Electronic Journal of Differential Equations, Vol. 2005(2005), No. 82, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

TWO POSITIVE SOLUTIONS FOR SECOND-ORDER FUNCTIONAL AND ORDINARY BOUNDARY-VALUE PROBLEMS

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Abstract. In this paper we use a fixed point theorem due to Avery and Henderson to prove, under appropriate conditions, the existence of at least two positive solutions for a second-order functional and ordinary boundary-value problem.

1. Introduction

Throughout the recent years, an increasing interest has been observed in finding conditions that guarantee the existence of positive solutions for boundary-value problems. The well known Guo-Krasnoselskii fixed point theorem [6, 12] has been proved to be a useful tool to achieve such conditions, while this same theorem can be applied repeatedly to prove the existence of multiple positive solutions (see [4, 5, 8, 10, 18, 20, 21] and the references therein). Besides this theorem, there are a number of others, referring to Banach spaces ordered by proper cones, that provide conditions such that certain boundary-value problems have multiple positive solutions, for example the Leggett-Williams fixed point theorem [9, 13] and the Avery-Henderson fixed point theorem [2].

At this point we would like to stress the recent increase in the number of papers dealing with functional boundary-value problems, usually specifically with the existence of positive solutions for these problems (see [8, 11, 14, 15, 19] and the references therein). Here we will first study a functional boundary-value problem and then a separate section will be devoted to briefly outlining the analogues of our results for the ordinary case, since even these analogues are novel. We will use the Avery-Henderson fixed point theorem ([2], see also [15, 17]); for other papers using this theorem we refer to [3, 7, 15, 16, 17]. It is remarkable that under certain conditions the Avery-Henderson fixed point theorem can give results similar to those of the Krasnoselskii fixed point theorem, i.e. existence of solutions whose norm is upper and lower bounded by specific constants. Corollaries 3.5 and 4.3 provide such results.

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$ and $I := [0, 1]$. Also, let $0 \leq q < 1$ and $J := [-q, 0]$. For every closed interval $B \subseteq J \cup I$ we denote by

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2000 *Mathematics Subject Classification.* 34K10, 34B18.
Key words and phrases. Boundary value problems; Positive solutions.

Submitted February 22, 2005. Published July 15, 2005.

$C(B)$ the Banach space of all continuous real functions $\psi : B \to \mathbb{R}$ endowed with the usual sup-norm
$$\|\psi\|_B := \sup\{|\psi(s)| : s \in B\}.$$
Also, we define the set
$$C^+(B) := \{\psi \in C(B) : \psi \geq 0\}.$$
If $x \in C(J \cup I)$ and $t \in I$, then we denote by x_t the element of $C(J)$ defined by
$$x_t(s) = x(t+s), \quad s \in J.$$
Now, consider the equation
$$(p(t)x'(t))' + f(t, x_t) = 0, \quad t \in I, \tag{1.1}$$
along with the boundary conditions
$$x_0 = \phi, \tag{1.2}$$
$$ax(1) + bp(1)x'(1) = 0, \tag{1.3}$$
where $f : \mathbb{R}^+ \times C^+(J) \to \mathbb{R}^+$, $p : I \to (0, +\infty)$ and $\phi : J \to \mathbb{R}^+$ are continuous functions, p is also nondecreasing and such that $0 < \int_0^1 \frac{1}{p(s)} ds < +\infty$, and $a, b \in \mathbb{R}$. Also assume that $\phi(0) = 0$.

This paper is motivated by and extends the results of [1, 8] and is organized as follows. In section 2 we present the definitions and the lemmas we are going to use, including the fixed point theorem, due to Avery and Henderson [2]. Section 3 contains the results for the functional case and section 4 the results for the ordinary case. Finally, in section 5 we give some applications of our results.

2. Preliminaries and some basic lemmas

Definition. A function $x \in C(J \cup I)$ is a solution of the boundary-value problem (1.1)–(1.3) if x satisfies equation (1.1), the boundary condition (1.3) and, moreover $x|J = \phi$.

