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HALF-LINEAR DYNAMIC EQUATIONS WITH MIXED DERIVATIVES

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Abstract. We investigate oscillatory properties of the second order half-linear dynamic equation on a time scale with mixed derivatives

$$(r(t)\Phi(x^{\Delta}))^{\nabla}+c(t)\Phi(x)=0,\quad \Phi(x)=|x|^{p-2}x,\quad p>1.$$

In particular, we establish the Roundabout theorem which relates oscillatory properties of this equation to the solvability of the associated Riccati-type dynamic equation and to the positivity of the corresponding energy functional. This result is then used to prove (non)oscillation criteria for the above equation.

1. Introduction

In this paper we investigate oscillatory properties of solutions of the half-linear second order dynamic equation with mixed derivatives

$$(r(t)\Phi(x^{\Delta}))^{\nabla}+c(t)\Phi(x)=0. \tag{1.1}$$

Recently, several papers dealing with the Sturm-Liouville second order dynamic equation of the form (which is the special case $p=2$ in (1.1))

$$(r(t)x^{\Delta})^{\nabla}+c(t)x=0 \tag{1.2}$$

appeared, see [3, 10] and also [5, Chap. IV], where the basic qualitative theory of (1.2) has been established. It was shown that qualitative properties of solutions of this equation are very similar to those of the "normal" Sturm-Liouville dynamic equation

$$(r(t)x^{\Delta})^{\Delta}+c(t)x^{\sigma}=0, \tag{1.3}$$

whose theory is now relatively deeply developed, see [4] and the references given therein.

Another motivation for our research is a series of papers [1, 12, 13], where the half-linear dynamic equation

$$(r(t)\Phi(x^{\Delta}))^{\Delta}+c(t)\Phi(x^{\sigma})=0,\quad \Phi(x):=|x|^{p-1}\operatorname{sgn}x,\ p>1, \tag{1.4}$$

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is investigated and the theory unifying the theory of half-linear differential and difference equations is established.

Recall that a time scale $\mathbb{T}$ is any closed subset of the set of real numbers $\mathbb{R}$ with the inherited topology, and that for a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f can be replaced by a Banach space) the differential and integral calculus is developed in such a way that it unifies the usual differential and integral calculus if $\mathbb{T} = \mathbb{R}$ and the calculus of finite differences and sums if $\mathbb{T} = \mathbb{Z}$. We suppose that the reader is familiar with the basic facts of the time scales calculus, nevertheless, some elements of this theory we recall in the next section.

The paper is organized as follows. In the next section we collect a preliminary material, including the elements of the time scale calculus, which we need in our treatment, and also basic facts of oscillation theories of (1.2) and (1.4) as established in the above mentioned papers. Section 3 is devoted to the half-linear time scale version of Picone's identity which relates oscillation properties of (1.1) to the solvability of the associated Riccati-type equation and to the positivity of the corresponding energetic p-degree functional. A statement of this kind is usually referred to as the Roundabout theorem. The last section contains some oscillation and nonoscillation criteria for (1.1) which are the extension to (1.1) of some criteria proved in [1] for (1.4).

2. Preliminary results

Let $\mathbb{T}$ be a time scale. The operators $\rho, \sigma : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} :\ s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} :\ s < t\},$$

(with the convention $\inf \emptyset = \sup \mathbb{T}$, $\sup \emptyset = \inf \mathbb{T}$) and are called the *right jump operator* and *left jump operator*, respectively. The quantities $\mu(t) = \sigma(t) - t, \nu(t) = t - \rho(t)$ are called the (forward) *graininess* and *backward graininess* of $\mathbb{T}$, respectively. A point $t \in \mathbb{T}$ is said to be *right* (*left*) *dense* if $\mu(t) = 0$ ($\nu(t) = 0$), and it is said to be *right* (*left*) *scattered* if $\mu(t) > t$ ($\rho(t) < t$), we will use the abbreviation rd, ld, rs, ls-point respectively. If $\mathbb{T}$ has a left-scattered maximum M (right-scattered minimum m), then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$ ($\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$), otherwise $\mathbb{T}^\kappa = \mathbb{T}$ ($\mathbb{T}_\kappa = \mathbb{T}$). If $f : \mathbb{T} \to \mathbb{R}$, the Δ-derivative is defined by

$$f^\Delta(t) := \begin{cases} \lim_{s\to t} \frac{f(s)-f(t)}{s-t} & \text{if } \mu(t) = 0, \\ \frac{f(\sigma(t))-f(t)}{\mu(t)} & \text{if } \mu(t) > 0, \end{cases}$$

and the ∇-derivative by

$$f^\nabla(t) := \begin{cases} \lim_{s\to t} \frac{f(s)-f(t)}{s-t} & \text{if } \nu(t) = 0, \\ \frac{f(t)-f(\rho(t))}{\nu(t)} & \text{if } \nu(t) > 0. \end{cases} \tag{2.1}$$

We have $f^\Delta(t) = f'(t) = f^\nabla(t)$ if $\mathbb{T} = \mathbb{R}$, and $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$, $f^\nabla(t) = f(t) - f(t-1)$ if $\mathbb{T} = \mathbb{Z}$.

By f^σ and f^ρ we denote the composition $f \circ \sigma$ and $f \circ \rho$, respectively. It holds

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \quad f^\rho(t) = f(t) - \nu(t) f^\nabla(t) \tag{2.2}$$

and the formulas for the ∇-derivative of a product and a ratio of two function read

$$(fg)^\nabla = f^\nabla g^\rho + f g^\nabla = f^\rho g^\nabla + f^\nabla g, \quad \Big(\frac{f}{g}\Big)^\nabla = \frac{f^\nabla g - f g^\nabla}{g g^\rho}. \tag{2.3}$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* (*ld-continuous*) if it is right continuous (left continuous) at all rd points (ld points) and the left limit (right limit) at ld points (rd points) exists (finite). If f is rd-continuous (ld-continuous) then there exists a Δ-differentiable function F (a ∇-differentiable function G) such that $F^\Delta(t) = f(t)$ ($G^\nabla(t) = f(t)$). Using these functions we define the integrals

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \int_a^b f(t)\nabla t = G(b) - G(a).$$

In some proofs we will also need the nabla version of integration by parts

$$\int_a^b f^\nabla(t) g^\rho(t)\nabla(t) = [f(t)g(t)]_a^b - \int_a^b f(t)g^\nabla(t)\nabla t. \tag{2.4}$$

In the theory of half-linear equations the frequently used tool is the Young inequality, see [8].

Lemma 2.1. *If $p > 1$ and $q > 1$ are mutually conjugate numbers, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, then for any $u, v \in \mathbb{R}$*

$$\frac{|u|^p}{p} + \frac{|v|^q}{q} \ge |uv|, \tag{2.5}$$

and equality holds if and only if $u = |v|^{q-2}v$.

The next lemma can be considered as a time scale version of the second mean value theorem of integral calculus. Its proof can be found in [12].

Lemma 2.2. *Let f be a function such that its Δ-derivative f^Δ is rd-continuous and f^Δ does not change its sign for $t \in [a, b]$. Then for any rd-continuous function g there exist $c, d \in [a, b]^\kappa$ such that*

$$\int_a^b f^\sigma(t)g(t)\Delta \le f(a)\int_a^c g(t)\Delta t + f(b)\int_c^b g(t)\Delta t$$

and

$$\int_a^b f^\sigma(t)g(t)\Delta \ge f(a)\int_a^d g(t)\Delta t + f(b)\int_d^b g(t)\Delta t.$$

Next we recall the relationship between the delta and nabla derivatives. The proof of this statement can be found in [5, Chap. 4].

Lemma 2.3. *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^\kappa$ and f^Δ is rd-continuous on $\mathbb{T}^\kappa$, then f is ∇-differentiable on $\mathbb{T}_\kappa$, and*

$$f^\nabla(t) = \begin{cases} \lim_{s\to t-} f^\Delta(s) & t \text{ is ld and rs,} \\ f^\Delta(\rho(t)) & \text{otherwise.} \end{cases}$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}_\kappa$ and g^∇ is ld-continuous on $\mathbb{T}_\kappa$, then g is Δ-differentiable on $\mathbb{T}_\kappa$, and

$$g^\Delta(t) = \begin{cases} \lim_{s\to t+} g^\nabla(s) & t \text{ is ls and rd,} \\ g^\nabla(\sigma(t)) & \text{otherwise.} \end{cases}$$

The previous statement, applied to the Δ-integral and ∇-integral gives the following result.

