞ and, provided that $N > 0$, the first N derivatives of every function in H tend to zero at ∞, and, for each $\epsilon > 0$, there exists a $T_\epsilon \geq T$ such that, for all functions h in H,

$$|h(t) - \lim_{s\to\infty} h(s)| < \epsilon \quad \text{for every } t \geq T_\epsilon$$

and, provided that $N > 0$,

$$|h^{(k)}(t)| < \epsilon \quad \text{for every } t \geq T_\epsilon \quad (k = 1,\dots,N).$$

Let $(BC)^N_\ell([T,\infty),\mathbb{R})$ be the subspace of $(BC)^N([T,\infty),\mathbb{R})$ consisting of all functions h in $(BC)^N([T,\infty),\mathbb{R})$ such that $\lim_{t\to\infty} h(t)$ exists in $\mathbb{R}$ and, provided that $N > 0$,

$$\lim_{t\to\infty} h^{(k)}(t) = 0 \quad (k = 1,\dots,N).$$

Note that $(BC)^N_\ell([T,\infty),\mathbb{R})$ is a closed subspace of $(BC)^N([T,\infty),\mathbb{R})$.

Consider the Banach space $C([0,1],\mathbb{R})$ of all continuous real-valued functions on the interval $[0,1]$, endowed with the sup-norm $\|\cdot\|^0$:

$$\|h\|^0 = \sup_{0\le t\le 1} |h(t)| \quad \text{for } h \in C([0,1],\mathbb{R}).$$

Consider, also, the set $C^N([0,1],\mathbb{R})$ defined as follows: $C^0([0,1],\mathbb{R})$ coincides with $C([0,1],\mathbb{R})$; for $N>0$, $C^N([0,1],\mathbb{R})$ is the set of all N-times continuously differentiable real-valued functions on the interval $[0,1]$. Clearly, $C^N([0,1],\mathbb{R})$ is a Banach space endowed with the norm $\|\ \|_N^0$ defined by

$$\|h\|_N^0 = \max_{k=0,1,\dots,N} \|h^{(k)}\|^0 \quad \text{for } h \in C^N([0,1],\mathbb{R}).$$

By the Arzelà-Ascoli theorem, a subset of the Banach space $C^N([0,1],\mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous. Note that a subset H_0 of $C^N([0,1],\mathbb{R})$ is called uniformly bounded if there exists a positive constant M such that, for all functions h_0 in H_0,

$$|h_0^{(k)}(t)| \le M \quad \text{for every } t\in[0,1] \quad (k=0,1,\dots,N).$$

Also, a subset H_0 of $C^N([0,1],\mathbb{R})$ is said to be equicontinuous if, for each $\epsilon>0$, there exists a $\delta\equiv\delta(\epsilon)>0$ such that, for all functions h_0 in H_0,

$$|h_0^{(k)}(t_1) - h_0^{(k)}(t_2)| < \epsilon$$

for every $t_1,t_2\in[0,1]$ with $|t_1-t_2|<\delta$ $(k=0,1,\dots,N)$.

Next, we consider the function $\Phi : (BC)_\ell^N([T,\infty),\mathbb{R}) \to C^N([0,1],\mathbb{R})$ defined by the formula

$$(\Phi x)(t) = \begin{cases} x(T+\frac{t}{1-t}), & \text{if } 0\le t<1\\ \lim_{s\to\infty} x(s), & \text{if } t=1. \end{cases}$$

It is not difficult to check that Φ is a homeomorphism between the Banach spaces $(BC)_\ell^N([T,\infty),\mathbb{R})$ and $C^N([0,1],\mathbb{R})$. So, it follows that a subset of the space $(BC)_\ell^N([T,\infty),\mathbb{R})$ is relatively compact if and only if it is uniformly bounded, equicontinuous and equiconvergent at ∞.

Now, assume that the sets $H^{(k)}$ $(k=0,1,\dots,N)$ are uniformly bounded, equicontinuous and equiconvergent at ∞. Then H is uniformly bounded, equicontinuous and equiconvergent at ∞, in $(BC)^N([T,\infty),\mathbb{R})$. Thus, H is a relatively compact subset of $(BC)_\ell^N([T,\infty),\mathbb{R})$. Since $(BC)_\ell^N([T,\infty),\mathbb{R})$ is a closed subspace of $(BC)^N([T,\infty),\mathbb{R})$, we can be led to the conclusion that H is also relatively compact in $(BC)^N([T,\infty),\mathbb{R})$. The proof of the lemma is complete. □

Proof of Theorem 2.1. Set

$$P_m(t) = c_0 + c_1 t + \dots + c_m t^m \equiv \sum_{i=0}^m c_i t^i \quad \text{for } t\in\mathbb{R}.$$

We have

$$P_m^{(j)}(t) = \sum_{i=j}^m i(i-1)...(i-j+1)c_i t^{i-j} \quad \text{for } t\in\mathbb{R} \quad (j=1,...,m)$$

and, provided that $m<n-1$,

$$P_m^{(\lambda)}(t) = 0 \quad \text{for } t\in\mathbb{R} \quad (\lambda = m+1,\dots,n-1).$$

Furthermore, we see that the substitution $y = x - P_m$ transforms the differential equation (1.3) into the equation

$$y^{(n)}(t) = f(t, y(t) + P_m(t), y'(t) + P'_m(t), \dots, y^{(N)}(t) + P_m^{(N)}(t)). \tag{2.9}$$

We observe that

$$y(t) = x(t) - (c_0 + c_1 t + \dots + c_m t^m),$$
$$y^{(j)}(t) = x^{(j)}(t) - \sum_{i=j}^{m} i(i-1)\dots(i-j+1)c_i t^{i-j} \quad (j = 1, \dots, m),$$

and, provided that $m < n-1$,

$$y^{(\lambda)}(t) = x^{(\lambda)}(t) \quad (\lambda = m+1, \dots, n-1).$$

So, by taking into account (2.6), (2.7) and (2.8), we can be led to the conclusion that what we have to prove is that the differential equation (2.9) has a solution y on the interval $[T, \infty)$, which satisfies

$$\lim_{t\to\infty} y^{(\rho)}(t) = 0 \quad (\rho = 0, 1, \dots, n-1). \tag{2.10}$$

Let E denote the Banach space $(BC)^N([T,\infty), \mathbb{R})$ endowed with the norm $\|\cdot\|_N$, and let us define

$$Y = \{y \in E : \ \|y\|_N \le K\}.$$

It is clear that Y is a nonempty convex and closed subset of E.

Consider, now, an arbitrary function y in Y. Then $|y(t)| \le K$ for every $t \ge T$ and, provided that $N > 0$,

$$|y^{(\ell)}(t)| \le K \quad \text{for every } t \ge T \quad (\ell = 1, \dots, N).$$

Thus, for every $t \ge T$, we obtain

$$\frac{|y(t) + P_m(t)|}{t^m} \le \frac{|y(t)|}{t^m} + \sum_{i=0}^{m} \frac{|c_i|}{t^{m-i}} \le \frac{K}{T^m} + \sum_{i=0}^{m} \frac{|c_i|}{T^{m-i}}$$

and, provided that $N > 0$,

$$\begin{aligned}\frac{|y^{(\ell)}(t) + P_m^{(\ell)}(t)|}{t^{m-\ell}} &\le \frac{|y^{(\ell)}(t)|}{t^{m-\ell}} + \sum_{i=\ell}^{m} \frac{i(i-1)\dots(i-\ell+1)|c_i|}{t^{m-i}} \\ &\le \frac{K}{T^{m-\ell}} + \sum_{i=\ell}^{m} \frac{i(i-1)\dots(i-\ell+1)|c_i|}{T^{m-i}} \quad (\ell = 1, \dots, N).\end{aligned}$$