Define $P : I \to \mathbb{R}^+$ and $A : I \to \mathbb{R}$, as
$$P(t) := \int_0^t \frac{1}{p(s)} ds, \quad t \in I, \quad \text{and} \quad A(t) := a \int_t^1 \frac{1}{p(s)} ds + b, \quad t \in I.$$
At this point we make our first assumption:

(H1) For the constants a, b and the function p,
$$A(0) = aP(1) + b \neq 0 \quad \text{and} \quad aA(0) = a(aP(1) + b) \leq 0.$$

We remark that assumption (H1) is equivalent to
$$a \neq -\frac{b}{P(1)} \quad \text{and} \quad \min\{0, -\frac{b}{P(1)}\} \leq a \leq \max\{0, -\frac{b}{P(1)}\}.$$

Lemma 2.1. *A function $x \in C(J \cup I)$ is a solution of the boundary-value problem* (1.1)–(1.3) *if and only if x is a fixed point of the operator $T : C(J \cup I) \to C(J \cup I)$, with*
$$Tx(t) = \begin{cases} \phi(t), & t \in J \\ \int_0^1 G(t,s) f(s, x_s) ds, & t \in I, \end{cases}$$
where
$$G(t,s) = \frac{1}{A(0)} \begin{cases} P(t)A(s), & 0 \leq t \leq s \leq 1 \\ P(s)A(t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.1}$$

Proof. It is well known (see [1]) that the Green's function for the homogenous boundary-value problem consisting of the equation $(p(t)x'(t))' = 0$ and the boundary conditions $x(0) = 0$ and (1.3) is given by the formula (2.1). Therefore, since $\phi(0) = 0$, the proof is obvious. $\square$

In the sequel, we need the following definitions:

Let $\mathbb{E}$ be a real Banach space. A cone in $\mathbb{E}$ is a nonempty, closed set $\mathbb{P} \subset \mathbb{E}$ such that

(i) $\kappa u + \lambda v \in \mathbb{P}$ for all u, $v \in \mathbb{P}$ and all κ, $\lambda \geq 0$
(ii) u, $-u \in \mathbb{P}$ implies $u = 0$.

Let $\mathbb{P}$ be a cone in a real Banach space $\mathbb{E}$. A functional $\psi : \mathbb{P} \to \mathbb{E}$ is said to be increasing on $\mathbb{P}$ if $\psi(x) \leq \psi(y)$, for any x, $y \in \mathbb{P}$ with $x \leq y$, where $\leq$ is the partial ordering induced to the Banach space by the cone $\mathbb{P}$, i.e.

$$x \leq y \quad \text{if and only if} \quad y - x \in \mathbb{P}.$$

Let ψ be a nonnegative functional on a cone $\mathbb{P}$. For each $d > 0$ we denote by $\mathbb{P}(\psi, d)$ the set

$$\mathbb{P}(\psi, d) := \{x \in \mathbb{P} : \psi(x) < d\}.$$

At this point we can present Theorem 2.2, due to Avery and Henderson ([2], see also [15, 17]). This theorem will be used to show that our boundary-value problem (1.1)–(1.3) has at least two distinct positive solutions and, moreover, for each of these solutions, we have an upper bound at some specific point of its domain and a lower bound at some other specific point of its domain. Also, both solutions are concave and nondecreasing on I.

Theorem 2.2. *Let $\mathbb{P}$ be a cone in a real Banach space $\mathbb{E}$. Let α and γ be increasing, nonnegative, continuous functionals on $\mathbb{P}$, and let θ be a nonnegative, continuous functional on $\mathbb{P}$ with $\theta(0) = 0$ such that, for some $c > 0$ and $\Theta > 0$, $\gamma(x) \leq \theta(x) \leq \alpha(x)$ and $\|x\| \leq \Theta\gamma(x)$, for all $x \in \overline{\mathbb{P}(\gamma, c)}$. Moreover, suppose there exists a completely continuous operator $T : \overline{\mathbb{P}(\gamma, c)} \to \mathbb{P}$ and $0 < a < b < c$ such that*