Lemma 2.4. *Let f be a ld-continuous function and let*

$$\hat{f}(t) = \begin{cases} \lim_{s\to t+} f(s) & \text{if } t \text{ is ls and rd point,} \\ f^\sigma(t) & \text{otherwise.} \end{cases}$$

Then

$$\int_a^b f(t)\,\nabla t = \int_a^b \hat{f}(t)\,\Delta t.$$

Proof. Let F be the ∇-antiderivative of f, i.e. $F^\nabla = f$. Then by Lemma 2.3 we have

$$F^\Delta(t) = \begin{cases} \lim_{s\to t+} F^\nabla(s) = \lim_{s\to t+} f(s) & t \text{ is ls and rd,} \\ F^\nabla(\sigma(t)) = f^\sigma(t) & \text{otherwise.} \end{cases}$$

Hence, $F^\Delta(t) = \hat{f}(t)$, and thus

$$\int_a^b \hat{f}(t)\,\Delta t = F(t)|_a^b = \int_a^b f(t)\,\nabla t.$$

□

Finally, we present a formula for the ∇-derivative of a composite function, the proof of this statement is the same as for Δ-derivative and it is based on the Lagrange Mean Value Theorem.

Lemma 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $g : \mathbb{T} \to \mathbb{R}$ be ∇-differentiable. Then we have*

$$[f(g(t))]^\nabla = f'(\xi)g^\nabla(t),$$

where ξ is between $g^\rho(t)$ and $g(t)$.

Now we recall some results of the above mentioned papers [10] and [1] dealing with equations (1.2) and (1.4). These results are summarized in statements which are usually referred to as the Reid roundabout theorem. Recall that by a solution of (1.2) it is understood a function x which is Δ-differentiable, rx^Δ is ∇-differentiable and (1.2) is satisfied. A solution of (1.4) is defined in a similar way. We use the standard notation for time scale intervals. An interval $[a, b]$ actually means $\{t \in \mathbb{T} : a \le t \le b\}$, open and half-open intervals have the same meaning.

Proposition 2.6 ([10], [5, Chap. 4])**.** *Suppose that the function c is ld-continuous, r is continuous and $r(t) > 0$. Then the following statements are equivalent.*

(i) *Equation (1.2) is disconjugate on an interval $[\rho(a), \sigma(b)]$, i.e., the solution x of (1.2) given by the initial condition $x^\rho(a) = 0$, $(rx^\Delta)^\rho(a) = 1$ has no generalized zero in $(\rho(a), \sigma(b)]$, i.e., it satisfies $x^\rho(t)x(t) > 0$ for $t \in (\rho(a), \sigma(b)]$.*
(ii) *There exists a solution of (1.2) having no generalized zero in $[\rho(a), \sigma(b)]$.*
(iii) *The quadratic functional*

$$\mathcal{F}(y) = \int_{\rho(a)}^{\sigma(b)} \left[r^\rho(t)(y^\nabla)^2 - c(t)y^2\right] \nabla t > 0$$

over nontrivial $y : [\rho(a), \sigma(b)] \to \mathbb{R}$ for which y^∇ exists, it is ld-continuous, and $y^\rho(a) = 0 = y^\sigma(b)$.

(iv) *There exists a solution of the Riccati equation*

$$z^{\nabla} + c(t) + \frac{(z^{\rho}(t))^2}{r^{\rho}(t) + \nu(t) z^{\rho}(t)} = 0,$$

related to (1.2) *by the substitution* $z = \frac{r(t)x^{\Delta}}{x}$, *which is defined on* $[\rho(a), \sigma(b)]$ *and satisfies there* $r^{\rho}(t) + \nu(t) z^{\rho}(t) > 0$.

Note that it is supposed in [3] that *both* functions c, r in (1.2) are continuous. However, under this assumption the ∇-derivative $(r(t)x^{\Delta})^{\nabla}$ is continuous, in particular, ld-continuous, hence applying the forward jump operator to (1.2), using Lemma 2.3 we get the equation

$$(r(t)x^{\Delta})^{\Delta} + c^{\sigma}(t) x^{\sigma} = 0$$

which is just the equation of the form (1.3) and the above formulated Proposition 2.6 can be essentially deduced from a corresponding statement for (1.3), see [4]. Also, a statement analogous to Proposition 2.6 can be formulated without positivity assumption on the function r, however, as showed e.g. in [7] where (1.3) is investigated, a "reasonable" oscillation criteria can be derived only under some sign restrictions on the function r, we refer to [7] for details. Finally note that our presentation of Proposition 2.6 follows exactly the presentation of [5] and [10]. Later, in Section 3, we give a similar result for half-linear equation (1.1), but instead of the interval $[\rho(a), \sigma(b)]$ considered in Proposition 2.6, we formulate our results for $t \in [a, b]$.

Now we turn our attention to the roundabout theorem for (1.4), see [12].

Proposition 2.7. *Suppose that the functions* r, c *are rd-continuous and* $r(t) \neq 0$. *Then the following statements are equivalent.*

(i) *Equation* (1.4) *is disconjugate on a time scale interval* $[a, b]$, *i.e., the solution* x *given by the initial condition* $x(a) = 0$, $r(a)\Phi(x^{\Delta}(a)) = 1$ *has no generalized zero in* $(a, b]$, *i.e.,* $r(t)\Phi(x(t))\Phi(x^{\sigma}(t)) > 0$ *for* $t \in (a, b]$.

(ii) *There exists a solution of* (1.4) *having no generalized zero in* $[a, b]$.

(iii) *The functional*

$$\mathcal{F}(y) = \int_a^b [r(t)|y^{\Delta}|^p - c(t)|y^{\sigma}|^p] \Delta t > 0$$

for every nontrivial y *whose* Δ*-derivative is piecewise rd-continuous and at endpoins* $y(a) = 0 = y(b)$.

(iv) *There exists a solution of the Riccati-type equation (related to* (1.4) *by the substitution* $w = r\Phi(x^{\Delta}/x)$*)*

$$w^{\Delta} + c(t) = \begin{cases} -(p-1) r^{1-q}(t) |w|^q & \text{if } \sigma(t) = t, \\ -\frac{w}{\mu(t)} \left(1 - \frac{r(t)}{\Phi(\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w))}\right) & \text{if } \sigma(t) > t \end{cases}$$

which is defined for $t \in [a, b]$ *and satisfies* $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t))) > 0$ *in this interval.*

3. Picone's identity and roundabout theorem

Before passing to the main subject of this section which are basic necessary statements for the elaboration of the oscillation theory of (1.1), let us note that we are not concerned with the existence and uniqueness problem for (1.1) in this paper. This result can be proved using the time scales induction essentially in the same way as in [3, Theorem 3.1] and [12, Section 3].

Throughout the paper we suppose that

(H) $r(t)$ is continuous, $c(t)$ is ld-continuous, and $r(t) \neq 0$

in a time scale interval under consideration. Under this asumption, System (1.1) can be written as a 2×2 system

$$x^{\nabla} = \Phi^{-1}(u^{\rho}/r^{\rho}(t)), \quad u^{\nabla} = -c(t)\Phi(x^{\rho} + \nu(t)\Phi^{-1}(u^{\rho}/r^{\rho}(t)))$$

and the existence and uniqueness problem for (1.1) is investigated via this first order system. We have the same statement as [3, Theorem 3.1], namely that a solution of (1.1) is uniquely determined by the initial condition $x(t_0) = x_0$, $x^{\nabla}(t_0) = x_1$, $t_0 \in \mathbb{T}$, $x_0, x_1 \in \mathbb{R}$, it exists on any interval where the hypotheses (H) are satisfied and depends continuously on the initial condition. We conjucture, that the results of our paper hold under the weaker assumption that r is only ld-continuous, but under this weaker assumption we have some difficulties with the existence problem for (1.1).

We start with the Riccati substitution for (1.1).