Hence, we have

$$g_0\Big(\frac{|y(t) + P_m(t)|}{t^m}\Big) \le \Theta_0(c_0, c_1, \dots, c_m; T; K) \quad \text{for } t \ge T,$$

where $\Theta_0(c_0, c_1, \dots, c_m; T; K)$ is defined by (2.4); moreover, provided that $N > 0$, we have

$$g_\ell\Big(\frac{|y^{(\ell)}(t) + P_m^{(\ell)}(t)|}{t^{m-\ell}}\Big) \le \Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K) \quad \text{for } t \ge T \quad (\ell = 1, \dots, N),$$

where $\Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K)$ are defined by (2.5). But, from (2.1) it follows that

$$\begin{aligned}&\big|f\big(t, y(t) + P_m(t), y'(t) + P'_m(t), \dots, y^{(N)}(t) + P_m^{(N)}(t)\big)\big| \\ &\le p_0(t) g_0\Big(\frac{|y(t) + P_m(t)|}{t^m}\Big) + p_1(t) g_1\Big(\frac{|y'(t) + P'_m(t)|}{t^{m-1}}\Big)\end{aligned}$$

$$+\dots + p_N(t)g_N\big(\frac{|y^{(N)}(t)+P_m^{(N)}(t)|}{t^{m-N}}\big) + q(t)$$

for all $t \ge T$. So, we have

$$\begin{aligned}&|f(t, y(t)+P_m(t), y'(t)+P_m'(t), \dots, y^{(N)}(t)+P_m^{(N)}(t))| \\ &\le \sum_{\ell=0}^{N} p_\ell(t)\Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K) \quad \text{for every } t \ge T. \end{aligned} \tag{2.11}$$

This inequality, together with (2.2), guarantee that

$$\int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} f(s, y(s)+P_m(s), y'(s)+P_m'(s), \dots, y^{(N)}(s)+P_m^{(N)}(s))ds$$

exists in $\mathbb{R}$. More generally, for each $\rho \in \{0, 1, \dots, n-1\}$,

$$\int_T^\infty \frac{(s-T)^{n-1-\rho}}{(n-1-\rho)!} f(s, y(s)+P_m(s), y'(s)+P_m'(s), \dots, y^{(N)}(s)+P_m^{(N)}(s))ds$$

exists in $\mathbb{R}$. Next, we use (2.11) to obtain, for any $k \in \{0, 1, \dots, N\}$ and for every $t \ge T$,

$$\begin{aligned}&\Big|\int_t^\infty \frac{(s-t)^{n-1-k}}{(n-1-k)!} f\Big(s, y(s)+P_m(s), y'(s)+P_m'(s), \dots, y^{(N)}(s)+P_m^{(N)}(s)\Big)ds\Big| \\ &\le \int_t^\infty \frac{(s-t)^{n-1-k}}{(n-1-k)!}\Big|f\Big(s, y(s)+P_m(s), y'(s)+P_m'(s), \dots, y^{(N)}(s)+P_m^{(N)}(s)\Big)\Big|ds \\ &\le \int_T^\infty \frac{(s-T)^{n-1-k}}{(n-1-k)!}\Big|f\Big(s, y(s)+P_m(s), y'(s)+P_m'(s), \dots, y^{(N)}(s)+P_m^{(N)}(s)\Big)\Big|ds \\ &\le \sum_{\ell=0}^{N}\Big[\int_T^\infty \frac{(s-T)^{n-1-k}}{(n-1-k)!}p_\ell(s)ds\Big]\Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K) \\ &\quad + \int_T^\infty \frac{(s-T)^{n-1-k}}{(n-1-k)!}q(s)ds\,.\end{aligned}$$

Hence, by using (2.3), we have

$$\begin{aligned}&\Big|\int_t^\infty \frac{(s-t)^{n-1-k}}{(n-1-k)!} f\big(s, y(s)+P_m(s), y'(s)+P_m'(s), \dots, y^{(N)}(s)+P_m^{(N)}(s)\big)ds\Big| \\ &\le K \quad \text{for all } t \ge T \quad (k = 0, 1, \dots, N).\end{aligned} \tag{2.12}$$

We have thus proved that every function y in Y is such that (2.12) holds. So, it is not difficult to check that the formula

$$\begin{aligned}(Sy)(t) &= (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \\ &\quad\times f\big(s, y(s)+P_m(s), y'(s)+P_m'(s), \dots, y^{(N)}(s)+P_m^{(N)}(s)\big)ds \quad \text{for } t \ge T\end{aligned}$$

defines a mapping S of Y into itself. Our purpose is to apply the Schauder theorem for this mapping. We shall prove that S satisfies the assumptions of the Schauder theorem.

We will show that SY is relatively compact. Define $(SY)^{(0)} = SY$ and, provided that $N > 0$, $(SY)^{(k)} = \{(Sy)^{(k)} : y \in Y\}$ for $k = 1, \dots, N$. By the given compactness criterion, in order to show that SY is relatively compact, it suffices to establish that each one of the sets $(SY)^{(k)}$ ($k = 0, 1, \dots, N$) is uniformly bounded, equicontinuous, and equiconvergent at ∞. Let k be an arbitrary integer in $\{0, 1, \dots, N\}$. Since $SY \subseteq Y$, we obviously have $\|(Sy)^{(k)}\| \le K$ for all $y \in Y$, and consequently $(SY)^{(k)}$ is uniformly bounded. Moreover, in view of (2.11), we obtain, for any function $y \in Y$ and every $t \ge T$,

$$\begin{aligned}
|(Sy)^{(k)}(t) - 0| &= \Big| \int_t^\infty \frac{(s-t)^{n-1-k}}{(n-1-k)!} \\
&\quad \times f\Big(s, y(s) + P_m(s), y'(s) + P_m'(s), \dots, y^{(N)}(s) + P_m^{(N)}(s)\Big) ds \Big| \\
&\le \int_t^\infty \frac{(s-t)^{n-1-k}}{(n-1-k)!} \\
&\quad \times \Big| f\Big(s, y(s) + P_m(s), y'(s) + P_m'(s), \dots, y^{(N)}(s) + P_m^{(N)}(s)\Big) \Big| ds \\
&\le \sum_{\ell=0}^{N} \Big[\int_t^\infty \frac{(s-t)^{n-1-k}}{(n-1-k)!} p_\ell(s) ds \Big] \Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K) \\
&\quad + \int_t^\infty \frac{(s-t)^{n-1-k}}{(n-1-k)!} q(s) ds.
\end{aligned}$$

Thus, by (2.2), it follows easily that $(SY)^{(k)}$ is equiconvergent at ∞. Furthermore, by using again (2.11), for any $y \in Y$ and for every t_1, t_2 with $T \le t_1 < t_2$, we have:

$$\begin{aligned}
&|(Sy)^{(n-1)}(t_2) - (Sy)^{(n-1)}(t_1)| \\
&= \Big| \int_{t_2}^\infty f\Big(s, y(s) + P_m(s), y'(s) + P_m'(s), \dots, y^{(n-1)}(s) + P_m^{(n-1)}(s)\Big) ds \\
&\quad - \int_{t_1}^\infty f\Big(s, y(s) + P_m(s), y'(s) + P_m'(s), \dots, y^{(n-1)}(s) + P_m^{(n-1)}(s)\Big) ds \Big| \\
&= \Big| - \int_{t_1}^{t_2} f\Big(s, y(s) + P_m(s), y'(s) + P_m'(s), \dots, y^{(n-1)}(s) + P_m^{(n-1)}(s)\Big) ds \Big| \\
&\le \int_{t_1}^{t_2} f\Big(s, y(s) + P_m(s), y'(s) + P_m'(s), \dots, y^{(n-1)}(s) + P_m^{(n-1)}(s)\Big) \Big| ds \\
&\le \sum_{\ell=0}^{n-1} \Big[\int_{t_1}^{t_2} p_\ell(s) ds \Big] \Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K) + \int_{t_1}^{t_2} q(s) ds,
\end{aligned}$$

if $k = n-1$ (and so $N = n-1$); and

$$\begin{aligned}
&|(Sy)^{(k)}(t_2) - (Sy)^{(k)}(t_1)| \\
&= \Big| \int_{t_2}^\infty \frac{(s-t_2)^{n-1-k}}{(n-1-k)!} f\Big(s, y(s) + P_m(s), y'(s) + P_m'(s), \dots, y^{(N)}(s) + P_m^{(N)}(s)\Big) ds \\
&- \int_{t_1}^\infty \frac{(s-t_1)^{n-1-k}}{(n-1-k)!} f\Big(s, y(s) + P_m(s), y'(s) + P_m'(s), \dots, y^{(N)}(s) + P_m^{(N)}(s)\Big) ds \Big| \\
&= \Big| \int_{t_2}^\infty \Big[\int_r^\infty \frac{(s-r)^{n-2-k}}{(n-2-k)!} f(s, y(s) + P_m(s), y'(s) + P_m'(s), \\
&\quad \dots, y^{(N)}(s) + P_m^{(N)}(s)) ds \Big] dr
\end{aligned}$$