$$\theta(\lambda x) \leq \lambda\theta(x), \quad \textit{for} \quad 0 \leq \lambda \leq 1 \quad \textit{and} \quad x \in \partial\mathbb{P}(\theta, b),$$

and

(i) *$\gamma(Tx) > c$, for all $x \in \partial\mathbb{P}(\gamma, c)$,*
(ii) *$\theta(Tx) < b$, for all $x\partial \in \mathbb{P}(\theta, b)$,*
(iii) *$\mathbb{P}(\alpha, a) \neq \emptyset$, and $\alpha(Tx) > a$, for all $x \in \partial\mathbb{P}(\alpha, a)$,*

or

(i') *$\gamma(Tx) < c$, for all $x \in \partial\mathbb{P}(\gamma, c)$,*
(ii') *$\theta(Tx) > b$, for all $x \in \partial\mathbb{P}(\theta, b)$,*
(iii') *$\mathbb{P}(\alpha, a) \neq \emptyset$, and $\alpha(Tx) < a$, for all $x \in \partial\mathbb{P}(\alpha, a)$.*

Then T has at least two fixed points x_1 and x_2 belonging to $\overline{\mathbb{P}(\gamma, c)}$ such that

$$a < \alpha(x_1) \quad \textit{and} \quad \theta(x_1) < b,$$

and

$$b < \theta(x_2) \quad \textit{and} \quad \gamma(x_2) < c.$$

3. Main Results

Define the set

$$\mathbb{K} := \{x \in C(J \cup I) : x(t) \geq 0, \quad t \in J \cup I,\ x|I \text{ is nondecreasing and concave}\},$$

which is a cone in $C(J \cup I)$. The following lemma (see [11]) will be needed later.

Lemma 3.1. *Let $x : I \to \mathbb{R}$ be a nonnegative, nondecreasing and concave function. Then, $x(t) \geq t\|x\|_I$, $t \in I$.*

Proof. For any $t \in I$, since x is nonnegative, nondecreasing and concave, we have

$$x(t) = x((1-t)0 + t) \geq (1-t)x(0) + tx(1) \geq tx(1) = t\|x\|_I.$$

□

Now let $0 < r_1 \leq r_2 \leq r_3 \leq 1$ and consider the following functionals:

$$\begin{aligned} \gamma(x) &= x(r_1), \quad x \in \mathbb{K}, \\ \theta(x) &= x(r_2), \quad x \in \mathbb{K}, \\ \alpha(x) &= x(r_3), \quad x \in \mathbb{K}. \end{aligned}$$

It is easy to see that α, γ are nonnegative, increasing and continuous functionals on $\mathbb{K}$, θ is nonnegative and continuous on $\mathbb{K}$ and $\theta(0) = 0$. Also, it is straightforward to see that

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \quad x \in \mathbb{K}, \tag{3.1}$$

since $x \in \mathbb{K}$ is nondecreasing on I. Furthermore, for any $x \in \mathbb{K}$, by Lemma 3.1 we have $\gamma(x) = x(r_1) \geq r_1\|x\|_I$. So

$$\|x\|_I \leq \frac{1}{r_1}\gamma(x), \quad x \in \mathbb{K}. \tag{3.2}$$

Additionally, by the definition of θ we obtain

$$\theta(\lambda x) = \lambda\theta(x), \quad 0 \leq \lambda \leq 1, \quad x \in \mathbb{K}.$$

At this point, we state the following assumptions:

(H2) There exist $M > 0$, continuous function $u : I \to \mathbb{R}^+$ and a function $L : \mathbb{R}^+ \to \mathbb{R}^+$, which is nondecreasing on $[0, M]$, such that

$$f(t, y) \leq u(t)L(\|y\|_J), \quad t \in I,\ y \in C^+(J),$$
$$L(M)\int_0^1 G(r_2, s)u(s)ds < Mr_2.$$