Lemma 3.1. *Suppose that x is a solution of* (1.1) *such that $x(t) \neq 0$ in a time scale interval $I = [a, b]$. Then $w = r\Phi(x^{\Delta}/x)$ is a solution of the Riccati-type equation*

$$w^{\nabla} + c(t) = \begin{cases} -(p-1)\frac{|w|^q}{\Phi^{-1}(r(t))} & \text{if } t = \rho(t), \\ -\frac{w^{\rho}}{\nu(t)}\Big(1 - \frac{r^{\rho}(t)}{\Phi(\Phi^{-1}(r^{\rho}(t)) + \nu(t)\Phi^{-1}(w^{\rho}))}\Big) & \text{if } \quad \rho(t) < t. \end{cases} \tag{3.1}$$

Moreover, if

$$r^{\rho}(t)x(t))x^{\rho}(t) > 0 \quad \text{for} \quad t \in [a, b]_{\kappa}, \tag{3.2}$$

holds, then

$$\Phi^{-1}(r^{\rho}(t)) + \nu(t)\Phi^{-1}(w^{\rho}(t)) > 0 \tag{3.3}$$

for $t \in [a, b]_{\kappa}$.

Proof. Let $w = r\Phi(\frac{x^{\Delta}}{x})$. Then using (2.2), (2.3) and Lemma 2.3 we have (suppressing the argument t)

$$\begin{aligned} w^{\nabla} &= \frac{(r\Phi(x^{\Delta}))^{\nabla}\Phi(x) - r\Phi(x^{\Delta})\Phi^{\nabla}(x)}{\Phi(x^{\rho})\Phi(x)} \\ &= \frac{(r\Phi(x^{\Delta}))^{\nabla}(\Phi(x^{\rho}) + \nu\Phi^{\nabla}(x)) - r\Phi(x^{\Delta})\Phi^{\nabla}(x)}{\Phi(x^{\rho})\Phi(x)} \\ &= -c + \frac{[\nu(r\Phi(x^{\Delta}))^{\nabla} - r\Phi(x^{\Delta})]\Phi^{\nabla}(x)}{\Phi(x^{\rho})\Phi(x)} \\ &= -c - \frac{(r\Phi(x^{\Delta}))^{\rho}\Phi^{\nabla}(x)}{\Phi(x^{\rho})\Phi(x)} = -c - \frac{w^{\rho}\Phi^{\nabla}(x)}{\Phi(x)}. \end{aligned}$$

Now we have to distinguish two cases:

(i) Suppose that t is left dense. Then the nabla derivative reduces to the "normal derivative" ($\Phi^\nabla(x) = \Phi'(x)$) and the ρ-operator has no effect, so that

$$\begin{aligned} w^\nabla &= -c - w\frac{\Phi'(x)}{\Phi(x)} = -c - w\frac{(p-1)|x|^{p-2}x'}{\Phi(x)} = -c - (p-1)w\frac{x'}{x}\frac{\Phi^{-1}(r)}{\Phi^{-1}(r)} \\ &= -c - (p-1)w\frac{\Phi^{-1}(w)}{\Phi^{-1}(r)} = -c - (p-1)\frac{|w|^q}{\Phi^{-1}(r)} \end{aligned}$$

which is equation (3.1).
(ii) Suppose that t is left scattered. Then because of (2.2)

$$\begin{aligned} \frac{\Phi^\nabla(x)}{\Phi(x)} &= \frac{\Phi(x)-\Phi(x^\rho)}{\nu\Phi(x)} = \frac{1}{\nu}\Big(1 - \frac{\Phi(x^\rho)}{\Phi(x)}\Big) = \frac{1}{\nu}\Big(1 - \Phi\Big(\frac{x^\rho}{x^\rho + \nu(x^\Delta)^\rho}\Big)\Big) \\ &= \frac{1}{\nu}\Big(1 - \frac{1}{\Phi(1+\nu(\frac{x^\Delta}{x})^\rho)}\Big) = \frac{1}{\nu}\Big(1 - \frac{r^\rho}{\Phi(\Phi^{-1}(r^\rho)+\nu\Phi^{-1}(w^\rho))}\Big) \end{aligned}$$

which implies the second case of relation (3.1). The last fact we need to prove is the inequality $\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho) > 0$ for $t \in [a,b]_\kappa$. But

$$\begin{aligned} \Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho) &= \Phi^{-1}(r^\rho) + \nu\Phi^{-1}(r^\rho)\frac{x^{\Delta\rho}}{x^\rho} = \Phi^{-1}(r^\rho)\Big(1 + \nu\frac{x^{\Delta\rho}}{x^\rho}\Big) \\ &= \Phi^{-1}(r^\rho)\frac{x^\rho + \nu x^{\Delta\rho}}{x^\rho} = \Phi^{-1}(r^\rho)\frac{x^\rho + \nu x^\nabla}{x^\rho} = \Phi^{-1}(r^\rho)\frac{x}{x^\rho} \end{aligned}$$

and the last expression is positive if and only if (3.2) holds. ∎

In the next statement and also later we will denote by C^1_{ld} the class of functions $y : [a,b] \subset \mathbb{T} \to \mathbb{R}$ such that y^∇ exists and it is ld-continuous.

Theorem 3.2 (Picone's Identity). *Assume that w is a solution of the Riccati equation* (3.1) *on $[a,b]$. Let $y \in C^1_{ld}[a,b]$. Then for $t \in [a,b]$ (suppressing the argument)*

$$(w|y|^p)^\nabla = r^\rho|y^\nabla|^p - c|y|^p - G(y,w), \tag{3.4}$$

holds, where

$$G(y,w) = \begin{cases} \frac{p}{\Phi^{-1}(r)}\Big[\frac{|\Phi^{-1}(r)y^\nabla|^p}{p} - w\Phi(y)\Phi^{-1}(r)y^\nabla + \frac{|w\Phi(y)|^q}{q}\Big] & \text{if } \rho(t) = t \\ |r^\rho y^\nabla|^p - \frac{w^\rho r^\rho}{\nu\Phi(\Phi^{-1}(r^\rho)+\nu\Phi^{-1}(w^\rho))}|y^\rho + \nu y^\nabla|^p + \frac{w^\rho}{\nu}|y^\rho|^p & \text{if } \rho(t) < t. \end{cases} \tag{3.5}$$

Proof. First suppose that t is left dense, i.e. $\rho(t) = t$. Then

$$\begin{aligned} (w|y|^p)^\nabla &= w^\nabla|y|^p + w(|y|^p)^\nabla = \Big(-c - (p-1)\frac{|w|^q}{\Phi^{-1}(r)}\Big)|y|^p + pw\Phi(y)y^\nabla \\ &= r|y^\nabla|^p - c|y|^p - p\Big\{\frac{r|y^\nabla|^p}{p} - w\Phi(y)y^\nabla + \frac{1}{q}\frac{|w|^q|y|^p}{\Phi^{-1}(r)}\Big\} \\ &= r|y^\nabla|^p - c|y|^p - \frac{p}{\Phi^{-1}(r)}\Big\{\frac{|\Phi^{-1}(r)y^\nabla|^p}{p} - w\Phi(y)\Phi^{-1}(r)y^\nabla + \frac{|w\Phi(y)|^q}{q}\Big\}. \end{aligned}$$

For ls-point t we have (using (2.2) and (3.1))

$$\begin{aligned}
&(w|y|^p)^\nabla \\
&= w^\nabla |y|^p + w(|y|^p)^\nabla = \big[-c - \frac{w^\rho}{\nu}\big(1 - \frac{r^\rho}{\Phi(\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho))}\big)\big]|y|^p \\
&\quad + w^\rho \frac{|y|^p - |y^\rho|^p}{\nu} \\
&= r^\rho |y^\nabla|^p - c|y|^p + \frac{w^\rho r^\rho}{\nu\Phi(\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho))}|y|^p - \frac{w^\rho}{\nu}|y^\rho|^p - r^\rho|y^\nabla|^p \\
&= r^\rho |y^\nabla|^p - c|y|^p - \big\{r^\rho|y^\nabla|^p - \frac{w^\rho r^\rho}{\nu\Phi(\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho))}|y|^p + \frac{w^\rho}{\nu}|y^\rho|^p\big\};
\end{aligned}$$

i.e., (3.5) holds since $y = y^\rho + \nu y^\nabla$. □

Theorem 3.3. *Let the assumptions of the previous theorem be satisfied and, in addition, suppose that $y \in C^1[a,b]$ satisfies $\{r^\rho y y^\rho\}(t) > 0$ for $t \in \mathbb{T}_\kappa$. Then $G(y,w)(t) \geq 0$ for $t \in [a,b]_\kappa$, where the equality holds if and only if $w\Phi(y) = r\Phi(y^\Delta)$.*