$$
\begin{aligned}
&-\int_{t_1}^{\infty}\Big[\int_r^{\infty}\frac{(s-r)^{n-2-k}}{(n-2-k)!}f(s,y(s)+P_m(s),y'(s)+P'_m(s),\\
&\dots,y^{(N)}(s)+P_m^{(N)}(s))ds\Big]dr\Big|\\
&=\Big|-\int_{t_1}^{t_2}\Big[\int_r^{\infty}\frac{(s-r)^{n-2-k}}{(n-2-k)!}f\big(s,y(s)+P_m(s),y'(s)+P'_m(s),\\
&\dots,y^{(N)}(s)+P_m^{(N)}(s)\big)ds\Big]dr\Big|\\
&\le\int_{t_1}^{t_2}\Big[\int_r^{\infty}\frac{(s-r)^{n-2-k}}{(n-2-k)!}\Big|f(s,y(s)+P_m(s),y'(s)+P'_m(s),\\
&\dots,y^{(N)}(s)+P_m^{(N)}(s))\Big|ds\Big]dr\\
&\le\sum_{\ell=0}^{N}\Big\{\int_{t_1}^{t_2}\Big[\int_r^{\infty}\frac{(s-r)^{n-2-k}}{(n-2-k)!}p_\ell(s)ds\Big]dr\Big\}\Theta_\ell(c_\ell,c_{\ell+1},\dots,c_m;T;K)\\
&\quad+\int_{t_1}^{t_2}\Big[\int_r^{\infty}\frac{(s-r)^{n-2-k}}{(n-2-k)!}q(s)ds\Big]dr,
\end{aligned}
$$

if $k<n-1$. Hence, it is not difficult to verify that the set $(SY)^{(k)}$ is equicontinuous. We have thus proved that SY is relatively compact.

It remains to prove that the mapping S is continuous. To this end, let us consider a $y\in Y$ and an arbitrary sequence $(y_\nu)_{\nu\ge1}$ in Y with

$$\|\cdot\|_N-\lim_{\nu\to\infty}y_\nu=y.$$

Then we obviously have $\|\cdot\|-\lim_{\nu\to\infty}y_\nu=y$ and, provided that $N>0$,

$$\|\cdot\|-\lim_{\nu\to\infty}y_\nu^{(k)}=y^{(k)}\quad(k=1,\dots,N).$$

On the other hand, by (2.11), we have, for all $\nu\ge1$,

$$
\begin{aligned}
&\Big|f\Big(t,y_\nu(t)+P_m(t),y'_\nu(t)+P'_m(t),\dots,y_\nu^{(N)}(t)+P_m^{(N)}(t)\Big)\Big|\\
&\le\sum_{\ell=0}^{N}p_\ell(t)\Theta_\ell(c_\ell,c_{\ell+1},\dots,c_m;T;K)+q(t)\quad\text{for every } t\ge T.
\end{aligned}
$$

So, because of (2.2), one can apply the Lebesgue dominated convergence theorem to obtain, for every $t\ge T$,

$$
\begin{aligned}
&\lim_{\nu\to\infty}\int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}f(s,y_\nu(s)+P_m(s),y'_\nu(s)+P'_m(s),\dots,y_\nu^{(N)}(s)+P_m^{(N)}(s))ds\\
&=\int_t^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}f(s,y(s)+P_m(s),y'(s)+P'_m(s),\dots,y^{(N)}(s)+P_m^{(N)}(s))ds.
\end{aligned}
$$

This ensures the pointwise convergence $\lim_{\nu\to\infty}(Sy_\nu)(t)=(Sy)(t)$ for $t\ge T$. Next, we establish that

$$\|\cdot\|_N-\lim_{\nu\to\infty}Sy_\nu=Sy. \tag{2.13}$$

For this purpose, we consider an arbitrary subsequence $(Sy_{\mu_\nu})_{\nu\ge1}$ of $(Sy_\nu)_{\nu\ge1}$. Since SY is relatively compact, there exists a subsequence $(Sy_{\mu_{\lambda_\nu}})_{\nu\ge1}$ of $(Sy_{\mu_\nu})_{\nu\ge1}$ and a $u\in E$ so that

$$\|\cdot\|_N-\lim_{\nu\to\infty}Sy_{\mu_{\lambda_\nu}}=u.$$

Since the $\|\cdot\|_N$−convergence implies the uniform convergence and, in particular, the pointwise one to the same limit function, we must have $u = Sy$. This means that (2.13) holds true. We have thus proved that the mapping S is continuous.

Now, by applying the Schauder theorem, we conclude that there exists a $y \in Y$ with $y = Sy$. That is,

$$y(t) = (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \times f\Big(s, y(s) + P_m(s), y'(s) + P_m'(s), \dots, y^{(N)}(s) + P_m^{(N)}(s)\Big) ds \quad \text{for } t \ge T.$$

This yields

$$y^{(n)}(t) = f(t, y(t) + P_m(t), y'(t) + P_m'(t), \dots, y^{(N)}(t) + P_m^{(N)}(t)) \quad \text{for } t \ge T$$

and so y is a solution on $[T,\infty)$ of the differential equation (2.9). Furthermore, for each $\rho = 0, 1, \dots, n-1$, we have

$$(-1)^{n-\rho} y^{(\rho)}(t) = \int_t^\infty \frac{(s-t)^{n-1-\rho}}{(n-1-\rho)!} f\Big(s, y(s) + P_m(s), y'(s) + P_m'(s), \dots, y^{(N)}(s) + P_m^{(N)}(s)\Big) ds$$

for all $t \ge T$. Thus, it follows that the solution y satisfies (2.10). The proof of the theorem is now complete. □

Proof of Corollary 2.2. Let $c_0, c_1, \dots, c_m$ be given real numbers. By taking into account the hypothesis that g_k $(k = 0, 1, \dots, N)$ are not identically zero on $[0,\infty)$, we can consider a positive constant K so that

$$\Theta_0^0 \equiv \sup\Big\{g_0(z) : \ 0 \le z \le K + \sum_{i=0}^m |c_i|\Big\} > 0$$

and, provided that $N > 0$,

$$\Theta_\ell^0 \equiv \sup\Big\{g_\ell(z) : \ 0 \le z \le K + \sum_{i=0}^m i(i-1)\dots(i-\ell+1)|c_i|\Big\} > 0 \quad (\ell = 1, \dots, N).$$

Furthermore, by (2.2), we can choose a point $T \ge \max\{t_0, 1\}$ such that

$$\int_T^\infty \frac{(s-T)^{n-1-k}}{(n-1-k)!} p_\ell(s) ds \le \frac{K}{2(N+1)\Theta_\ell^0} \quad (k, \ell = 0, 1, \dots, N)$$

and

$$\int_T^\infty \frac{(s-T)^{n-1-k}}{(n-1-k)!} q(s) ds \le \frac{K}{2} \quad (k = 0, 1, \dots, N).$$