(H3) There exist constants $\delta \in (0, 1)$, $\eta_1, \eta_3 > 0$ and functions $\tau : I \to [0, q]$, continuous $v : I \to \mathbb{R}^+$ and nondecreasing $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \geq v(t)w(y(-\tau(t))), \quad t \in X,\ y \in C^+(J),$$

where $X := \{t \in I : \delta \leq t - \tau(t) \leq 1\}$, $\sup\{v(t) : t \in X\} > 0$,

$$w(\eta_i)\int_X G(r_i, s)v(s)ds > \frac{\eta_i}{\delta}, \quad i \in \{1, 3\},$$

$0 < \eta_3 < M\delta r_2 < \eta_1$, and M is defined in (H2).

The following lemma can be found in [11]. The proof is provided for the sake of completeness.

Lemma 3.2. *If $x : I \to \mathbb{R}$ is a differentiable function with $x'(t) \geq 0$, $t \in I$, and $p : I \to \mathbb{R}$ is a positive and nondecreasing function such that $(p(t)x'(t))' \leq 0$, $t \in I$, then x is concave.*

Proof. Let $t_1, t_2 \in I$, with $t_1 \leq t_2$. Since $(p(t)x'(t))' \leq 0$, $t \in I$, px' is a nonincreasing function on I, so $p(t_1)x'(t_1) \geq p(t_2)x'(t_2)$. Therefore, we have

$$\begin{aligned}
p(t_1)(x'(t_2) - x'(t_1)) &= p(t_1)x'(t_2) - p(t_1)x'(t_1) \\
&\leq p(t_1)x'(t_2) - p(t_2)x'(t_2) \\
&= x'(t_2)(p(t_1) - p(t_2)).
\end{aligned}$$

Since p is nondecreasing on I, we have $p(t_1) - p(t_2) \leq 0$, so

$$p(t_1)(x'(t_2) - x'(t_1)) \leq x'(t_2)(p(t_1) - p(t_2)) \leq 0.$$

However $p(t_1) \geq 0$, therefore we get that $x'(t_1) \geq x'(t_2)$, which implies that x' is nonincreasing on I. Consequently, x is concave. □

The following lemma will be needed for the proof of Theorem 3.4.

Lemma 3.3. *Suppose that* (H1) *holds. Then*

(i) $\frac{A(t)}{A(0)} > 0$, $t \in I$.
(ii) $\frac{A'(t)}{A(0)} \geq 0$, $t \in I$.
(iii) $Tx(t) \geq 0$, $t \in I$, $x \in \mathbb{K}$.
(iv) $(Tx)'(t) \geq 0$, $t \in I$, $x \in \mathbb{K}$.
(v) $T(\mathbb{K}) \subseteq \mathbb{K}$.

Proof. (i) For any $t \in I$, keeping in mind that $aA(0) \leq 0$, we get

$$\begin{aligned}
\frac{A(t)}{A(0)} &= \frac{a\int_t^1 \frac{1}{p(s)}ds + b}{A(0)} \\
&= \frac{a}{A(0)}\int_t^1 \frac{1}{p(s)}ds + \frac{b}{A(0)} \\
&\geq \frac{a}{A(0)}\int_0^1 \frac{1}{p(s)}ds + \frac{b}{A(0)} \\
&= \frac{a\int_0^1 \frac{1}{p(s)}ds + b}{A(0)} = 1.
\end{aligned}$$

Therefore, $\frac{A(t)}{A(0)} \geq 0$, $t \in I$.
(ii) Since $p(t) > 0$, $t \in I$, and, by (H1), $\frac{a}{A(0)} \leq 0$, for any $t \in I$ we have

$$\frac{A'(t)}{A(0)} = \frac{-\frac{a}{p(t)}}{A(0)} = -\frac{a}{A(0)}\frac{1}{p(t)} \geq 0.$$

(iii) By the definition of T, we get

$$Tx(t) = \frac{A(t)}{A(0)}\int_0^t P(s)f(s,x_s)ds + P(t)\int_t^1 \frac{A(s)}{A(0)}f(s,x_s)ds, \;\; t \in I.$$

Moreover, if $x \in \mathbb{K}$ then $x_t \geq 0$, $t \in I$. By the definition of f we have $f(t,x_t) \geq 0$, $t \in I$. So using (i) we conclude that $Tx(t) \geq 0$, $t \in I$.