Proof. Again first suppose that t is left dense. This implies $0 < \{r^\rho y y^\rho\}(t) = \{ry^2\}(t)$, so that $r(t) > 0$ and $\Phi^{-1}(r(t)) > 0$. We have

$$G(y,w) = \frac{p}{\Phi^{-1}(r)}\big\{\frac{|\Phi^{-1}(r)y^\nabla|^p}{p} - w\Phi(y)\Phi^{-1}(r)y^\nabla + \frac{|w\Phi(y)|^q}{q}\big\}.$$

This case is very easy to prove, because the expression in brackets is nonnegative according to Young's inequality (Lemma 2.1 with $u = \Phi^{-1}(r)y^\nabla, v = w\Phi(y)$). Equality occurs if and only if $v = \Phi(u)$, i.e., if and only if $w\Phi(y) = r\Phi(y^\Delta)$. And this equality holds iff w is related to y by the Riccati substitution.

Now suppose that t is ls-point. If we set $\alpha = \nu y^\nabla, \beta = y^\rho$, we can write the function G in variables α, β as

$$G(\alpha,\beta) = \frac{1}{\nu}\big\{\frac{r^\rho}{\nu^{p-1}}|\alpha|^p - \frac{w^\rho r^\rho}{\Phi(\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho))}|\alpha+\beta|^p + w^\rho|\beta|^p\big\}.$$

Note that the case $\alpha = -\beta$ cannot happen, otherwise the assumption $r^\rho y y^\rho > 0$ would be violated. Our aim is to prove that

$$\frac{r^\rho \nu^{1-p}|\alpha|^p + w^\rho|\beta|^p}{|\alpha+\beta|^p} \geq \frac{w^\rho r^\rho}{\Phi(\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho))}. \tag{3.6}$$

Left-hand side of the last inequality is homogeneous in variables α, β, i.e., it does not change by the transformation $\alpha \mapsto k\alpha, \beta \mapsto k\beta$ for any $k \in \mathbb{R} \setminus \{0\}$. For this reason, we can assume that $\alpha + \beta = \pm 1$, for example $\alpha + \beta = 1$. We will show that the minimum of the function $\widetilde{G}(\alpha,\beta) := \frac{r^\rho}{\nu^{p-1}}|\alpha|^p + w^\rho|\beta|^p$, provided $\alpha + \beta = 1$, is equal to the right-hand side of the inequality (3.6).

First we will express $\widetilde{G}$ as a function of one variable only using the condition $\alpha + \beta = 1$. So we have

$$\widetilde{G}(\alpha) = \frac{r^\rho}{\nu^{p-1}}|\alpha|^p + w^\rho|1-\alpha|^p.$$

The derivative of this function is

$$\widetilde{G}'(\alpha) = p\big\{\frac{r^\rho}{\nu^{p-1}}\Phi(\alpha) - w^\rho\Phi(1-\alpha)\big\}$$

with the only stationary point

$$\alpha^* = \frac{\nu\Phi^{-1}(w^\rho)}{\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho)}.$$

The second derivative is given by

$$\widetilde{G}''(\alpha) = p(p-1)\left\{\frac{r^\rho}{\nu^{p-1}}|\alpha|^{p-2} + w^\rho|1-\alpha|^{p-2}\right\} \tag{3.7}$$

and at the stationary point α^* satisfies

$$\begin{aligned}
&\widetilde{G}''(\alpha^*)\\
&= p(p-1)\frac{1}{|\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho)|^{p-2}}\big\{\frac{r^\rho}{\nu^{p-1}}|\nu\Phi^{-1}(w^\rho)|^{p-2} + w^\rho|\Phi^{-1}(r^\rho)|^{p-2}\big\}\\
&= p(p-1)\frac{1}{\nu|\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho)|^{p-2}}\left\{r^\rho|w^\rho|^{2-q} + \nu w^\rho|r^\rho|^{2-q}\right\}\\
&= p(p-1)\frac{|r^\rho w^\rho|^{2-q}}{\nu|\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho)|^{p-2}}\left\{\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho)\right\}
\end{aligned}$$

so that the sign of $\widetilde{G}''(\alpha^*)$ depends only on the last bracket. But its positiveness is equivalent to our assumption according to Lemma 3.1. This implies that α^* is a local minimum point of the function $\widetilde{G}$ and one can directly verify that the value $\widetilde{G}(\alpha^*)$ is just the expression on the right side of inequality (3.6). Finally, using the fact that $\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho) > 0$ and (3.7), by a similar computation as above one can verify that $\widetilde{G}''(\alpha) \geq 0$, i.e. $\widetilde{G}$ is convex and hence α^* is also the global minimum of $\widetilde{G}$. □

Oscillatory properties of (1.1) are defined via the concept of a generalized zero of a solution of this equation. We say that a solution x of equation (1.1) has a *generalized zero* at t if $x(t) = 0$ or, if t is ls-point, if $(r^\rho x x^\rho)(t) < 0$. We say that equation (1.1) is *disconjugate* on an interval $[a, b]$ if the nontrivial solution y satisfying $y(a) = 0$ has no generalized zero in $(a, b]$ and any other nontrivial solution of (1.1) has at most one generalized zeros in $[a, b]$.

Now, let us define $\mathbb{A}$ to be the set of functions

$$\mathbb{A} := \{y \in C^1_{ld}([a, b], \mathbb{R}) : y(a) = y(b) = 0\}.$$

Define the p-degree functional $\mathcal{F}$ on $\mathbb{A}$ by

$$\mathcal{F}(y; a, b) = \int_a^b \{r^\rho(t)|y^\nabla|^p - c(t)|y|^p\}\nabla t. \tag{3.8}$$

We say $\mathcal{F}$ is *positive definite* (and write $\mathcal{F} > 0$) on $\mathbb{A}$ provided $\mathcal{F}(y) \geq 0$ for all $y \in \mathbb{A}$ and $\mathcal{F}(y) = 0$ if and only if $y \equiv 0$.

The next theorem establishes basic methods of the oscillation theory of (1.1) and relates disconjugacy of this equation to the solvability of the Riccati equation (3.1) and positivity of the energy functional (3.8).

Theorem 3.4 (Roundabout Theorem). *The following statements are equivalent:*

(i) *Equation* (1.1) *is disconjugate on* $[a, b]$.
(ii) *Equation* (1.1) *has a positive solution on* $[a, b]$.
(iii) *The Riccati equation* (3.1) *has a solution* w *satisfying for all* $t \in [a, b]_\kappa$ *the inequality* $\{\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho)\}(t) > 0$.