Since $T \ge 1$, we have

$$\frac{K}{T^m} + \sum_{i=0}^m \frac{|c_i|}{T^{m-i}} \le K + \sum_{i=0}^m |c_i|$$

and, provided that $N > 0$,

$$\frac{K}{T^{m-\ell}} + \sum_{i=\ell}^m \frac{i(i-1)\dots(i-\ell+1)|c_i|}{T^{m-i}} \le K + \sum_{i=\ell}^m i(i-1)\dots(i-\ell+1)|c_i|$$

for $\ell = 1, \dots, N$. Consequently,

$$\Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K) \le \Theta_\ell^0 \quad (\ell = 0, 1, \dots, N),$$

where $\Theta_0(c_0, c_1, \dots, c_m; T; K)$ is defined by (2.4) and, in the case where $N > 0$, $\Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K)$ $(\ell = 1, \dots, N)$ are defined by (2.5). Now, we obtain

$$
\begin{aligned}
&\max_{k=0,1,\dots,N} \Big\{ \sum_{\ell=0}^{N} \Big[\int_T^\infty \frac{(s-T)^{n-1-k}}{(n-1-k)!} p_\ell(s)ds \Big] \Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K) \\
&+ \int_T^\infty \frac{(s-T)^{n-1-k}}{(n-1-k)!} q(s)ds \Big\} \\
&\le \sum_{\ell=0}^{N} \frac{K}{2(N+1)\Theta_\ell^0} \Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K) + \frac{K}{2} \\
&= \sum_{\ell=0}^{N} \frac{K}{2(N+1)} \cdot \frac{\Theta_\ell(c_\ell, c_{\ell+1}, \dots, c_m; T; K)}{\Theta_\ell^0} + \frac{K}{2} \\
&\le \frac{K}{2(N+1)}(N+1) + \frac{K}{2} = K,
\end{aligned}
$$

which implies (2.3). Hence, the corollary follows from Theorem 2.1. □

3. Sufficient Conditions for all Solutions to be Asymptotic to Polynomials at Infinity

Our results in this section are formulated as a proposition and a theorem. Our proposition is interesting of its own as a new result. Moreover, this proposition will be used, in a basic way, in proving Theorem 3.2.

Proposition 3.1. *Assume that*

$$
|f(t, z_0, z_1, \dots, z_N)| \le \sum_{k=0}^{N} p_k(t) g_k\big(\frac{|z_k|}{t^{n-1-k}}\big) + q(t)
$$

$$
\text{for all } (t, z_0, z_1, \dots, z_N) \in [t_0, \infty) \times \mathbb{R}^{N+1}, \quad (3.1)
$$

where p_k $(k = 0, 1, \dots, N)$ and q are nonnegative continuous real-valued functions on $[t_0, \infty)$ such that

$$
\int_{t_0}^\infty p_k(t)dt < \infty \quad (k = 0, 1, \dots, N), \quad \text{and} \quad \int_{t_0}^\infty q(t)dt < \infty; \tag{3.2}
$$

and g_k $(k = 0, 1, \dots, N)$ are continuous real-valued functions on $[0, \infty)$, which are positive and increasing on $(0, \infty)$ and such that

$$
\int_1^\infty \frac{dz}{\sum_{k=0}^{N} g_k(z)} = \infty. \tag{3.3}
$$

Then every solution x on an interval $[T, \infty)$, $T \ge t_0$, of the differential equation (1.3) *satisfies*

$$
x^{(j)}(t) = \frac{c}{(n-1-j)!} t^{n-1-j} + o(t^{n-1-j}) \quad \text{as } t \to \infty \quad (j = 0, 1, \dots, n-1), \tag{3.4}
$$

where c is some real number (depending on the solution x).

Theorem 3.2. *Assume that* (3.1) *is satisfied, where* p_k $(k = 0, 1, \dots, N)$ *and* q *are as in Theorem* 2.1*, i.e., nonnegative continuous real-valued functions on* $[t_0, \infty)$ *such that*

$$\int_{t_0}^{\infty} t^{n-1} p_k(t)dt < \infty \quad (k = 0, 1, \dots, N), \quad \text{and} \quad \int_{t_0}^{\infty} t^{n-1} q(t)dt < \infty, \tag{3.5}$$

and g_k $(k = 0, 1, \dots, N)$ *are as in Proposition* 3.1*. Then every solution* x *on an interval* $[T, \infty)$, $T \ge t_0$, *of the differential equation* (1.3) *is asymptotic to a polynomial* $c_0 + c_1 t + \dots + c_{n-1} t^{n-1}$ *as* $t \to \infty$*; i.e.,*

$$x(t) = c_0 + c_1 t + \dots + c_{n-1} t^{n-1} + o(1) \quad \text{as} \quad t \to \infty, \tag{3.6}$$

and satisfies

$$x^{(j)}(t) = \sum_{i=j}^{n-1} i(i-1)\dots(i-j+1)c_i t^{i-j} + o(1) \quad \text{as} \quad t \to \infty \quad (j = 1, \dots, n-1), \tag{3.7}$$

where $c_0, c_1, \dots, c_{n-1}$ *are real numbers* (*depending on the solution* x)*. More precisely, every solution* x *on an interval* $[T, \infty)$, $T \ge t_0$, *of* (1.3) *satisfies*

$$x(t) = C_0 + C_1(t-T) + \dots + C_{n-1}(t-T)^{n-1} + o(1) \quad \text{as} \quad t \to \infty \tag{3.8}$$

and

$$x^{(j)}(t) = \sum_{i=j}^{n-1} i(i-1)\dots(i-j+1)C_i(t-T)^{i-j} + o(1) \quad \text{as} \quad t \to \infty \quad (j = 1, \dots, n-1), \tag{3.9}$$

where

$$C_i = \frac{1}{i!}\Big[x^{(i)}(T) + (-1)^{n-1-i}\int_T^{\infty} \frac{(s-T)^{n-1-i}}{(n-1-i)!} f(s, x(s), x'(s), \dots, x^{(N)}(s))ds\Big] \quad (i = 0, 1, \dots, n-1). \tag{3.10}$$

Combining Corollary 2.2 and Theorem 3.2, we obtain the following result.

Assume that (3.1) *is satisfied, where* p_k $(k = 0, 1, \dots, N)$ *and* q *are nonnegative continuous real-valued functions on* $[t_0, \infty)$ *such that* (3.5) *holds, and* g_k $(k = 0, 1, \dots, N)$ *are nonnegative continuous real-valued functions on* $[0, \infty)$ *which are not identically zero. Then, for any real polynomial of degree at most* $n-1$*, the differential equation* (1.3) *has a solution defined for all large* t*, which is asymptotic at* ∞ *to this polynomial and such that the first* $n-1$ *derivatives of the solution are asymptotic at* ∞ *to the corresponding first* $n-1$ *derivatives of the given polynomial. Moreover, if, in addition,* g_k $(k = 0, 1, \dots, N)$ *are positive and increasing on* $(0, \infty)$ *and such that* (3.3) *holds, then every solution defined for all large* t *of the differential equation* (1.3) *is asymptotic at* ∞ *to a real polynomial of degree at most* $n-1$ (*depending on the solution*)*and the first* $n-1$ *derivatives of the solution are asymptotic at* ∞ *to the corresponding first* $n-1$ *derivatives of this polynomial.*

The following lemma plays an important role in proving our proposition. This lemma is the well-known Bihari's lemma (see Bihari [2]; see, also, Corduneanu [6]) in a simple form which suffices for our needs.

Lemma 3.3 (Bihari). *Assume that*

$$h(t) \le M + \int_{T_0}^{t} \mu(s) g(h(s)) ds \quad \text{for } t \ge T_0,$$

where M is a positive constant, h and μ are nonnegative continuous real-valued functions on $[T_0, \infty)$, and g is a continuous real-valued function on $[0, \infty)$, which is positive and increasing on $(0, \infty)$ and such that

$$\int_{1}^{\infty} \frac{dz}{g(z)} = \infty.$$

Then

$$h(t) \le G^{-1}\Big(G(M) + \int_{T_0}^{t} \mu(s) ds\Big) \quad \text{for } t \ge T_0,$$

where G is a primitive of $1/g$ on $(0, \infty)$ and G^{-1} is the inverse function of G.