(iv) By the definition of T, for every $t \in I$ we get

$$\begin{aligned}
(Tx)'(t) &= A'(t)\int_0^t \frac{1}{A(0)}P(s)f(s,x_s)ds + A(t)\frac{1}{A(0)}P(t)f(t,x_t) \\
&\quad + P'(t)\int_t^1 \frac{1}{A(0)}A(s)f(s,x_s)ds - P(t)\frac{1}{A(0)}A(t)f(t,x_t) \\
&= \frac{A'(t)}{A(0)}\int_0^t P(s)f(s,x_s)ds + P'(t)\int_t^1 \frac{A(s)}{A(0)}f(s,x_s)ds.
\end{aligned}$$

Therefore, using (ii) and the easily provable facts that $P'(t) \geq 0$, $t \in I$, and $f(t,x_t) \geq 0$, $t \in I$, $x \in \mathbb{K}$, we conclude that $(Tx)'(t) \geq 0$, $t \in I$.
(v) From (1.1), for every $t \in I$ and $x \in \mathbb{K}$ we have

$$(p(t)x'(t))' = -f(t,x_t) \leq 0.$$

So, since for $x \in \mathbb{K}$ we have $x_t \geq 0$, $t \in I$, by the definition of f and, according to Lemma 3.2, we have that Tx is concave. This, along with (iii) and (iv), completes the proof. □

Theorem 3.4. *Suppose that assumptions* (H1)–(H3) *hold and* $\|\phi\|_J < M$. *Then the boundary-value problem* (1.1)–(1.3) *has at least two solutions* x_1, x_2, *which are concave and nondecreasing on* I, *positive on* $J \cup I$ *and such that* $x_1(r_3) > \frac{\eta_3}{\delta}$, $x_1(r_2) < Mr_2$, $x_2(r_2) > Mr_2$ *and* $x_2(r_1) < \frac{\eta_1}{\delta}$.

Proof. First of all, we observe that, because of (H1), $f(t,\cdot)$ maps bounded sets into bounded sets. Therefore T is a completely continuous operator. Additionally, according to Lemma 3.3 we have $T : \overline{\mathbb{K}(\gamma,c)} \to \mathbb{K}$.

Now we set $\beta_1 = \frac{\eta_1}{\delta}$, $\beta_2 = Mr_2$ and $\beta_3 = \frac{\eta_3}{\delta}$. Let $x \in \partial\mathbb{K}(\gamma,\beta_1)$. Then $\gamma(x) = x(r_1) = \beta_1$ and so $\|x\|_I \geq \beta_1$. Having in mind assumption (H_3), we get

$$\begin{aligned}
\gamma(Tx) &= (Tx)(r_1) \\
&= \int_0^1 G(r_1,s)f(s,x_s)ds \\
&\geq \int_X G(r_1,s)f(s,x_s)ds \\
&\geq \int_X G(r_1,s)v(s)w(x_s(-\tau(s)))ds \\
&= \int_X G(r_1,s)v(s)w(x(s-\tau(s)))ds \\
&\geq \int_X G(r_1,s)v(s)w(x(\delta))ds.
\end{aligned}$$

Additionally, by assumption (H3), the definition of $\mathbb{K}$ and Lemma 3.1, we have

$$\begin{aligned}\gamma(Tx) &\geq \int_X G(r_1,s)v(s)w(\delta\|x\|_I)ds\\ &\geq w(\delta\beta_1)\int_X G(r_1,s)v(s)ds\\ &= w(\eta_1)\int_X G(r_1,s)v(s)ds\\ &> \frac{\eta_1}{\delta} = \beta_1.\end{aligned}$$

This means that condition (i) of Theorem 2.2 is satisfied.