(iv) *the p-degree functional $\mathcal{F}$ is positive definite on $\mathbb{A}$.*

Proof. We show the following four implications:
(i)⇒(ii): Let $\widetilde{y}$ be the solution of (1.1) satisfying the initial conditions $\widetilde{y}(a) = 0, \widetilde{y}^{\nabla}(a) = 1$. From (i) we get that $(r^\rho \widetilde{y}\widetilde{y}^\rho)(t) > 0$ for $t \in (a, b]$. Consider a solution y_ε given by the initial conditions (with $\varepsilon > 0$)

$$y_\varepsilon(a) = \varepsilon, \quad y_\varepsilon^{\Delta\rho}(a) = y_\varepsilon^{\nabla}(a) = \widetilde{\nu}(a)\Big(\frac{\varepsilon r^\rho(a) - 1}{r^\rho(a)} - \nu(a)\Big) + 1,$$

where $\widetilde{\nu} = 0$ if $\nu = 0$ and $\widetilde{\nu} = \frac{1}{\nu}$ if $\nu > 0$. Then $y_\varepsilon \to \widetilde{y}$ for $\varepsilon \to 0$. Hence, if we choose $\varepsilon > 0$ sufficiently small, then $y \equiv y_\varepsilon$ fulfills $(r^\rho y y^\rho)(t) > 0$ for $t \in (a, b]$. Moreover, for ls-point a we get $(r^\rho y y^\rho)(a) = \varepsilon (r^\rho(a))^2 > 0$ because

$$y^{\nabla}(a) = \big(\frac{y - y^\rho}{\nu}\big)(a) = \frac{\varepsilon r^\rho(a) - 1}{\nu(a) r^\rho(a)}$$

by (2.1). In the case when a is ld-point we have

$$(r^\rho y y^\rho)(a) = (r y^2)(a) = r(a)\varepsilon^2,$$

which is positive if and only if $r(a) > 0$. Suppose conversely that $r(a) < 0$. Consider a solution $\widehat{y}$ that satisfies the initial conditions $\widehat{y}(d) = 0, \widehat{y}^{\Delta}(d) = 1$, where $d \in (a, b]$. The disconjugacy of the equation (1.1) implies $(r^\rho \widehat{y}\widehat{y}^\rho)(a) > 0$. Since a is left dense, we get $r(a) > 0$ which is contradiction. Altogether, y is the solution of (1.1) with $(r^\rho y y^\rho)(t) > 0$ for $t \in [a, b]$, so that (ii) holds.
(ii)⇒(iii): This implication is the Riccati substitution already proved in Lemma 3.1.
(iii)⇒(iv): Suppose that w is a solution of Riccati equation (3.1) satisfying the inequality $\{\Phi^{-1}(r^\rho) + \nu\Phi^{-1}(w^\rho)\}(t) > 0$ for $t \in [a, b]_\kappa$. Let $y \in \mathbb{A}$, i.e. $y(a) = y(b) = 0$. From the Picone identity we have

$$r^\rho(t)|y^{\nabla}|^p - c(t)|y|^p = (w(t)|y|^p)^{\nabla} + G(y, w)$$

and by integrating from a to b we obtain

$$\begin{aligned}\mathcal{F}(y; a, b) &= \int_a^b \{r^\rho(t)|y^{\nabla}|^p - c(t)|y|^p\}\nabla t \\ &= [w(t)|y|^p]_a^b + \int_a^b G(y, w)\nabla t = \int_a^b G(y, w)\nabla t.\end{aligned}$$

Hence $\mathcal{F}(y; a, b) \geq 0$ because of Theorem 3.3 and, moreover, the case $\mathcal{F}(y; a, b) = 0$ can occur if and only if $w\Phi(y) = r\Phi(y^{\Delta})$, i.e. $y^{\Delta} = \Phi^{-1}(w/r)y$. But since $y(a) = 0$, the initial value problem admits only the trivial solution. Consequently, $\mathcal{F}(y; a, b) > 0$ for all nontrivial $y \in \mathbb{A}$.
(iv)⇒(i): Suppose, by contradiction, that $\mathcal{F}(y; a, b) > 0$ and (1.1) is not disconjugate on $[a, b]$. Then either the nontrivial solution $\tilde{y}$ of (1.1) given by the initial condition $y(a) = 0$ has a generalized zero in $(a, b]$ or there is a nontrivial solution y of (1.1) such that y has at least two generalized zeros in $(a, b]$. Consider the latter possibility, the former one can be treated in a similar way. Let $\alpha, \beta \in (a, b]$, where $\alpha < \rho(\beta)$, be two smallest generalized zeros of y in $(a, b]$. There are four possibilities according to whether α, β are ld- or ls-points. We consider here the case when β ld-point (i.e., $\rho(\beta) = \beta$) and we construct a nontrivial piecewise C^1_{ld} continuous function y with $y(a) = 0 = y(b)$, such that $\mathcal{F}(y; a, b) \leq 0$. If the remaining two

possibilities happen ($\rho(\beta) < \beta$), we proceed in a similar way as in the remaining part of the proof.

First suppose that α is ls-point and define

$$u(t) = \begin{cases} 0 & \text{for } t \in [a, \alpha) \\ y(t) & \text{for } t \in [\alpha, \beta] \\ 0 & \text{for } t \in (\beta, b] \end{cases}$$

which implies $u \in \mathbb{A}$ and $u(t) \neq 0$ for $t \in (\alpha, \beta)$.

Using integration by parts, the definition of u, the fact that $\int_{\rho(t)}^{t} f(s)\nabla s = f(t)\nu(t)$, and that $(r\Phi(y^\Delta))(\alpha) = (r\Phi(y^\Delta))^\rho(\alpha) + \nu(\alpha)(r\Phi(y^\Delta))^\nabla(\alpha)$, we have

$$\begin{aligned}
&\mathcal{F}(u; a, b) \\
&= \int_a^b \left[r^\rho(t)|u^\nabla|^p - c(t)|u|^p\right] \nabla t \\
&= \int_{\rho(\alpha)}^{\alpha} \left[r^\rho(t)|u^\nabla|^p - c(t)|u|^p\right] \nabla t + \int_\alpha^\beta \left[r^\rho(t)|u^\nabla|^p - c(t)|u|^p\right] \nabla t \\
&= \left\{(r^\rho|u^\nabla|^p - c|u|^p)\nu\right\}(\alpha) + ur\Phi(u^\Delta)\big|_\alpha^\beta - \int_\alpha^\beta u\left[(r(t)\Phi(u^\Delta))^\nabla + c(t)\Phi(u)\right] \nabla t \\
&= \left\{\nu r^\rho |u^\nabla|^p\right\}(\alpha) - u\left\{c\Phi(u)\nu + (r\Phi(u^\Delta))^\rho + \nu(r\Phi(u^\Delta))^\nabla\right\}(\alpha) \\
&= r^\rho(\alpha)\left|\frac{u(\alpha) - u^\rho(\alpha)}{\nu(\alpha)}\right|^p \nu(\alpha) - y(\alpha)r^\rho(\alpha)\Phi(y^\nabla(\alpha)) \\
&= \frac{y(\alpha)r^\rho(\alpha)}{\Phi(\nu(\alpha))}\left[\Phi(y(\alpha)) - \Phi(\nu(\alpha)y^\nabla(\alpha))\right].
\end{aligned}$$

Hence, it suffices to show that

$$\left\{yr^\rho\Phi(y) - yr^\rho\Phi(\nu y^\nabla)\right)(\alpha) \leq 0. \tag{3.9}$$

This inequality is equivalent to the inequality

$$\left\{\Phi^{-1}(yr^\rho)(y - \nu y^\nabla)\right\}(\alpha) = \left\{\Phi^{-1}(yr^\rho\Phi(y^\rho)\right\}(\alpha) \leq 0,$$

but this inequality is just definition of the generalized zero at α what is our assumption, so (3.9) holds and hence $\mathcal{F}(u; a, b) \leq 0$, a contradiction.

Now suppose that α is an ld-point, i.e. $\rho(\alpha) = \alpha$. Since $r(t) \neq 0$, the inequality $\{r^\rho y^\rho y\}(\alpha) \leq 0$ means that either $y(\alpha) = 0$ or $r(\alpha) < 0$. If $y(\alpha) = 0$, the same function u as in the previous part of the proof gives $\mathcal{F}(u; a, b) = 0$, a contradiction, so we suppose that $y(\alpha) \neq 0$ and $r^\rho(\alpha) < 0$. In this case we proceed in the same way as in the continuous case (see, e.g. [14]). Let $t_m \to \alpha-$, as $m \to \infty$, be the left-sequence for α and put

$$u_m(t) = \begin{cases} \frac{t - t_m}{(\alpha - t_m)^{1/p}} & \text{for } t \in [t_m, \alpha] \cap \mathbb{T}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, by the same computation as in [12, 13] yields

$$\mathcal{F}(u_m; a, b) \to r^\rho(\alpha) < 0 \quad \text{as } m \to \infty,$$

a contradiction. $\square$

Remark 3.5. (i) The previous theorem implies that the Sturm Comparison theorem extends verbatim to (1.1). In particular, let the equation