Proof of Proposition 3.1. Consider an arbitrary solution x on an interval $[T, \infty)$, $T \ge t_0$, of the differential equation (1.3). From (1.3) it follows that

$$x^{(k)}(t) = \sum_{i=k}^{n-1} \frac{(t-T)^{i-k}}{(i-k)!} x^{(i)}(T) + \int_{T}^{t} \frac{(t-s)^{n-1-k}}{(n-1-k)!} f\big(s, x(s), x'(s), \dots, x^{(N)}(s)\big) ds$$

$(k = 0, 1, \dots, N)$ for $t \ge T$. Therefore, in view of (3.1), for any $k \in \{0, 1, \dots, N\}$ and every $t \ge T$, we obtain

$$\begin{aligned}
&|x^{(k)}(t)| \\
&\le \sum_{i=k}^{n-1} \frac{(t-T)^{i-k}}{(i-k)!} |x^{(i)}(T)| + \int_{T}^{t} \frac{(t-s)^{n-1-k}}{(n-1-k)!} |f(s, x(s), x'(s), \dots, x^{(N)}(s))| ds \\
&\le \sum_{i=k}^{n-1} \frac{t^{i-k}}{(i-k)!} |x^{(i)}(T)| + t^{n-1-k} \int_{T}^{t} |f(s, x(s), x'(s), \dots, x^{(N)}(s))| ds \\
&\le \sum_{i=k}^{n-1} \frac{t^{i-k}}{(i-k)!} |x^{(i)}(T)| + t^{n-1-k} \int_{T}^{t} \Big[\sum_{\ell=0}^{N} p_\ell(s) g_\ell\Big(\frac{|x^{(\ell)}(s)|}{s^{n-1-\ell}}\Big) + q(s)\Big] ds \\
&\le \Big[\sum_{i=k}^{n-1} \frac{t^{i-k}}{(i-k)!} |x^{(i)}(T)| + t^{n-1-k} \int_{T}^{t} q(s) ds\Big] \\
&\quad + t^{n-1-k} \int_{T}^{t} \Big[\sum_{\ell=0}^{N} p_\ell(s) g_\ell\Big(\frac{|x^{(\ell)}(s)|}{s^{n-1-\ell}}\Big)\Big] ds.
\end{aligned}$$

Thus, for any $k \in \{0, 1, \dots, N\}$, we have

$$\begin{aligned}
&\frac{|x^{(k)}(t)|}{t^{n-1-k}} \\
&\le \Big[\sum_{i=k}^{n-1} \frac{1}{(i-k)! t^{n-1-i}} |x^{(i)}(T)| + \int_{T}^{\infty} q(s) ds\Big] + \int_{T}^{t} \Big[\sum_{\ell=0}^{N} p_\ell(s) g_\ell\Big(\frac{|x^{(\ell)}(s)|}{s^{n-1-\ell}}\Big)\Big] ds
\end{aligned}$$

for every $t \geq T$. So, by taking into account (3.2), we immediately conclude that, for each $k \in \{0, 1, \dots, N\}$, there exists a positive constant M_k such that

$$\frac{|x^{(k)}(t)|}{t^{n-1-k}} \leq M_k + \int_T^t \Big[\sum_{\ell=0}^{N} p_\ell(s) g_\ell\Big(\frac{|x^{(\ell)}(s)|}{s^{n-1-\ell}}\Big)\Big] ds \quad \text{for } t \geq T.$$

Hence, by setting $M = \max_{k=0,1,\dots,N} M_k$ (M is a positive constant), we obtain

$$\frac{|x^{(k)}(t)|}{t^{n-1-k}} \leq M + \int_T^t \Big[\sum_{\ell=0}^{N} p_\ell(s) g_\ell\Big(\frac{|x^{(\ell)}(s)|}{s^{n-1-\ell}}\Big)\Big] ds \quad \text{for} \quad t \geq T \quad (k = 0, 1, \dots, N).$$

That is,

$$\frac{|x^{(k)}(t)|}{t^{n-1-k}} \leq h(t) \quad \text{for every } t \geq T \quad (k = 0, 1, \dots, N), \tag{3.11}$$

where

$$h(t) = M + \int_T^t \Big[\sum_{\ell=0}^{N} p_\ell(s) g_\ell\Big(\frac{|x^{(\ell)}(s)|}{s^{n-1-\ell}}\Big)\Big] ds \quad \text{for } t \geq T.$$

Furthermore, by using (3.11) and the hypothesis that g_k ($k = 0, 1, \dots, N$) are increasing on $(0, \infty)$, we obtain for every $t \geq T$

$$\begin{aligned} h(t) &\equiv M + \int_T^t \Big[\sum_{\ell=0}^{N} p_\ell(s) g_\ell\Big(\frac{|x^{(\ell)}(s)|}{s^{n-1-\ell}}\Big)\Big] ds \\ &\leq M + \int_T^t \Big[\sum_{\ell=0}^{N} p_\ell(s) g_\ell(h(s))\Big] ds \\ &\leq M + \int_T^t \Big[\sum_{\ell=0}^{N} p_\ell(s)\Big]\Big[\sum_{\ell=0}^{N} g_\ell(h(s))\Big] ds. \end{aligned}$$

Consequently,

$$h(t) \leq M + \int_T^t \Big[\sum_{\ell=0}^{N} p_\ell(s)\Big] g(h(s)) ds \quad \text{for } t \geq T, \tag{3.12}$$

where $g = \sum_{\ell=0}^{N} g_\ell$. Clearly, g is a continuous real-valued function on $[0, \infty)$, which is positive and increasing on $(0, \infty)$. Moreover, because of (3.3), g is such that

$$\int_1^\infty \frac{dz}{g(z)} = \infty. \tag{3.13}$$

Next, we consider the function

$$G(z) = \int_M^z \frac{du}{g(u)} \quad \text{for } z \geq M.$$

We observe that G is a primitive of the function $1/g$ on $[M, \infty)$. It is obvious that $G(M) = 0$ and that G is strictly increasing on $[M, \infty)$. Also, by (3.13), we have $G(\infty) = \infty$. So, it follows that $G([M, \infty)) = [0, \infty)$. Thus, the inverse function G^{-1} of G is defined on $[0, \infty)$. Moreover, G^{-1} is also strictly increasing on $[0, \infty)$, and $G^{-1}([0, \infty)) = [M, \infty)$. Furthermore, we can take into account (3.12) and use the Bihari lemma to conclude that h satisfies

$$h(t) \leq G^{-1}\Big(G(M) + \int_T^t \Big[\sum_{\ell=0}^{N} p_\ell(s)\Big] ds\Big) = G^{-1}\Big(\sum_{\ell=0}^{N} \int_T^t p_\ell(s) ds\Big)$$

for $t \ge T$. Therefore, in view of (3.2), it follows that

$$h(t) \le G^{-1}\Big(\sum_{\ell=0}^{N}\int_T^{\infty} p_\ell(s)ds\Big) \quad \text{for every } t \ge T;$$

i.e., there exists a positive constant Λ such that $h(t) \le \Lambda$ for $t \ge T$. So, (3.11) yields

$$\frac{|x^{(k)}(t)|}{t^{n-1-k}} \le \Lambda \quad \text{for all } t \ge T \quad (k = 0, 1, \dots, N). \tag{3.14}$$

Now, by taking into account (3.1) and (3.14), we obtain for $t \ge T$

$$\begin{aligned}|f\big(t, x(t), x'(t), \dots, x^{(N)}(t)\big)| &\le \sum_{k=0}^{N} p_k(t) g_k\Big(\frac{|x^{(k)}(t)|}{t^{n-1-k}}\Big) + q(t)\\ &\le \sum_{k=0}^{N} p_k(t)\big[\sup_{0\le z\le\Lambda} g_k(z)\big] + q(t)\end{aligned}$$

and consequently, in view of (3.2),

$$\int_T^{\infty} f\big(s, x(s), x'(s), \dots, x^{(N)}(s)\big)ds \quad \text{exists in } \mathbb{R}.$$

On the other hand, from (1.3) it follows that

$$x^{(n-1)}(t) = x^{(n-1)}(T) + \int_T^t f(s, x(s), x'(s), \dots, x^{(N)}(s))ds \quad \text{for } t \ge T,$$

which gives

$$\lim_{t\to\infty} x^{(n-1)}(t) = x^{(n-1)}(T) + \int_T^{\infty} f\big(s, x(s), x'(s), \dots, x^{(N)}(s)\big)ds \equiv c,$$

where c is a real number (depending on the solution x). Finally, by applying the L'Hospital rule, we obtain

$$\lim_{t\to\infty}\frac{x^{(j)}(t)}{t^{n-1-j}} = \frac{1}{(n-1-j)!}\lim_{t\to\infty} x^{(n-1)}(t) = \frac{c}{(n-1-j)!} \quad (j = 0, 1, \dots, n-1),$$

which implies that x satisfies (3.4). The proof of the proposition is complete. □