Now let $x \in \partial\mathbb{K}(\theta,\beta_2)$. Then $\theta(x) = x(r_2) = \beta_2$ and so

$$\|x\|_I \leq \frac{1}{r_2}x(r_2) = \frac{1}{r_2}\beta_2 = M.$$

Also we assumed that $\|\phi\|_J \leq M$, so $\|x\|_{J\cup I} \leq M$. Now, by (H_2), we have

$$\begin{aligned}\theta(Tx) &= Tx(r_2)\\ &= \int_0^1 G(r_2,s)f(s,x_s)ds\\ &\leq \int_0^1 G(r_2,s)u(s)L(\|x_s\|_J)ds\\ &\leq \int_0^1 G(r_2,s)u(s)L(M)ds\\ &= L(M)\int_0^1 G(r_2,s)u(s)ds < Mr_2 = \beta_2.\end{aligned}$$

So condition (ii) of Theorem 2.2 is also satisfied.

Now, define the function $y : J\cup I \to \mathbb{R}$ with $y(t) = \frac{\beta_3}{2}$. Then it is obvious that $\alpha(y) = \frac{\beta_3}{2} < \beta_3$, so $\mathbb{K}(\alpha,\beta_3) \neq \emptyset$. Also, for any $x \in \partial\mathbb{K}(\alpha,\beta_3)$ we have $\alpha(x) = x(r_3) = \beta_3$. Therefore, $\|x\|_I \geq \beta_3$. Now, having in mind assumption (H_3) and as in the case of the functional γ above, we get

$$\alpha(Tx) = Tx(r_3) \geq \int_X G(r_3,s)v(s)w(x(\delta))ds\,.$$

Taking into account assumption (H3), the definition of $\mathbb{K}$ and Lemma 3.1, we have

$$\alpha(Tx) = w(\eta_3)\int_X G(r_3,s)v(s)ds > \frac{\eta_3}{\delta} = \beta_3.$$

Consequently, assumption (iii) of Theorem 2.2 is satisfied. The result can now be obtained by applying Theorem 2.2. $\square$

The solutions x_1, x_2 obtained in Theorem 3.4 are both nondecreasing. Thus, in the special case when $r_1 = r_2 = r_3 = 1$, we have that $x_i(r_j) = x_i(1) = \|x_i\|$, $i = 1,2$, $j = 1,2,3$. Therefore, we have the following corollary of Theorem 3.4.

Corollary 3.5. *Suppose that assumptions* (H1)–(H3) *hold for* $r_1 = r_2 = r_3 = 1$ *and furthermore* $\|\phi\|_J \leq M$. *Then the boundary-value problem* (1.1)–(1.3) *has at* *least two solutions x_1, x_2, which are concave and nondecreasing on I, positive on $J \cup I$ and such that*

$$\frac{\eta_3}{\delta} < \|x_1\| < M < \|x_2\| < \frac{\eta_1}{\delta}.$$

4. The Ordinary Case

Suppose that $q = 0$. Then $J = \{0\}$, so the boundary-value problem (1.1)–(1.3) is reformulated as follows

$$(p(t)x'(t))' + f(t, x(t)) = 0, \quad t \in I, \tag{4.1}$$

$$x(0) = 0, \tag{4.2}$$

$$ax(1) + bp(1)x'(1) = 0, \tag{4.3}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $p : I \to (0, +\infty)$ are continuous functions, p is also nondecreasing and lower bounded by a strictly positive number and $a, b \in \mathbb{R}$. Note that equation (4.1) is equivalent to the form

$$(p(t)x'(t))' + f(t, x_t(0)) = 0, \quad t \in I$$

and $C^+(\{0\}) \equiv \mathbb{R}^+$, so $f : \mathbb{R}^+ \times C^+(\{0\}) \to \mathbb{R}^+$.

Now, the analogue of Lemma 2.1 for this case is as follows.