$$(R(t)\Phi(x^{\Delta}))^{\nabla} + C(t)\Phi(x) = 0 \tag{3.10}$$

be a Sturmian majorant of (1.1) on $[a, b]$, i.e.,

$$0 < R(t) \le r(t), \quad C(t) \ge c(t), \quad t \in [a, b].$$

If (1.1) is not disconjugate on $[a, b]$, i.e., there exists a nontrivial function $y \in \mathbb{A}$ such that

$$\mathcal{F}_{rc}(y; a, b) = \int_a^b \left[r^{\rho}(t)|y^{\nabla}|^p - c(t)|y|^p\right] \nabla t \le 0,$$

then also

$$\mathcal{F}_{RC}(y; a, b) = \int_a^b \left[R^{\rho}(t)|y^{\nabla}|^p - C(t)|y|^p\right] \nabla t \le 0,$$

and hence (3.10) is not disconjugate as well. Conversely, if (3.10) is disconjugate on $[a, b]$, i.e. $\mathcal{F}_{RC}(y; a, b) > 0$ for every $0 \not\equiv y \in \mathbb{A}$, then $\mathcal{F}_{rc}(y; a, b) > 0$ and (1.1) is also disconjugate on $[a, b]$.
(ii) Theorem 3.4 also shows that (1.1) does not admit coexistence of a solution without generalized zero in $[a, b]$ and a solution having two or more generalized zeros in this interval. Indeed, the existence of a solution of (1.1) without a generalized zero in $[a, b]$ implies $\mathcal{F}_{rc}(y; a, b) > 0$ for $0 \not\equiv y \in \mathcal{A}$, while the existence of a solution with two or more generalized zeros enables to construct a function $0 \not\equiv \tilde{y} \in \mathbb{A}$ for which $\mathcal{F}_{RC}(\tilde{y}; a, b) \le 0$.
(iii) The previous remark also justifies the classification of (1.1) on time scales unbounded above as oscillatory and non-oscillatory in the same way as the classical linear Sturm-Liouville differential equations.

4. Oscillation and nonoscillation criteria

Throughout this section we suppose that a time scale under consideration is *unbounded above*; i.e., there exists a sequence $t_n \in \mathbb{T}$ such that $t_n \to \infty$.

Equation (1.1) is said to be *nonoscillatory* if there exists $\alpha \in \mathbb{T}$ such that (1.1) is disconjugate on $[\alpha, \beta]$ for every $\beta > \alpha$. In the opposite case, (1.1) is said to be *oscillatory.*

As a direct consequence of the equivalence (i) and (iv) in the Roundabout Theorem, we have the following statement.

Lemma 4.1. *Equation* (1.1) *is nonoscillatory if and only if there exists $a \in \mathbb{T}$ such that*

$$\mathcal{F}(y; a, \infty) = \int_a^{\infty} \{r^{\rho}|y^{\nabla}|^p - c|y|^p\}(t)\nabla t > 0$$

for every nontrivial $y : [a, \infty) \to \mathbb{R}$ with y^{∇} piecewise ld-continuous, satisfying $y(a) = 0$, and for which there exists $d > a$ with $y(t) \equiv 0$ for $t > d$.

Theorem 4.2 (Leighton-Wintner criterion)**.** *Suppose that $r(t) > 0$ for large t*

$$\int^{\infty} (r^{\rho}(t))^{1-q}\nabla t = \infty \quad \textit{and} \quad \int^{\infty} c(t)\nabla t = \infty. \tag{4.1}$$

Then equation (1.1) *is oscillatory.*

Proof. Let $a \in \mathbb{T}$ be arbitrary and $t_1, t_2, t_3, t_4 \in [a, \infty)$ be such that $a \le t_1 < t_2 < t_3 < t_4$. Define function y by

$$y(t) = \begin{cases} 0 & \text{for } t \in [a, t_1], \\ f(t) & \text{for } t \in [t_1, t_2], \\ 1 & \text{for } t \in [t_2, t_3], \\ g(t) & \text{for } t \in [t_3, t_4], \\ 0 & \text{for } t \in [t_4, \infty), \end{cases}$$

where f, g are given by the formulas

$$f(t) = \frac{\int_{t_1}^{t} (r^\rho(s))^{1-q} \nabla s}{\int_{t_1}^{t_2} (r^\rho(s))^{1-q} \nabla s}, \quad g(t) = \frac{\int_{t}^{t_3} (r^\rho(s))^{1-q} \nabla s}{\int_{t_3}^{t_4} (r^\rho(s))^{1-q} \nabla s};$$

i.e., they satisfy the boundary conditions $f(t_1) = 0, f(t_2) = 1, g(t_3) = 1, g(t_4) = 0$. This yields $y(t_1) = y(t_4) = 0, y(t) > 0$ for $t \in (t_1, t_4)$ and y^∇ is piecewise ld-continuous. It holds

$$f^\nabla(t) = \frac{(r^\rho(t))^{1-q}}{\int_{t_1}^{t_2} (r^\rho(s))^{1-q} \nabla s}, \quad g^\nabla(t) = \frac{(r^\rho(t))^{1-q}}{\int_{t_3}^{t_4} (r^\rho(s))^{1-q} \nabla s},$$

and consequently, using integration by parts,

$$\begin{aligned} \int_a^\infty r^\rho(t) |y^\nabla(t)|^p \nabla t &= \int_{t_1}^{t_4} r^\rho(t) |y^\nabla(t)|^p \nabla t \\ &= \int_{t_1}^{t_2} r^\rho(t) \Phi(f^\nabla(t)) f^\nabla(t) \nabla t + \int_{t_3}^{t_4} r^\rho(t) \Phi(g^\nabla(t)) g^\nabla(t) \nabla t \\ &= [r^\rho(t) \Phi(f^\nabla(t)) f(t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} (r^\rho(t) \Phi(f^\nabla(t)))^\nabla f^\rho \nabla t \\ &\quad + [r^\rho(t) \Phi(g^\nabla(t)) g(t)]_{t_3}^{t_4} - \int_{t_3}^{t_4} (r^\rho(t) \Phi(g^\nabla(t)))^\nabla g^\rho \nabla t \\ &= r^\rho(t_2) \Phi(f^\nabla(t_2)) f(t_2) - r^\rho(t_3) \Phi(g^\nabla(t_3)) g(t_3) \\ &= \Big(\int_{t_1}^{t_2} (r^\rho)^{1-q}(t) \nabla t \Big)^{1-p} + \Big(\int_{t_3}^{t_4} (r^\rho)^{1-q}(t) \nabla t \Big)^{1-p}. \end{aligned}$$

Now we compute the second term in $\mathcal{F}(y; a, \infty)$ by Lemma 2.4 (with $\hat{c}, \hat{g}$ defined in the same way as $\hat{f}$ in Lemma 2.4. We obtain

$$\int_{t_3}^{t_4} c(t) g^p(t) \, \nabla t = \int_{t_3}^{t_4} \hat{c}(t) \hat{g}^p(t) \, \Delta t = \int_{t_3}^{t_4} \hat{c}(t) g^p(\sigma(t)) \, \Delta t$$

since the function g is continuous. Using the second mean value theorem of integral calculus (Lemma 2.2) there exists $s_2 > t_3$ such that

$$\int_{t_3}^{t_4} \hat{c}(t) g^p(\sigma(t)) \, \Delta t = \int_{t_3}^{s_2} \hat{c}(t) \, \Delta t \le \int_{t_3}^{s_2} c(t) \, \nabla t.$$

By the same argument, there exists $s_1 \in (t_1, t_2)$ such that

$$\int_{t_1}^{t_2} c(t) f^p(t) \, \nabla t \le \int_{s_1}^{t_2} c(t) \, \nabla t.$$

Summarizing the previous computations, we get

$$\mathcal{F}(y;a,\infty) \le \Big(\int_{t_1}^{t_2} (r^\rho)^{1-q}(t)\nabla t\Big)^{1-p} + \Big(\int_{t_3}^{t_4} (r^\rho)^{1-q}(t)\nabla t\Big)^{1-p} - \int_{s_1}^{s_2} c^\rho(t)\nabla t.$$

Now, if t_1, t_2 are fixed, for sufficiently large t_3, t_4 the assumptions (4.1) of this theorem imply that $\mathcal{F}(y;a,\infty) < 0$. ☐

When the assumption of the previous theorem concerning the divergence of the integral $\int^\infty c(t)\nabla t$ is violated, the next criterion applies.