Proof of Theorem 3.2. Let x be an arbitrary solution on an interval $[T, \infty)$, $T \ge t_0$, of the differential equation (1.3). Since (3.5) implies (3.2), as in the proof of Proposition 3.1, we can be led to the conclusion that (3.14) holds, where Λ is some positive constant. This conclusion is also a consequence of Proposition 3.1 itself; in fact, from this proposition it follows that, for each $k = 0, 1, \dots, N$, $\lim_{t\to\infty}[x^{(k)}(t)/t^{n-1-k}]$ exists (as a real number). By using (3.1) and (3.14), we obtain

$$\begin{aligned}|f\big(t, x(t), x'(t), \dots, x^{(N)}(t)\big)| &\le \sum_{k=0}^{N} p_k(t) g_k\Big(\frac{|x^{(k)}(t)|}{t^{n-1-k}}\Big) + q(t)\\ &\le \sum_{k=0}^{N} p_k(t)\big[\sup_{0\le z\le\Lambda} g_k(z)\big] + q(t)\end{aligned}$$

for every $t \geq T$. This, together with (3.5), guarantee that

$$L_i \equiv \int_T^\infty \frac{(s-T)^{n-1-i}}{(n-1-i)!} f\big(s, x(s), x'(s), \dots, x^{(N)}(s)\big) ds \quad (i = 0, 1, \dots, n-1)$$

are real numbers. Now, (1.3) gives, for $t \geq T$,

$$x(t) = \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!} x^{(i)}(T) + \int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f\big(s, x(s), x'(s), \dots, x^{(N)}(s)\big) ds. \tag{3.15}$$

Following the same procedure as in the proof of the corresponding theorem in Philos, Purnaras and Tsamatos [25], we can show that

$$\begin{aligned} &\int_T^t \frac{(t-s)^{n-1}}{(n-1)!} f\big(s, x(s), x'(s), \dots, x^{(N)}(s)\big) ds \\ &= \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!} (-1)^{n-1-i} L_i + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\big(s, x(s), x'(s), \dots, x^{(N)}(s)\big) ds \end{aligned}$$

for all $t \geq T$. So, (3.15) becomes

$$\begin{aligned} x(t) &= \sum_{i=0}^{n-1} \frac{(t-T)^i}{i!} \big[x^{(i)}(T) + (-1)^{n-1-i} L_i\big] \\ &\quad + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\big(s, x(s), x'(s), \dots, x^{(N)}(s)\big) ds \quad \text{for } t \geq T. \end{aligned}$$

Taking into account the definition of L_i $(i = 0, 1, \dots, n-1)$ as well as (3.10), we see that the above equation can be written as

$$x(t) = \sum_{i=0}^{n-1} C_i (t-T)^i + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\big(s, x(s), x'(s), \dots, x^{(N)}(s)\big) ds \tag{3.16}$$

for all $t \geq T$. We have

$$\lim_{t\to\infty} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f\big(s, x(s), x'(s), \dots, x^{(N)}(s)\big) ds = 0$$

and thus (3.16) implies that the solution x satisfies (3.8). Furthermore, (3.16) gives

$$\begin{aligned} x^{(j)}(t) &= \sum_{i=j}^{n-1} i(i-1)\dots(i-j+1) C_i (t-T)^{i-j} \\ &\quad + (-1)^{n-j} \int_t^\infty \frac{(s-t)^{n-1-j}}{(n-1-j)!} f(s, x(s), x'(s), \dots, x^{(N)}(s)) ds \\ &\quad \text{for } \quad t \geq T \quad (j = 1, \dots, n-1). \end{aligned} \tag{3.17}$$

Since

$$\lim_{t\to\infty} \int_t^\infty \frac{(s-t)^{n-1-j}}{(n-1-j)!} f(s, x(s), x'(s), \dots, x^{(N)}(s)) ds = 0 \quad (j = 1, \dots, n-1),$$

it follows from (3.17) that the solution x satisfies, in addition, (3.9). Finally, we observe that

$$C_0 + C_1(t-T) + \dots + C_{n-1}(t-T)^{n-1} \equiv c_0 + c_1 t + \dots + c_{n-1} t^{n-1}$$

for some real numbers $c_0, c_1, \dots, c_{n-1}$. So, the solution x satisfies (3.6) and (3.7). The proof is complete. $\square$

4. Application of the Results to Second Order Nonlinear Ordinary Differential Equations

This section is devoted to the application of the results to the special case of the second order nonlinear ordinary differential equations (1.2) and (1.4).

In the case of the differential equation (1.2), Theorem 2.1, Corollary 2.2, Proposition 3.1, and Theorem 3.2 are formulated as follows:

Theorem 4.1. *Assume that*

$$|f(t,z)| \le p(t)g\Big(\frac{|z|}{t}\Big) + q(t) \quad \textit{for all } (t,z) \in [t_0,\infty) \times \mathbb{R}, \tag{4.1}$$

where p and q are nonnegative continuous real-valued functions on $[t_0,\infty)$ such that

$$\int_{t_0}^{\infty} tp(t)dt < \infty \quad \textit{and} \quad \int_{t_0}^{\infty} tq(t)dt < \infty;$$

and g is a nonnegative continuous real-valued function on $[0,\infty)$ which is not identically zero. Let c_0, c_1 be real numbers and T be a point with $T \ge t_0$, and suppose that there exists a positive constant K so that

$$\Big[\int_T^{\infty} (s-T)p(s)ds\Big] \sup\Big\{g(z) :\ 0 \le z \le \frac{K}{T} + \frac{|c_0|}{T} + |c_1|\Big\} + \int_T^{\infty} (s-T)q(s)ds \le K.$$

Then the differential equation (1.2) has a solution x on the interval $[T,\infty)$, which is asymptotic to the line $c_0 + c_1 t$ as $t \to \infty$; i.e.,

$$x(t) = c_0 + c_1 t + o(1) \quad \textit{as} \quad t \to \infty, \tag{4.2}$$

and satisfies

$$x'(t) = c_1 + o(1) \quad \textit{as} \quad t \to \infty. \tag{4.3}$$

Corollary 4.2. *Assume that (4.1) is satisfied, where p, q, and g are as in Theorem 4.1. Then, for any real numbers c_0, c_1, the differential equation (1.2) has a solution x on an interval $[T,\infty)$ (where $T \ge \max\{t_0, 1\}$ depends on c_0, c_1), which is asymptotic to the line $c_0 + c_1 t$ as $t \to \infty$; i.e., (4.2) holds, and satisfies (4.3).*

Proposition 4.3. *Assume that (4.1) is satisfied, where p and q are nonnegative continuous real-valued functions on $[t_0,\infty)$ such that*

$$\int_{t_0}^{\infty} p(t)dt < \infty \quad \textit{and} \quad \int_{t_0}^{\infty} q(t)dt < \infty,$$

and g is a continuous real-valued function on $[0,\infty)$, which is positive and increasing on $(0,\infty)$ and such that

$$\int_1^{\infty} \frac{dz}{g(z)} = \infty.$$

Then every solution x on an interval $[T,\infty)$, $T \ge t_0$, of the differential equation (1.2) satisfies

$$x(t) = ct + o(t) \quad \textit{and} \quad x'(t) = c + o(1), \quad \textit{as} \quad t \to \infty,$$

where c is some real number (depending on the solution x).

Theorem 4.4. *Assume that* (4.1) *is satisfied, where* p *and* q *are as in Theorem* 4.1*, and* g *is as in Proposition* 4.3*. Then every solution* x *on an interval* $[T,\infty)$*,* $T \ge t_0$*, of the differential equation* (1.2) *is asymptotic to a line* $c_0 + c_1 t$ *as* $t \to \infty$*; i.e.,* (4.2) *holds, and satisfies* (4.3)*, where* c_0, c_1 *are real numbers* (*depending on the solution* x)*. More precisely, every solution* x *on an interval* $[T,\infty)$*,* $T \ge t_0$*, of* (1.2) *satisfies*

$$x(t) = C_0 + C_1(t-T) + o(1) \quad \text{and} \quad x'(t) = C_1 + o(1), \quad \text{as} \quad t \to \infty,$$

where

$$C_0 = x(T) - \int_T^\infty (s-T) f(s, x(s)) ds \quad \text{and} \quad C_1 = x'(T) + \int_T^\infty f(s,x(s)) ds.$$

The above results have also been obtained in Philos, Purnaras and Tsamatos [25] (as consequences of the main results given therein). Here, these results are stated for the sake of completeness.