Lemma 4.1. *A function $x \in C(I)$ is a solution of the boundary-value problem* (4.1)–(4.3) *if and only if x is a fixed point of the operator $\widehat{T} : C(I) \to C(I)$, with*

$$\widehat{T}x(t) = \int_0^1 G(t, s)f(s, x(s))ds, \quad t \in I,$$

where G is defined in Lemma 2.1*.*

Assumptions (H2), (H3), for the special case $q = 0$, are stated as follows:

(H2') There exist $M > 0$, continuous function $u : I \to \mathbb{R}^+$ and a function $L : \mathbb{R}^+ \to \mathbb{R}^+$ which is nondecreasing on $[0, M]$, such that

$$f(t, y) \le u(t)L(y) \quad \text{for all} \quad t \in I \quad \text{and} \quad y \in \mathbb{R}^+$$

and

$$L(M) \int_0^1 G(r_2, s)u(s)ds < Mr_2.$$

(H3') There exist constants $\delta \in (0, 1)$, $\eta_1, \eta_3 > 0$ and functions $v : I \to \mathbb{R}^+$ continuous and $w : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing, such that

$$f(t, y) \ge v(t)w(y) \quad \text{for all} \quad t \in Z := [\delta, 1] \quad \text{and} \quad y \in \mathbb{R}^+$$

and

$$w(\eta_i) \int_Z G(r_i, s)v(s)ds > \frac{\eta_i}{\delta}, \quad i \in \{1, 3\},$$

where $0 < \eta_3 < M\delta r_2 < \eta_1$, and M is defined in (H2').

Therefore, we have the following results, which are the analogue of Theorem 3.4 and Corollary 3.5 respectively.

Theorem 4.2. *Suppose that assumptions* (H1), (H2)', (H3') *hold. Then the boundary-value problem* (4.1)–(4.3) *has at least two solutions x_1, x_2, which are concave, nondecreasing and positive on I, such that $x_1(r_3) > \frac{\eta_3}{\delta}$, $x_1(r_2) < Mr_2$, $x_2(r_2) > Mr_2$ and $x_2(r_1) < \frac{\eta_1}{\delta}$.*

Corollary 4.3. *Suppose that* (H1), (H2'), (H3') *hold for* $r_1 = r_2 = r_3 = 1$. *Then the boundary-value problem* (4.1)–(4.3) *has at least two solutions* x_1, x_2*, which are concave, nondecreasing and positive on* I*, such that*

$$\frac{\eta_3}{\delta} < \|x_1\| < M < \|x_2\| < \frac{\eta_1}{\delta}.$$

5. Applications

1. Consider the boundary-value problem

$$(e^t x'(t))' + \exp\big(\frac{t+5}{20}x(t-\frac{1}{2})\big) = 0, \quad t \in I := [0,1] \tag{5.1}$$

$$x_0(t) = \phi(t) := |t|, \quad t \in J := [-\frac{1}{2}, 0], \tag{5.2}$$

$$x(1) - 2ex'(1) = 0. \tag{5.3}$$

Obviously, $f(t,y) := \exp((t+5)y/20)$ is positive on $\mathbb{R}^+ \times C^+(J)$, ϕ is positive on J and $p(t) := e^t$ is positive and nondecreasing on I. Also we have $a = 1$ and $b = -2$, so $P(t) = 1 - e^{-t}$, $t \in I$, $A(t) = e^{-t} - e^{-1} - 2$, $t \in I$, $A(0) = -(1+e^{-1}) \neq 0$, $aA(0) = -(1+e^{-1}) \leq 0$ and

$$G(t,s) = \begin{cases} -\frac{(1-e^{-t})(e^{-s}-e^{-1}-2)}{1+e^{-1}}, & 0 \leq t \leq s \leq 1 \\ -\frac{(1-e^{-s})(e^{-t}-e^{-1}-2)}{1+e^{-1}}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Set $r_1 = 1/3$, $r_2 = 1/2$ and $r_3 = 2/3$. Define $L(t) = e^{0.3t}$, $t \in \mathbb{R}^+$, and $u(t) = 1$, $t \in I$. Since

$$e^{0.3M} + \frac{1+e^{-1}}{2e^{-1} + e^{-0.5} - e^{-1.5}} M < 0$$

for $M = 2$, assumption (H2) is satisfied.