Theorem 4.3. *Suppose that $r(t) > 0$ for large t,*

$$\int^\infty (r^\rho(t))^{1-q}\nabla t = \infty,$$

and

$$\liminf_{t\to\infty} \Big(\int_a^t (r^\rho)^{1-q}(s)\nabla s\Big)^{p-1}\Big(\int_t^\infty c(s)\nabla s\Big) > 1. \tag{4.2}$$

Then equation (1.1) *is oscillatory.*

Proof. Define the function y in the same way as in the previous proof. Then $\mathcal{F}(y;a,\infty)$ satisfies

$$\begin{aligned}\mathcal{F}(y;a,\infty) &\le \Big(\int_{t_1}^{t_2} (r^\rho(t))^{1-q}\,\nabla t\Big)^{1-p} + \Big(\int_{t_3}^{t_4} (r^\rho(t))^{1-q}\,\nabla t\Big)^{1-p} - \int_{s_1}^{s_2} c(t)\,\nabla t\\ &= \Big(\int_{t_1}^{t_2} (r^\rho(t))^{1-q}\,\nabla t\Big)^{1-p}\\ &\quad\times\Big[1 - \Big(\int_{t_1}^{t_2} (r^\rho(t))^{1-q}\,\nabla t\Big)^{p-1}\int_{s_1}^{s_2} c(t)\nabla t + \Big(\frac{\int_{t_1}^{t_2}(r^\rho)^{1-q}(t)\nabla t}{\int_{t_3}^{t_4}(r^\rho(t))^{1-q}\,\nabla t}\Big)^{p-1}\Big].\end{aligned}$$

It is not so difficult to show that if (4.2) holds, then the expression in brackets is negative for sufficiently large $t_2 < t_3 < t_4$. This proof is exactly the same as for differential equation, i.e. $\mathbb{T} = \mathbb{R}$, see [6]. ☐

In the proof of the next nonoscillation criterion for (1.1) we will need the following refinement of the Riccati equivalence of (i) and (iii) in Theorem 3.4. We will denote by $\mathcal{R}[w]$ the so-called Riccati operator (compare (3.1)), i.e.,

$$\mathcal{R}[w] := \begin{cases} w^\nabla + c(t)(p-1)r^{1-q}(t)|w|^q & \text{if } t = \rho(t),\\ w^\nabla + c(t) - \frac{w^\rho}{\nu(t)}\big(1 - \frac{r^\rho(t)}{\Phi(\Phi^{-1}(r^\rho(t))+\nu(t)\Phi^{-1}(w^\rho))}\big) & \text{if } \rho(t) < t.\end{cases} \tag{4.3}$$

and by $\mathcal{L}(x)$ the left-hand side of (1.1), i.e.,

$$\mathcal{L}(x) := (r(t)\Phi(x^\Delta))^\nabla + c(t)\Phi(x).$$

The proof of the next lemma follows the same idea as in the continuous case, but for the reader's convenience we present here the main ideas of this proof.

Lemma 4.4. *Equation* (1.1) *is nonoscillatory if and only if there exists a ∇-differentiable function w satisfying* (3.3) *such that $\mathcal{R}[w] \le 0$ for large t.*

Proof. The implication "⇒" is trivial since it is only a restatement of the Riccati equivalence (i) $\iff$ (iii) for large t. To prove the opposite implication, suppose that there exists a function w satisfying assumptions of the lemma on an interval $[T, \infty)$. To prove that (1.1) is nonoscillatory, we will construct a nonoscillatory majorant of this equation in such a way that w is a solution of the Riccati equation associated to this majorant equation.

Let y be the solution of the initial value problem

$$y^{\Delta} = r^{1-q}(t)\Phi^{-1}(w(t))y, \quad y(T) = 1,$$

where T is sufficiently large. Using the computation at the beginning of Lemma 3.1 we have

$$\mathcal{R}[w] = w^{\nabla} + c(t) + \frac{r^{\rho}\Phi^{\rho}(y^{\Delta})(\Phi(y))^{\nabla}}{\Phi(y^{\rho})\Phi(y)}.$$

Then we have, again following the computation in the proof of Lemma 3.1, in particular, splitting the cases $\rho(t) < t$ and $\rho(t) = t$,

$$0 \geq y^{p}\mathcal{R}[w] = y^{p}\Big[w^{\nabla} + c(t) + \frac{r^{\rho}\Phi^{\rho}(y^{\Delta})(\Phi(y))^{\nabla}}{\Phi(y^{\rho})\Phi(y)}\Big] = y\mathcal{L}(y).$$

Now, let $\tilde{c}(t) := c(t) - \frac{y(t)\mathcal{L}[y](t)}{y^{p}(t)}$. Then $\tilde{c}(t) \geq c(t)$ and y is a solution of the equation (which is a Sturmian majorant of (1.1))

$$(r(t)\Phi(y^{\Delta}))^{\nabla} + \tilde{c}(t)\Phi(y) = 0 \tag{4.4}$$

for which $r^{\rho}(t)y^{\rho}(t)y(t) > 0$ for large t, i.e., (4.4) is nonoscillatory and hence (1.1) is nonoscillatory as well. □

Now we apply Lemma 4.4 to prove the Hille-Nehari-type nonoscillation criterion for (1.1). The idea of the proof is the same as in the continuous case $\mathbb{T} = \mathbb{R}$, but the particularities of time scale calculus require some additional assumptions (which are automatically satisfied for $\mathbb{T} = \mathbb{R}$) and also some technical modifications, compare the proof of [6, Theorem 2.1].

Theorem 4.5. *Suppose that $r(t) > 0$ for large t, $\int^{\infty} (r^{\rho}(t))^{1-q}\,\nabla t = \infty$, the integral $\int^{\infty} c(t)\,\nabla t$ is convergent,*

$$\lim_{t\to\infty} \frac{\nu(t)\,[r^{\rho}(t)]^{1-q}}{\int_a^{\rho(t)} [r^{\rho}(s)]^{1-q}\,\nabla s} = 0, \tag{4.5}$$

$$\liminf_{t\to\infty} \Big(\int_{t_0}^{\rho(t)} (r^{\rho}(s))^{1-q}\nabla s\Big)^{p-1}\Big(\int_{\rho(t)}^{\infty} c(s)\,\nabla s\Big) > -\frac{2p-1}{p}\big(\frac{p-1}{p}\big)^{p-1} \tag{4.6}$$

$$\limsup_{t\to\infty} \Big(\int_{t_0}^{\rho(t)} (r^{\rho}(s))^{1-q}\nabla s\Big)^{p-1}\Big(\int_{\rho(t)}^{\infty} c(s)\,\nabla s\Big) < \frac{1}{p}\big(\frac{p-1}{p}\big)^{p-1}, \tag{4.7}$$

then (1.1) *is nonoscillatory.*

Proof. By the previous lemma, we will construct a function w such that $\mathcal{R}[w](t) \leq 0$ for large t. To this end, we denote (for the notational convenience)

$$\tilde{r} := r^{\rho}, \quad \tilde{w} = w^{\rho},$$

we also denote

$$\mathcal{A}(t) := \Big(\int_{t_0}^{t} \tilde{r}^{1-q}(s)\nabla s\Big)^{p-1}\Big(\int_{t}^{\infty} c(s)\,\nabla s\Big).$$

Let

$$w(t)=\Big(\frac{p-1}{p}\Big)^{p-1}\Big(\int_{t_0}^t \tilde r^{1-q}(s)\,\nabla s\Big)^{1-p}+\int_t^\infty c(s)\,\nabla s.$$

Using Lemma 2.5 (a ∇-chain rule for differentiation) we have

$$\Big[\Big(\int_{t_0}^t \tilde r^{1-q}(t)\nabla t\Big)^{1-p}\Big]^\nabla=(1-p)\tilde r^{1-q}(t)\theta^{-p}(t),$$

where

$$\int_{t_0}^{\rho(t)} \tilde r^{1-q}(t)\,\nabla t\le\theta(t)\le\int_{t_0}^t \tilde r^{1-q}(t)\nabla t.$$