Now, we concentrate on the differential equation (1.4). By applying Theorem 2.1, Corollary 2.2, Proposition 3.1, and Theorem 3.2 to the differential equation (1.4), we obtain following results:

Theorem 4.5. *Assume that*

$$|f(t,z_0,z_1)| \le p_0(t) g_0\big(\frac{|z_0|}{t}\big) + p_1(t) g_1(|z_1|) + q(t) \quad \text{for all } (t,z_0,z_1) \in [t_0,\infty) \times \mathbb{R}^2, \tag{4.4}$$

where p_0*,* p_1*, and* q *are nonnegative continuous real-valued functions on* $[t_0,\infty)$ *such that*

$$\int_{t_0}^\infty t p_0(t) dt < \infty, \quad \int_{t_0}^\infty t p_1(t) dt < \infty, \quad \text{and} \quad \int_{t_0}^\infty t q(t) dt < \infty;$$

and g_0 *and* g_1 *are nonnegative continuous real-valued functions on* $[0,\infty)$ *which are not identically zero. Let* c_0, c_1 *be real numbers and* T *be a point with* $T \ge t_0$*, and suppose that there exists a positive constant* K *so that*

$$\begin{aligned} &\Big[\int_T^\infty (s-T) p_0(s) ds\Big] \sup\big\{g_0(z) : \ 0 \le z \le \frac{K}{T} + \frac{|c_0|}{T} + |c_1|\big\} \\ &+ \Big[\int_T^\infty (s-T) p_1(s) ds\Big] \sup\big\{g_1(z) : \ 0 \le z \le K + |c_1|\big\} + \int_T^\infty (s-T) q(s) ds \le K \end{aligned}$$

and

$$\begin{aligned} &\Big[\int_T^\infty p_0(s) ds\Big] \sup\big\{g_0(z) : \ 0 \le z \le \frac{K}{T} + \frac{|c_0|}{T} + |c_1|\big\} \\ &+ \Big[\int_T^\infty p_1(s) ds\Big] \sup\big\{g_1(z) : \ 0 \le z \le K + |c_1|\big\} + \int_T^\infty q(s) ds \le K\,. \end{aligned}$$

Then the conclusion of Theorem 4.1 *holds for the differential equation* (1.4)*.*

Corollary 4.6. *Assume that* (4.4) *is satisfied, where* p_0, p_1*, and* q*, and* g_0 *and* g_1 *are as in Theorem* 4.5*. Then the conclusion of Corollary* 4.2 *holds for the differential equation* (1.4)*.*

Proposition 4.7. *Assume that* (4.4) *is satisfied, where* p_0, p_1, q *are nonnegative continuous real-valued functions on* $[t_0, \infty)$ *such that*

$$\int_{t_0}^{\infty} p_0(t)dt < \infty, \quad \int_{t_0}^{\infty} p_1(t)dt < \infty, \quad \text{and} \quad \int_{t_0}^{\infty} q(t)dt < \infty;$$

and g_0 *and* g_1 *are continuous real-valued functions on* $[0, \infty)$, *which are positive and increasing on* $(0, \infty)$ *and such that*

$$\int_{1}^{\infty} \frac{dz}{g_0(z) + g_1(z)} = \infty.$$

Then the conclusion of Proposition 4.3 *holds for the differential equation* (1.4).

Theorem 4.8. *Assume that* (4.4) *is satisfied, where* p_0, p_1, q *are as in Theorem* 4.5, *and* g_0 *and* g_1 *are as in Proposition* 4.7. *Then the conclusion of Theorem* 4.4 *holds for the differential equation* (1.4) *with*

$$C_0 = x(T) - \int_{T}^{\infty} (s-T) f(s, x(s), x'(s))ds, \quad C_1 = x'(T) + \int_{T}^{\infty} f(s, x(s), x'(s))ds.$$

5. Examples

Example 5.1 (Philos, Purnaras, Tsamatos [25]). Consider the second order superlinear Emden-Fowler equation

$$x''(t) = a(t)[x(t)]^2 \operatorname{sgn} x(t), \quad t \geq t_0 > 0, \tag{5.1}$$

where *a is a continuous real-valued function on* $[t_0, \infty)$.

Applying Theorem 2.1 (or, in particular, Theorem 4.1), we obtain the following result:

Assume that

$$\int_{t_0}^{\infty} t^3 |a(t)| dt < \infty. \tag{5.2}$$

Let c_0, c_1 *be real numbers and* T *be a point with* $T \geq t_0$, *and suppose that there exists a positive constant* K *so that*

$$A(T)\Big(\frac{K}{T} + \frac{|c_0|}{T} + |c_1|\Big)^2 \leq K, \tag{5.3}$$

where

$$A(T) = \int_{T}^{\infty} (s-T) s^2 |a(s)| ds. \tag{5.4}$$

Then (5.1) *has a solution* x *on the interval* $[T, \infty)$, *which is asymptotic to the line* $c_0 + c_1 t$ *as* $t \to \infty$; *i.e.,*

$$x(t) = c_0 + c_1 t + o(1) \quad \text{as} \quad t \to \infty, \tag{5.5}$$

and satisfies

$$x'(t) = c_1 + o(1) \quad \text{as} \quad t \to \infty\,. \tag{5.6}$$

Now, assume that (5.2) is satisfied, and let c_0, c_1 be given real numbers and $T \geq t_0$ be a fixed point. Moreover, let $A(T)$ be defined by (5.4). As it has been proved in [25], there exists a positive constant K so that (5.3) holds if and only if

$$A(T)(|c_0| + |c_1|T) \leq \frac{T^2}{4}. \tag{5.7}$$

Thus, we have the following result:

Assume that (5.2) *is satisfied, and let* c_0, c_1 *be real numbers and* $T \geq t_0$ *be a point so that* (5.7) *holds, where* $A(T)$ *is defined by* (5.4)*. Then* (5.1) *has a solution* x *on the interval* $[T, \infty)$*, which satisfies* (5.5)*and* (5.6)*.*

In particular, let us consider the differential equation (5.1) with $a(t) = t^{\sigma}\mu(t)$ for $t \geq t_0$, where σ is a real number and μ is a continuous and bounded real-valued function on $[t_0, \infty)$. In this case, there exists a positive constant θ so that

$$|a(t)| \leq \theta t^{\sigma} \quad \text{for every } t \geq t_0.$$

We see that (5.2) is satisfied if $\sigma < -4$. Furthermore, assume that $\sigma < -4$ and let c_0, c_1 be real numbers and $T \geq t_0$ be a point. Then (see [25]) it follows that (5.7) holds if

$$T^{\sigma+2}(|c_0| + |c_1|T) \leq \frac{(\sigma+3)(\sigma+4)}{4\theta}.$$

Example 5.2. Consider the n-th order ($n > 1$) sublinear Emden-Fowler equation

$$x^{(n)}(t) = a(t)|x(t)|^{1/2}\operatorname{sgn} x(t), \quad t \geq t_0 > 0, \tag{5.8}$$

where *a is a continuous real-valued function on* $[t_0, \infty)$.