Additionally, set $\delta = 1/5$, $\tau(t) = 1/2$, $t \in I$, $v(t) = 1$, $t \in I$ and $w(t) = e^{0.25t}$, $t \in \mathbb{R}^+$. Then, $X = [\frac{7}{10}, 1]$ and the inequalities in assumption (H3) take the forms

$$e^{0.25\eta_1} + \frac{5(1+e^{-1})}{(1-e^{-1/3})(e^{-0.7} - 1.3e^{-1}) - 0.6}\eta_1 > 0,$$

$$e^{0.25\eta_3} + \frac{5(1+e^{-1})}{(1-e^{-2/3})(e^{-0.7} - 1.3e^{-1}) - 0.6}\eta_3 > 0,$$

which are satisfied for $\eta_1 = 29$ and $\eta_3 = 0.004$.

Finally, it is obvious that $0 < 0.004 < 0.2 < 29$ and $\|\phi\|_J \leq 2$, so we can apply Theorem 3.4 to get that the boundary-value problem (5.1)–(5.3) has at least two concave and nondecreasing on $[0,1]$ and positive on $[-\frac{1}{2}, 1]$ solutions x_1, x_2, such that

$$x_1(\frac{2}{3}) > 0.02, \quad x_1(\frac{1}{2}) < 1, \quad x_2(\frac{1}{2}) > 1, \quad x_2(\frac{1}{3}) < 145.$$

2. Consider the boundary-value problem

$$x''(t) + \big(x(t) - \frac{4}{5}\big)^5 + 1 = 0, \quad t \in I := [0,1], \tag{5.4}$$

$$x(0) = 0, \tag{5.5}$$

$$2x'(1) - x(1) = 0. \tag{5.6}$$

Obviously, $f(t,y) := (y-\frac{4}{5})^5+1$ is positive on $\mathbb{R}^+\times\mathbb{R}^+$ and $p(t) := 1$ is positive and nondecreasing on I. Also we have $a=-1$ and $b=2$, so $P(t)=t$, $t\in I$, $A(t)=t+1$, $t\in I$, $A(0)=1\neq 0$, $aA(0)=-1\leq 0$ and

$$G(t,s)=\begin{cases} t(s+1), & 0\leq t\leq s\leq 1\\ s(t+1), & 0\leq s\leq t\leq 1.\end{cases}$$

Set $r_1=2/5$, $r_2=3/5$ and $r_3=4/5$. Define $L(t) := (t-\frac{4}{5})^5+1$, $t\in\mathbb{R}^+$, and $u(t)=1$, $t\in I$. Since

$$\big(M-\frac{4}{5}\big)^5+1-\frac{5}{6}M<0$$

for $M=1.5$, assumption (H2') is satisfied.

Additionally, set $\delta=9/10$, $v(t)=1$, $t\in I$ and $w(t)=(t-\frac{4}{5})^5+1$, $t\in\mathbb{R}^+$. Then, $Z=[\frac{9}{10},1]$ and the inequalities in assumption (H3') take the forms

$$\big(\eta_1-\frac{4}{5}\big)^5+1-\frac{2500}{351}\eta_1>0,\quad \big(\eta_3-\frac{4}{5}\big)^5+1-\frac{2500}{351}\eta_3>0,$$

which are satisfied for $\eta_1=3$ and $\eta_3=0.1$. Finally, it is obvious that

$$0<\eta_3<\frac{27}{50}M<\eta_1,$$

so we can apply Theorem 4.2 to conclude that the boundary-value problem (5.4)–(5.6) has at least two solutions x_1, x_2, which are concave, nondecreasing and positive on $[0,1]$, such that

$$x_1(\frac{3}{4})>\frac{1}{9},\quad x_1(\frac{1}{2})<\frac{9}{10},\quad x_2(\frac{1}{2})>\frac{9}{10},\quad x_2(\frac{1}{4})<\frac{10}{3}.$$

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