Also, using the Lagrange mean value we have

$$\begin{aligned}\frac{\tilde w}{\nu}\Big(1-\frac{\tilde r}{\Phi\left(\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)\right)}\Big)&=\frac{\tilde w}{\nu}\frac{\Phi\left(\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)\right)-\Phi(\Phi^{-1}(\tilde r)}{\Phi\left(\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)\right)}\\&=(p-1)\frac{|\eta|^{p-2}|\tilde w|^q}{\Phi\left(\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)\right)},\end{aligned}$$

where η is between $\Phi^{-1}(\tilde r)$ and $\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)$. By adding $(\frac{p-1}{p})^p$ to the pair of inequalities

$$-\frac{2p-1}{p}\Big(\frac{p-1}{p}\Big)^{p-1}<\mathcal A^\rho(t)<\frac1p\Big(\frac{p-1}{p}\Big)^{p-1},$$

we obtain

$$-\Big(\frac{p-1}{p}\Big)^p<\mathcal A^\rho(t)+\Big(\frac{p-1}{p}\Big)^p<\Big(\frac{p-1}{p}\Big)^p.$$

More precisely, (4.6) implies the existence of $\varepsilon>0$ such that

$$\big|\mathcal A^\rho(t)+\Big(\frac{p-1}{p}\Big)^p\big|^q(1+\varepsilon)<\Big(\frac{p-1}{p}\Big)^p$$

for large t. Now we will estimate the quantity

$$|\tilde w|^q=\Big(\int_{t_0}^{\rho(t)} \tilde r^{1-q}(s)\,\nabla s\Big)^{-p}\big|\mathcal A^\rho(t)+\Big(\frac{p-1}{p}\Big)^p\big|^q.$$

(I) First consider the case that t is ld-point, i.e. $\rho(t)<t$. Using the previous computations, we obtain

$$\begin{aligned}\mathcal R[w]&=w^\nabla+c+\frac{\tilde w}{\nu}\Big(1-\frac{\tilde r}{\Phi\left(\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)\right)}\Big)\\&=-(p-1)\Big(\frac{p-1}{p}\Big)^p|\theta|^{-p}\tilde r^{1-q}-c+c+(p-1)\frac{|\eta|^{p-2}|\tilde w|^q}{\Phi\left(\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)\right)}\\&\le(p-1)\tilde r^{1-q}\Big[-\Big(\frac{p-1}{p}\Big)^p\Big(\int_{t_0}^t \tilde r^{1-q}(s)\,\nabla s\Big)^{-p}\\&\quad+\Big(\int_{t_0}^{\rho(t)} \tilde r^{1-q}(s)\,\nabla s\Big)^{-p}\frac{|\eta|^{p-2}\tilde r^{q-1}(t)\left|(\frac{p-1}{p})^p+\mathcal A^\rho(t)\right|^q}{\Phi\left(\Phi^{-1}(\tilde r)+\nu\Phi^{-1}(\tilde w)\right)}\Big]\\&=\frac{(p-1)\tilde r^{1-q}}{\Big(\int_{t_0}^t\tilde r^{1-q}(s)\,\nabla s\Big)^p}\Big[-\Big(\frac{p-1}{p}\Big)^p+\mathcal B(t)\Big|\mathcal A^\rho(t)+\Big(\frac{p-1}{p}\Big)^p\Big|^q\Big],\end{aligned}$$

where

$$\mathcal{B}(t) := \Big(\frac{\int_{t_0}^{t} \tilde{r}^{1-q}(s)\,\nabla s}{\int_{t_0}^{\rho(t)} \tilde{r}^{1-q}(s)\,\nabla s}\Big)^p \frac{|\eta|^{p-2}\tilde{r}^{q-1}}{\Phi\left(\Phi^{-1}(\tilde{r})+\nu\Phi^{-1}(\tilde{w})\right)} \to 1$$

as $t \to \infty$, in particular, for any $\varepsilon > 0$, $\mathcal{B}(t) < (1+\varepsilon)$ for large t. Indeed, consider the case $p > 2$, the case $p \in (1,2)$ can be treated analogously. Using the fact that

$$\Phi(\tilde{r}) - \nu|\Phi(\tilde{w})| \le \eta \le \ \Phi(\tilde{r}) + \nu|\Phi(\tilde{w})|,$$

and that

$$\begin{aligned}
&\nu(t)\Big|\frac{\tilde{w}(t)}{\tilde{r}(t)}\Big|^{q-1} \\
&= \nu(t)\frac{\Big|(\frac{p-1}{p})^p\Big[\int_{t_0}^{\rho(t)} \tilde{r}^{1-q}(s)\,\nabla s\Big]^{1-p} + \int_{\rho(t)}^{\infty} c(s)\,\nabla s\Big|^{q-1}}{\tilde{r}^{q-1}(t)} \\
&= \frac{\nu(t)\tilde{r}^{1-q}(t)}{\int_{t_0}^{\rho(t)} \tilde{r}^{1-q}(s)\nabla s}\Big|(\frac{p-1}{p})^p + \Big(\int_{t_0}^{\rho(t)} \tilde{r}^{-1}(s)\,\nabla s\Big)^{p-1}\Big(\int_{\rho(t)}^{\infty} c(s)\,\nabla s\Big)\Big|^{q-1} \to 0.
\end{aligned}$$

as $t \to \infty$, since the second term in the last expression is bounded (see (4.6)) and the first one goes to zero by (4.5). Hence

$$\begin{aligned}
|\mathcal{B}(t)| &\le \left(\frac{\int_{t_0}^{\rho(t)} \tilde{r}^{1-q}(s)\,\nabla s + \nu(t)\tilde{r}^{1-q}(t)}{\int_{t_0}^{\rho(t)} \tilde{r}^{1-q}(s)\,\nabla s}\right)^p \frac{|\Phi^{-1}(\tilde{r})+\nu\Phi^{-1}(\tilde{w})|^{p-2}\tilde{r}^{q-1}}{\Phi\left(\Phi^{-1}(\tilde{r})+\nu\Phi^{-1}(\tilde{w})\right)} \\
&= \left(1+\frac{\nu(t)\tilde{r}^{1-q}(t)}{\int_{t_0}^{\rho(t)} \tilde{r}^{1-q}(s)\,\nabla s}\right)^p \frac{\tilde{r}^{q-1+(p-2)(q-1)}\left|1+\nu\Phi^{-1}(\tilde{w}/\tilde{r})\right|^{p-2}}{\tilde{r}\Phi\left(1+\nu\Phi^{-1}(\tilde{w}/\tilde{r})\right)^{p-1}} \\
&= \left(1+\frac{\nu(t)\tilde{r}^{1-q}(t)}{\int_{t_0}^{\rho(t)} \tilde{r}^{1-q}(s)\,\nabla s}\right)^p \frac{1}{1+\Phi^{-1}(\tilde{w}/\tilde{r})} \to 1, \quad \text{as } t \to \infty.
\end{aligned}$$

Summarizing all estimates, if t_0 is so large that all statements claimed to hold for large t hold for $t \ge t_0$, we have

$$\mathcal{R}[w] \le \Big(\frac{\int_{t_0}^{t} \tilde{r}^{1-q}(s)\,\nabla s}{\int_{t_0}^{\rho(t)} \tilde{r}^{1-q}(s)\,\nabla s}\Big)^p \Big[-(\frac{p-1}{p})^p + \Big|(\frac{p-1}{p})^p + \mathcal{A}^{\rho}(t)\Big|^q(1+\varepsilon)\Big] < 0$$

for large t.
(II) Case $\rho(t) = t$. This case is now easy to treat, since then $\tilde{w} = w$,

$$w^{\nabla} = (1-p)\left(\frac{p-1}{p}\right)^{p-1}\Big(\int_{t_0}^{t} \tilde{r}^{1-q}(s)\,\nabla s\Big)^{-p}\tilde{r}^{1-q}(t) - c(t)$$

and an easy modification of the previous computation shows that

$$\mathcal{R}[w] = w^{\nabla} + c(t) + (p-1)r^{1-q}(t)|w|^q \le 0$$

for large t. $\square$

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