For the differential equation (5.8), we have the following result:
Let m *be an integer with* $1 \leq m \leq n-1$*, and assume that*

$$\int_{t_0}^{\infty} t^{n-1+(m/2)}|a(t)|dt < \infty. \tag{5.9}$$

Then, for any real numbers $c_0, c_1, \dots, c_m$*, the differential equation* (5.8) *has a solution* x *on the* (*whole*) *interval* $[t_0, \infty)$*, which is asymptotic to the polynomial* $c_0 + c_1 t + \dots + c_m t^m$ *as* $t \to \infty$*; i.e.,*

$$x(t) = c_0 + c_1 t + \dots + c_m t^m + o(1) \quad \text{as} \quad t \to \infty,$$

and satisfies

$$x^{(j)}(t) = \sum_{i=j}^{m} i(i-1)\dots(i-j+1)c_i t^{i-j} + o(1) \quad \text{as} \quad t \to \infty \quad (j = 1, \dots, m)$$

and, provided that $m < n-1$*,*

$$x^{(\lambda)}(t) = o(1) \quad \text{as} \quad t \to \infty \quad (\lambda = m+1, \dots, n-1).$$

To prove the above result, we assume that (5.9) is satisfied and we consider arbitrary real numbers $c_0, c_1, \dots, c_m$. By Theorem 2.1, it is sufficient to show that there exists a positive constant K such that

$$A(t_0)\Big(\frac{K}{t_0^m} + \sum_{i=0}^{m} \frac{|c_i|}{t_0^{m-i}}\Big)^{1/2} \leq K, \tag{5.10}$$

where

$$A(t_0) = \int_{t_0}^{\infty} \frac{(s-t_0)^{n-1}}{(n-1)!} s^{m/2}|a(s)|ds.$$

In the trivial case $A(t_0) = 0$, (5.10) holds for any positive constant K. So, in the sequel, we suppose that $A(t_0) > 0$. We see that (5.10) is equivalent to

$$K^2 - \frac{[A(t_0)]^2}{t_0^m}K - [A(t_0)]^2 \sum_{i=0}^{m} \frac{|c_i|}{t_0^{m-i}} \geq 0. \tag{5.11}$$

Let us consider the quadratic equation

$$\Omega(\omega) \equiv \omega^2 - \frac{[A(t_0)]^2}{t_0^m}\omega - [A(t_0)]^2 \sum_{i=0}^{m} \frac{|c_i|}{t_0^{m-i}} = 0$$

in the complex plane. The discriminant of this equation is

$$\Delta = \Big[-\frac{[A(t_0)]^2}{t_0^m} \Big]^2 - 4\Big[-[A(t_0)]^2 \sum_{i=0}^{m} \frac{|c_i|}{t_0^{m-i}} \Big].$$

We see that $\Delta > 0$ and so the equation $\Omega(\omega) = 0$ has two real roots:

$$\omega_1 = \frac{[A(t_0)]^2}{2t_0^m} - \frac{\sqrt{\Delta}}{2}, \quad \omega_2 = \frac{[A(t_0)]^2}{2t_0^m} + \frac{\sqrt{\Delta}}{2}$$

with $\omega_1 < \omega_2$. Clearly, $\omega_2 > 0$. We have $\Omega(\omega) \geq 0$ for all $\omega \geq \omega_2$. Hence, (5.11) (or, equivalently, (5.10)) is satisfied for every positive constant K with $K \geq \omega_2 > 0$. We have thus proved that, in both cases where $A(t_0) = 0$ or $A(t_0) > 0$, there exists a positive constant K so that (5.10) holds. So, our result has been proved.

Example 5.3. Consider the second order Emden-Fowler equation

$$x''(t) = a(t)|x(t)|^\gamma \operatorname{sgn} x(t) + b(t)|x'(t)|^\delta \operatorname{sgn} x'(t), \quad t \geq t_0 > 0, \tag{5.12}$$

where *a and b are continuous real-valued functions on* $[t_0, \infty)$, *and* γ *and* δ *are positive real numbers.*

By applying Theorem 2.1 (or, in particular, Theorem 4.5) to the differential equation (5.12), we arrive at the next result:

Assume that

$$\int_{t_0}^{\infty} t^{1+\gamma}|a(t)|dt < \infty \quad \textit{and} \quad \int_{t_0}^{\infty} t|b(t)|dt < \infty. \tag{5.13}$$

Let c_0, c_1 *be real numbers and* T *be a point with* $T \geq t_0$, *and suppose that there exists a positive constant* K *so that*

$$\Big[\int_T^{\infty} (s-T)s^\gamma|a(s)|ds\Big]\Big(\frac{K}{T} + \frac{|c_0|}{T} + |c_1|\Big)^\gamma + \Big[\int_T^{\infty} (s-T)|b(s)|ds\Big](K + |c_1|)^\delta \\ \leq K$$

and

$$\Big[\int_T^{\infty} s^\gamma|a(s)|ds\Big]\Big(\frac{K}{T} + \frac{|c_0|}{T} + |c_1|\Big)^\gamma + \Big[\int_T^{\infty} |b(s)|ds\Big](K + |c_1|)^\delta \leq K.$$

Then (5.12) *has a solution* x *on the interval* $[T, \infty)$, *which is asymptotic to the line* $c_0 + c_1 t$ *as* $t \to \infty$; *i.e.,* (5.5) *holds, and satisfies* (5.6).

Moreover, an application of Corollary 2.2 (or, in particular, of Corollary 4.6) to the differential equation (5.12) leads to the following result:

Assume that (5.13) *is satisfied. Then, for any real numbers* c_0, c_1, (5.12) *has a solution* x *on an interval* $[T, \infty)$ (*where* $T \geq \max\{t_0, 1\}$ *depends on* c_0, c_1), *which satisfies* (5.5) *and* (5.6).

Also, we can apply Proposition 3.1 (or, in particular, Proposition 4.7) for the differential equation (5.12) to obtain the result:

If

$$\int_{t_0}^{\infty} t^\gamma|a(t)|dt < \infty \quad \textit{and} \quad \int_{t_0}^{\infty} |b(t)|dt < \infty, \tag{5.14}$$

and $\gamma \le 1$ *and* $\delta \le 1$, *then every solution* x *on an interval* $[T,\infty)$, $T \ge t_0$, *of* (5.12) *satisfies*

$$x(t) = ct + o(t) \quad \textit{and} \quad x'(t) = c + o(1), \quad \textit{as} \quad t \to \infty,$$

where c *is some real number (depending on the solution* x*).*

Furthermore, applying Theorem 3.2 (or, in particular, Theorem 4.8) to the differential equation (5.12), we obtain the following result:

Assume that (5.13) *is satisfied, and that* $\gamma \le 1$ *and* $\delta \le 1$. *Then every solution* x *on an interval* $[T,\infty)$, $T \ge t_0$, *of* (5.12) *is asymptotic to a line* $c_0 + c_1 t$ *as* $t \to \infty$; *i.e.,* (5.5) *holds, and satisfies* (5.6), *where* c_0, c_1 *are real numbers (depending on the solution* x*). More precisely, every solution* x *on an interval* $[T,\infty)$, $T \ge t_0$, *of* (5.12) *satisfies*

$$x(t) = C_0 + C_1(t-T) + o(1) \quad \textit{and} \quad x'(t) = C_1 + o(1), \quad \textit{as} \quad t \to \infty,$$

where

$$C_0 = x(T) - \int_T^\infty (s-T)a(s)|x(s)|^\gamma \operatorname{sgn} x(s)ds - \int_T^\infty (s-T)b(s)|x'(s)|^\delta \operatorname{sgn} x'(s)ds,$$

$$C_1 = x'(T) + \int_T^\infty a(s)|x(s)|^\gamma \operatorname{sgn} x(s)ds + \int_T^\infty b(s)|x'(s)|^\delta \operatorname{sgn} x'(s)ds.$$

Before ending this example, we concentrate on the Emden-Fowler equation (5.12) with

$$a(t) = t^\sigma \mu(t) \quad \text{for } t \ge t_0, \quad \text{and} \quad b(t) = t^\tau \nu(t) \quad \text{for } t \ge t_0,$$

where σ and τ are real numbers, and μ and ν are continuous and bounded real-valued functions on $[t_0,\infty)$. In this case, we have

$$|a(t)| \le \theta t^\sigma \quad \text{for } t \ge t_0, \quad \text{and} \quad |b(t)| = \xi t^\tau \quad \text{for } t \ge t_0,$$

where θ and ξ are positive constants. We see that (5.13) is satisfied if $\gamma + \sigma < -2$ and $\tau < -2$. Moreover, we observe that (5.14) holds if $\gamma + \sigma < -1$ and $\tau < -1$.

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Christos G. Philos
Department of Mathematics, University of Ioannina, P. O. Box 1186, 451 10 Ioannina, Greece
E-mail address: cphilos@cc.uoi.gr

Panagiotis Ch. Tsamatos
Department of Mathematics, University of Ioannina, P. O. Box 1186, 451 10 Ioannina, Greece
E-mail address: ptsamato@cc.uoi.gr